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Forward commodity trading with private information

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We consider the use of forward contracts to reduce risk for firms operating in a spot market. Firms have private information on the distribution of prices in the spot market. We discuss different ways in which firms may agree on a bilateral forward contract: either through direct negotiation or through a broker. We introduce a form of supply-function equilibrium in which two firms each offer a supply function and the clearing price and quantity for the forward contracts are determined from the intersection. In this context a firm can use the offer of the other player to augment its own information about the future price.

Key words: Forward contracts, Nash bargaining, Supply-function equilibrium, Wholesale electricity markets

1. Introduction

Forward contracting is a common arrangement between firms who seek to minimize the risk of price variation when trading commodities. A firm that wishes to sell a commodity that is delivered in the future might arrange a forward price with the buyer so that both have some certainty on what will be exchanged when the contract is settled. In this paper we are interested in such contracts that are negotiated over the counter by a seller and a buyer, as opposed to being traded in an exchange.

Such bilateral contracting is a common feature of many wholesale electricity markets that are characterized by occasional very high prices in the spot market caused by restrictions on the storage of electricity. It is no surprise that in such markets forward contracts are signed between retailers (purchasers in the spot market) and generators (sellers in the spot market) in order to reduce the risk that arises from price spikes. The most common contracts take the form of financial instruments with payments depending on the price of electricity. Retailers sell power at a fixed price, but buy at the spot price, so a forward contract that fixes the price for the contract quantity helps to protect retailers from price spikes. Generators have an opposite set of incentives, with a forward contract guaranteeing their income against a situation where prices drop. Thus the behavior of the forward market will depend on the degree of risk aversion of the participants.

A popular form of contract in electricity markets is called a *contract for differences* (CfD). If a contract trades at a price f then a firm i buying an amount Q of this contract from a firm j agrees to pay an amount $Q(f-p)$ to firm j (and receives this amount if is negative) where p is the average price over a specified period of time. The average price p may be calculated over all hours, over peak hours only, or on the basis of a profile of average demand. The period of time in question is often of three months duration, or shorter. A CfD may be traded in a futures market (with daily mark-to-market payments being made), or it may be traded through a broker of some sort, or it may be an over-the-counter agreement between two firms. More details on forward market arrangements in different jurisdictions can be found in the literature (e.g. for Australia [2], for the UK [14], for the US [9], [17], for Nordpool [6]).

In the electricity market literature there has been relatively little attention paid to the process by which contracts are negotiated, and the results of such negotiations. Although they can be traded in an exchange, most CfD contracts are bilateral, and involve a seller and a buyer settling a contract quantity Q and contract price f through some bargaining process. For example, this may happen when the energy manager of a large consumer contacts her counterpart in an electricity generator and seeks to settle a contract price and quantity over the telephone. In such a negotiation the participants may have different views of future price outcomes. If negotiators are risk neutral then different beliefs about $\mathbb{E}[p]$ would result in infinite contract quantities (since these are financial contracts without any physical delivery), so some form of risk aversion is needed to ensure finite outcomes. Contracting increases welfare by reducing risk, and by settling on Q and f, the agents arrive at a value for this surplus and how it should be distributed. The classical solution (based on an axiomatic approach) is the *Nash bargaining solution* [24].

Instead of direct contact between buyer and seller, contracts can be arranged by a broker who mediates between them. The simplest approach, which we call the *broker mechanism with non-strategic offers,* involves no explicit negotiation. The seller constructs an increasing supply curve $Q(f)$ of contract quantities that maximizes his expected utility, and the buyer constructs a decreasing demand curve $D(f)$ of contract purchases that maximizes her expected utility. They supply these curves to a broker, and agree with the broker to settle on a contract price f for which supply equals demand.

In this paper we compare the broker mechanism with a *direct negotiation model* in which firms negotiate with each other, without using a broker, through discussion of potential price quantity pairs at which they may agree a contract. The model we have in mind involves two firms each with a type determined by a one-dimensional parameter ρ, that represents the firm's forecast of future prices. There are repeated alternating offers made by the two parties until a contract price is agreed (with some probability of breakdown at any stage). In contrast to the broker mechanisms, all private information of the firms (including ρ) is common knowledge. Binmore et al. [5] show how this leads to a Nash bargaining solution in the limit as the breakdown probability approaches zero and the number of rounds of negotiation increases. In this context we need to know each player's expected utility in the case that there is no agreement. In any case if we assume that there is ample opportunity for discussion between the two firms about to enter into a contract, then it is natural to assume that any agreed price quantity pair will be Pareto optimal, in the sense that there is no alternative contract that would produce improved expected utilities for both players. We show (for our limited model) that the direct negotiation model and the broker model yield very similar outcomes.

Observe that the broker mechanism with non-strategic offers does not yield a Nash equilibrium. Since offered supply and demand curves vary with type, sellers and buyers might anticipate the other firm's type, and alter their own offer curve. A seller will know that all buyer types will buy less of a contract as the price goes up, and this information can be used to improve his outcome. If the function $D(f)$ offered by the purchaser were known, then the seller would choose a point on this function that maximizes his own utility and then offer a supply function $Q(f)$ that goes through this point. Similarly the purchaser will respond to her conjectures about the function $Q(f)$ offered by the seller. This strategic interaction gives rise to a game we call the *broker mechanism with strategic offers*. As in the broker mechanism with non-strategic offers, each firm makes an offer that is a curve of acceptable Q and f pairs, and the broker determines the clearing price. The difference is that the curves offered by each firm can, under some conditions, be chosen to be optimal for each possible type of the other firm.

This approach opens up a further possibility. Each firm's type ρ defines their forecast of future prices. Since the firms make optimal choices for every possible type of the counterparty, it is possible to use the other firm's value of ρ, embodied in the price, as an additional input to the forecast information. We can expect that combining the values of ρ of the two firms will give a better price forecast than either firm can achieve individually. In other words, even when private information on a firm's spot price forecasts is not shared explicitly, deductions can be made from the firm's readiness to buy or sell contracts at different prices. In simple terms we can say that if a seller is ready to sell significant numbers of contracts at a low price then the buyer can deduce that the seller anticipates low prices in the spot market and adjust her own forecast accordingly. We will explore what happens to the equilibrium when firms make use of these deductions.

The main contributions of the paper are as follows:

1. We put forward the first model in the electricity contracting literature to explore the dependence of contracting outcomes on conjectures of future price distributions held by risk-averse agents.

2. We establish a relationship between Nash bargaining and a broker model with non-strategic offers, when firms have different expectations about the distribution of future prices.

3. We show how expectations of future prices can influence contract outcomes. For CARA utilities and a generator selling a fixed amount of power to a retailer, then both Nash bargaining and non-strategic offers made to a broker will result in contract quantities that are higher when buyers expect higher spot prices than sellers, and lower when buyers expect lower spot prices.

4. We define an equilibrium using supply functions to capture strategic behavior in a broker model, demonstrate that it exists, and show how to compute it in simple cases. The model is used to investigate the extent that behaving strategically affects outcomes for both players. In addition

we consider the impact of a player improving its own forecast using the information implicit in the other player's offer.

Our work falls into the area of economics that relates to bargaining between parties with private information. Much has been written about this, with examples often drawn from the field of legal disputes or wage negotiations. The review by Kennan and Wilson [18] gives a summary of this work, and draws attention to the way that careful specification of the procedure used in reaching agreement is necessary to determine the equilibria that may occur. We are interested in whether such behavior occurs in the particular case where agents with private information negotiate the forward trading of commodities in view of random future prices.

In reviewing the literature, we begin by considering work that looks at forward prices in electricity markets, since that is a primary area of application. A standard approach in modeling forward contracting is to invoke an arbitrage argument so that forward prices will match actual prices in expectation. The problem then reduces to a careful consideration of the detailed stochastic behavior of spot prices, on which much has been written in the electricity market literature (see e.g. [19]). In practice, it has been observed that forward prices rarely match actual prices in expectation, and there is considerable discussion in this literature of the sign of the difference between forward and spot prices. This is an empirical question which is not straightforward to answer since the sign of the forward premium will depend on circumstances. An important paper in this area is Bessembinder and Lemmon [4] who analyze the premium as reflecting a supply and demand imbalance as risk averse players attempt to optimize their utilities using the contract market. Their empirical results are based on day ahead prices obtained from the PJM market. Recent work has identified similar premia in the day-ahead markets operated by the Mid-West ISO[7], and the New England ISO [15].

Similar interest has emerged in Europe. Bunn and Chen [8] give a helpful discussion of the various factors that influence the British market looking at both the day ahead and month ahead data and using a vector autoregressive technique for estimating spot and premiums for both peak and non-peak prices. They show that premiums are positive for peak prices and negative for non-peak. There are significant behavioral effects where high peak premiums and high peak prices tend to induce higher premiums in the future. Related work can be found in Weron and Zator [29], who look at the Nord Pool market with weekly contracts between 1 and 6 weeks ahead. More generally, in the absence of risk-neutral arbitrageurs, premia in bilateral forward contracts will depend on the risk attitudes of the agents as well as their private information. Indeed, hedging risks is a key driver for bilateral forward contract arrangements. The same argument has been made by Dong and Liu [12] who discuss a supply chain context for a non-storable commodity and use a Nash bargaining approach. They use a mean-variance utility function.

The use of Nash bargaining as a model for bilateral contract negotiation in an electricity market is found in Yu et al [31] who use a CVaR-based utility function to measure risk aversion, and Sreekumaran and Liu [27] who choose a CVaR type measure but based on cash flow. There are alternatives to Nash bargaining. For example Wu et al. [30] consider a contract quantity bid in a Cournot framework and use a mean-variance approach to allow for risk aversion. Similarly, Downward et al [13] consider a differentiated products model for fixed-price electricity contracting by retailers who use a risk measure that combines expectation and CVaR.

We have a particular interest in private information, where different players have different expectations of future prices. This is a question which is of importance in practice, particularly in wholesale electricity markets where large price spikes make the use of hedging contracts very common, but where there are differences in the forecasts of average future spot prices. In this environment traders seeking to hedge their risk exposure will at the same time attempt to profit from their private information. Sanda, Olsen and Fleten [26] discuss the hedging behavior of hydro producers in the Nord Pool region. They find that large forward positions are taken with bilateral negotiation used for a significant fraction of these contracts (while the rest are traded in an exchange): some companies are able to use superior market forecasts to make considerable profits from these derivative contracts.

A number of authors have considered the negotiation process between a buyer and seller when each have private valuations. Myerson and Satterthwaite [23] consider mechanisms where a buyer and a seller have independent valuations for an individual item, and each submit their valuations to a broker. Individuals know their own valuations and the distribution of possible valuations for the other. The broker then determines whether there is a trade and the payment to be made by the buyer. These authors show that in general it is not possible to find a mechanism that is ex-post efficient. McKelvey and Page [21] extend this discussion to a divisible good and concave utility functions. Thus the bargaining outcome is both a quantity and a price, and these authors give conditions under which there is no ex-post efficient mechanism with independent valuations. The efficiency here relates to the social utility - being the sum of the utilities for the two players. McAfee and Reny [20] consider similar models with a correlation between buyer and seller valuations and show that this can have significant effects on the end result. They show how an efficient outcome can become possible, provided the broker acts as a budget balancer, with payments balancing on average but not for every realization. Our problem is similar to that considered by McKelvey and Page, except that we are concerned only with financial exchanges, and utility in our framework is used to reflect risk preferences, rather than utility from a good.

The Nash bargaining solution is derived for a case where all information is common knowledge. There have been several papers looking at Nash bargaining with incomplete information. Harsanyi and Selton [16] considered an extension of the Nash bargaining solution in which there is a fixed set of types and a distribution over the pairs of types for the two players. Myerson [22] also considers an extension of Nash bargaining looking at solutions that retain incentive compatibility. There are also papers addressing the mechanism question more directly by considering a sequence of alternating offers: see Chatterjee and Samuelson [10] and Cramton [11]. At each stage a player can accept the offer of the other player or make their own offer in response. These repeated offer games are usually formulated either with a penalty at each stage reflecting the impatience of the players who wish to reach agreement, or with a possibility that the negotiations are randomly interrupted at any stage with a small probability. The difference from our case is that the analysis of these games involves a single price offer at each stage, whereas our setting will require both price and quantity offers. We have not found a way to translate the results from these papers into our framework.

The calculation of supply-function equilibria in which players deduce information on the other player's forecast and use this to improve their own forecast parallels the discussion in Vives [28]. He considers a case where the type of the player relates to their cost function, but structurally this is similar to our approach in which we can expect to see a correlation between types. We can think of the final price outcome as a "common value" about which both players receive signals. The supply-function equilibria in Vives are restricted to be linear and are uniquely defined.

The paper is laid out as follows. In the next section we define a model of contracting under uncertainty, and then derive a broker mechanism with non-strategic offers for negotiating a contract. The outcomes of this mechanism are explored under various assumptions on the problem data. The following section then describes the Nash bargaining solution, and compares it with the broker mechanism. Section 4 introduces a model in which agents make conjectures on the beliefs of the counterparty and offer supply functions that respond to these. We conclude the paper with a general discussion in section 5. Proofs of all results are given in Appendix A.

2. The model

We consider a model in which there are two firms, a buyer (firm 1) and a seller (firm 2), who trade in a single divisible commodity. We will assume that the buyer and seller have strictly concave, increasing utility functions $U_1(z)$ and $U_2(z)$. Each firm views the spot price as a univariate random variable W which we assume has support in the bounded interval $[a,b]$. The probability distribution of W assumed by each firm depends on a univariate parameter ρ. Thus firm 1 trades assuming W has probability distribution $\mathbb{P}(\rho_1)$ and firm 2 assumes $\mathbb{P}(\rho_2)$. In most of what follows we study contract outcomes when ρ_1 and ρ_2 are fixed, and so to simplify notation we will denote probability distributions by $\mathbb{P}_1$ and $\mathbb{P}_2$. In each each price outcome w, firm i earns an operating profit $R_i(w)$. There may be different scenarios that result in the same spot price, but have different operating profits, and in this case $R_i(w)$ will be the expected operating profit given w.

The firms wish to arrange a forward contract quantity Q and contract price f. This is a purely financial contract under which the buyer (firm 1) buys this contract quantity and the seller (firm 2) sells this quantity. Under the contract terms a payment is made by the seller to the buyer of the difference between the spot price and strike price, f. Thus the total expected profit made by firm 1 in price outcome w is $R_1(w)+Q(w-f)$ and the total expected profit made by firm 2 in price outcome w is $R_2(w)+Q(f-w)$. The expected utilities for the two firms if the contract quantity is Q and the contract price is f, are respectively

$$\Pi_1(Q,f) = \int_a^b U_1\left(R_1(w)+Q(w-f)\right) d\mathbb{P}_1(w), \tag{1}$$

$$\Pi_2(Q,f) = \int_a^b U_2\left(R_2(w) + Q(f-w)\right) d\mathbb{P}_2(w). \tag{2}$$

This differs from the framework of [21] where a price is paid for a quantity of a divisible good, since we use $E_{\mathbb{P}_1}[U_1(R_1(w)+Q(w-f))]$, rather than $E_{\mathbb{P}_1}[U_1(R_1(w)+Qw)] - Qf$. If f is below the range of possible prices estimated by the buyer then $\Pi_1(Q,f)$ increases with Q and so has no maximizer. If f is above the range of possible prices estimated by the buyer then she would sell contracts and $\Pi_1(Q,f)$ increases as $Q \to -\infty$ and so has no maximizer. Similarly if f is outside the range of possible prices estimated by the seller then $\Pi_2(Q,f)$ has no maximizer. To avoid this, we impose the condition throughout the paper that f is chosen so that $\mathbb{P}_1([a,f))$, $\mathbb{P}_1((f,b])$, $\mathbb{P}_2([a,f))$, and $\mathbb{P}_2((f,b])$ are all strictly positive.

In what follows we will study the contracting outcomes (Q,f) that arise from a number of different negotiation procedures. The specific form of these outcomes will also depend on the problem data, so at different points we will make various assumptions on the problem data to simplify the analysis, while maintaining the essential structure of the negotiation process. The assumptions are as follows.

Assumption 1. *For each i, $U_i(z)$ is twice differentiable and strictly concave with* $\lim_{z\to\infty} U_i'(z) = 0$ *and* $\lim_{z\to-\infty} U_i'(z) = \infty$.

This assumption is satisfied by many utility functions, and implies that $U_i'(z) > 0$. We shall make Assumption 1 throughout the paper.

Assumption 2. *There are only two price outcomes w_L and w_H with $R_1(w_L) = R_2(w_H)$, $R_2(w_L) = R_1(w_H)$.*

Assumption 2 is used in nearly all the examples we consider.

Assumption 3. *For each i, $U_i''(\cdot)/U_i'(\cdot)$ is constant (CARA utility).*

The assumption of a CARA utility function enables us to prove that contract settlements are uniquely determined. Since it simplifies the analysis of contract outcomes, we will also resort to this choice of utility in the examples.

3. The broker mechanism with non-strategic offers

We now consider a broker mechanism in which each firm offers a supply function to the broker. This is done without knowledge of the other firm's supply function offer, and so we can regard the two offers as happening simultaneously. The broker then clears the market by finding the price at which the demand from one firm matches the supply from the other. Later, in our discussion of supply-function equilibria, we will discuss what happens when each firm acts strategically. However we begin by assuming that each firm acts without anticipating their rival's choice. This type of competitive behavior by the firms will also occur when many buyers and many suppliers all submit offers to a broker who clears the market. In this case a single participant can be too small to have a significant impact on the final outcome, and so decide that they will simply submit a supply function that gives their preferred quantity of contracts at each price. We analyze a bilateral arrangement with a single buyer and a single seller, but, as in the competitive case, in this section we concentrate on the case where offers are not strategic.

Given a fixed contract price f, the buyer (firm 1) seeks an optimal contract quantity Q to buy. This gives the following first order condition.

$$\frac{\partial}{\partial Q}\Pi_1(Q,f) = \int_a^b (w-f)U_1'\left(R_1(w)+Q(w-f)\right)d\mathbb{P}_1(w) = 0. \tag{3}$$

In the same way, we can find the first order condition for the seller determining the optimal contract quantity to sell:

$$\frac{\partial}{\partial Q}\Pi_2(Q,f) = \int_a^b (f-w)U_2'\left(R_2(w)+Q(f-w)\right)d\mathbb{P}_2(w) = 0. \tag{4}$$

Proposition 1. *Under Assumption 1 the first order conditions define unique supply functions $\widehat{Q}_1(f)$ and $\widehat{Q}_2(f)$ and the broker model with non-strategic offers has at least one solution (Q^*, f^*) where*

$$Q^* = \widehat{Q}_1(f^*) = \widehat{Q}_2(f^*).$$

This result leaves open the possibility of non-monotonic behavior of the optimal offer curves $\widehat{Q}_i(f)$ and hence more than one clearing price. However we can show that any solution obtained cannot be improved for one firm without making the other worse off.

PROPOSITION 2. *Any solution to the broker model with non-strategic offers is Pareto optimal, i.e. it satisfies*

$$\Pi_1(Q,f) > \Pi_1(Q^*,f^*) \Rightarrow \Pi_2(Q,f) < \Pi_2(Q^*,f^*),$$

$$\Pi_2(Q,f) > \Pi_2(Q^*,f^*) \Rightarrow \Pi_1(Q,f) < \Pi_1(Q^*,f^*).$$

By restricting attention to the case of CARA utility functions (i.e. those satisfying Assumption 3) we can demonstrate that the offer curves are monotonic and hence the clearing price and quantity is unique.

PROPOSITION 3. *Under Assumption 3 (CARA utility), the supply functions $\widehat{Q}_1(f)$ and $\widehat{Q}_2(f)$ are monotonic and there is a unique clearing price and quantity.*

We can be more specific about the clearing contract quantity, which is influenced by changes in revenue with price outcome. We define

$$\lambda_i = \inf_{w_1 \neq w_2} \{ \left| \frac{R_i(w_1) - R_i(w_2)}{w_1 - w_2} \right| : w_1, w_2 \in \mathrm{supp}(\mathbb{P}_i) \}$$

$$\mu_i = \sup_{w_1 \neq w_2} \{ \left| \frac{R_i(w_1) - R_i(w_2)}{w_1 - w_2} \right| : w_1, w_2 \in \mathrm{supp}(\mathbb{P}_i) \}.$$

If $\mathbb{P}_i$ is a continuous distribution on $[a,b]$ and R_i are differentiable then λ_i defines the minimum magnitude of the slope of R_i with respect to price, and μ_i defines the maximum magnitude of slope of R_i with respect to price. If $\mathbb{P}_i$ is a two-point distribution on $\{w_L, w_U\}$ then

$$\lambda_1 = \mu_1 = \frac{R_1(w_L) - R_1(w_U)}{w_U - w_L}, \quad \lambda_2 = \mu_2 = \frac{R_2(w_U) - R_2(w_L)}{w_U - w_L}.$$

If $R_1(w)$ is nonincreasing in w and $R_2(w)$ is nondecreasing in w and $\mathbb{P}_1 = \mathbb{P}_2$, then we can show (see Lemma 1 in Appendix A) that the solution satisfies

$$\min\{\lambda_1, \lambda_2\} \leq Q \leq \max\{\mu_1, \mu_2\}. \tag{5}$$

We now consider a special case where $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = \lambda$, say, so $R_1(w)$ and $R_2(w)$ are linear with slopes of the same magnitude but opposite signs. We can see an example of this in an electricity setting, with a base load generator (seller) and a retailer (buyer). The buyer in the wholesale market buys an amount q and receives price p from retail sales (per unit of power used) and has fixed cost of operation K_1, giving a profit when the spot price is w of

$$R_1(w) = pq - wq - K_1. \tag{6}$$

The seller also supplies an amount q and has fuel cost c and fixed cost of operation K_2. The seller's profit from spot market operations when the spot price is w is given by

$$R_2(w) = wq - cq - K_2. \tag{7}$$

The value of λ in this case is q.

When $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = \lambda$ and $\mathbb{P}_1 = \mathbb{P}_2$), (5) implies that the contract quantity Q is the same as λ. In the case where $\mathbb{P}_1 \neq \mathbb{P}_2$, we can show (see Lemma 2 in Appendix A) that the contract quantity satisfies

$$Q \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \lambda \iff \begin{Bmatrix} \mathbb{E}_{\mathbb{P}_1}[w] > \mathbb{E}_{\mathbb{P}_2}[w] \\ \mathbb{E}_{\mathbb{P}_1}[w] = \mathbb{E}_{\mathbb{P}_2}[w] \\ \mathbb{E}_{\mathbb{P}_1}[w] < \mathbb{E}_{\mathbb{P}_2}[w] \end{Bmatrix}. \tag{8}$$

In the electricity market context that is described by (6) and (7), λ is the amount q traded in the spot market, so $Q > q$ if the expected future price as anticipated by the retailer is greater than the expected future price as anticipated by the generator. It is worth remarking on the fact that (5) and (8) are independent of the utility functions for the two players, so they hold true even when the retailer is more risk averse than the generator. In particular, if $\lambda_1 = \lambda_2 = \mu_1 = \mu_2$ and $\mathbb{E}_{\mathbb{P}_1}[w] = \mathbb{E}_{\mathbb{P}_2}[w]$ then differences in risk aversion will have an impact only on the price f, and not on the quantity of contracts signed.

Example 1: Two price outcomes

To illustrate some of the above results, we consider the special case when the spot price W has two outcomes w_L and w_H. To simplify notation in the two-outcome case we will henceforth write $\rho_1 = \mathbb{P}_1(w_H)$ and $\rho_2 = \mathbb{P}_2(w_H)$, so $\mathbb{P}_1(w_L) = 1 - \rho_1$ and $\mathbb{P}_2(w_L) = 1 - \rho_2$. This gives an expected utility for each firm as follows.

$$\Pi_1(Q, f) = \rho_1 U_1 \left(R_1(w_H) + Q(w_H - f)\right) + (1 - \rho_1) U_1 \left(R_1(w_L) + Q(w_L - f)\right),$$

$$\Pi_2(Q, f) = \rho_2 U_2 \left(R_2(w_H) + Q(f - w_H)\right) + (1 - \rho_2) U_2 \left(R_2(w_L) + Q(f - w_L)\right).$$

The first order conditions (3) and (4) yield

$$(w_L - f)U_1'(R_1(w_L) + Q(w_L - f))(1 - \rho_1) + (w_H - f)U_1'(R_1(w_H) + Q(w_H - f))\rho_1 = 0, \quad (9)$$

$$(f - w_L)U_2'(R_2(w_L) + Q(f - w_L))(1 - \rho_2) + (f - w_H)U_2'(R_2(w_H) + Q(f - w_H))\rho_2 = 0. \quad (10)$$

We will also assume that

$$R_2(w_H) - R_2(w_L) = R_1(w_L) - R_1(w_H) = \Delta. \quad (11)$$

Identifying λ with $\Delta/(w_H - w_L)$, we can see that (8) implies that the quantity of the contract signed is greater than (less than)

$$Q^* = \frac{\Delta}{w_H - w_L}$$

when ρ_1 is greater than (less than) ρ_2 and is equal to Q^* when $\rho_1 = \rho_2$.

Let each agent $i = 1, 2$ have CARA utility functions with $U_i(0) = 0$, $U_i(\infty) = 1$, and $U_i(x) = 1 - e^{-\alpha_i x}$. We can then show (see Lemma 3, Appendix A) that the solution with two price outcomes w_L and w_H, and operating profits satisfying (11), is

$$Q = \frac{\Delta}{w_H - w_L} + \frac{1}{(\alpha_1 + \alpha_2)(w_H - w_L)} \log\left(\frac{\rho_1(1 - \rho_2)}{(1 - \rho_1)\rho_2}\right) \quad (12)$$

$$f = \frac{w_H + w_L\kappa}{1 + \kappa}, \quad (13)$$

where $\kappa = ((1 - \rho_1)/\rho_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}}((1 - \rho_2)/\rho_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}}$.

If $\rho_1 = \rho_2 = \rho$, then $Q = Q^*$ and $f = \rho w_H + (1 - \rho)w_L$, which is just the expected spot price. These will also be the values of f and Q if both firms disclose their private information (since by definition this implies both firms then have the same belief about ρ).

To illustrate Example 1 numerically, suppose $\alpha_1 = \alpha_2 = \alpha$, $w_H = 2$, $w_L = 1$, $R_2(w_H) = R_1(w_L) = 4$, and $R_2(w_L) = R_1(w_H) = 1$. The buyer curve corresponding to ρ_1 is

$$Q = 3 + \frac{1}{\alpha} \log\left(\frac{\rho_1(2 - f)}{(1 - \rho_1)(f - 1)}\right).$$

The seller curve corresponding to ρ_2 is

$$Q = 3 - \frac{1}{\alpha} \log\left(\frac{\rho_2(2 - f)}{(1 - \rho_2)(f - 1)}\right).$$

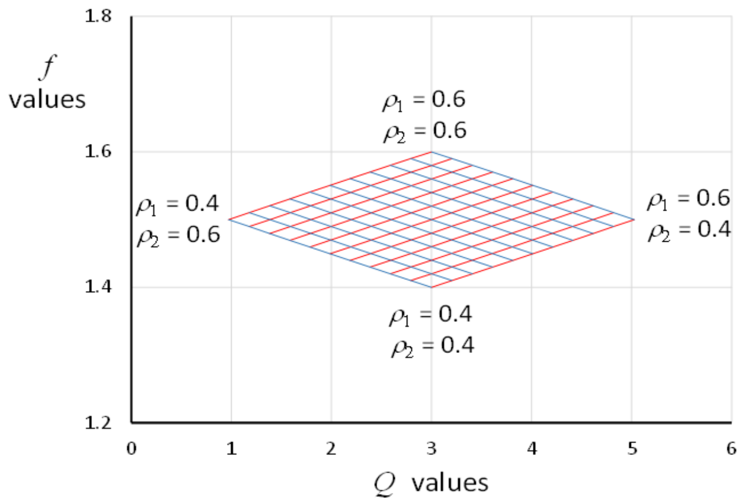

Figure 1 The grid of contract values for Example 1 with $\alpha = 0.2$ and ρ values that vary from 0.4 to 0.6. Seller offers are shown in red (ρ_2 decreases moving from left to right), purchaser offers in blue (ρ_1 increases moving from left to right).

For any ρ_1 and ρ_2 the market clears at $f = (2+\kappa)/(1+\kappa)$, $Q = 3 + (\log\sigma)/\alpha$. This gives a grid of possible (Q, f) outcomes depending on the ρ_1, ρ_2 values of the two firms. We show this in Figure 1 where we have allowed both ρ_1 and ρ_2 to vary between 0.4 and 0.6 in increments of 0.02. This is plotted for the case $\alpha = 0.2$.

4. Nash bargaining

The broker mechanism with non-strategic offers neglects the interaction we would expect between fully rational agents in finding a clearing price and quantity, since the non-strategic supply function is chosen in a way that would be appropriate only if the other player was using an unknown fixed price and was prepared to supply (demand) any amount of contracts at that price. From a conjectural variations perspective this would be an extreme view for a firm to hold. Knowing that a buyer will want to have a higher contract quantity if the price drops will allow the seller to achieve a better outcome by anticipating this broad behavior, even if the exact probability distribution of the buyer is unknown.

In the next section we will consider the results of strategic behavior by the agents in more detail. Here we consider another possibility and ask what will be the result of the two firms both revealing their probability distributions. In this setting we can consider negotiation taking place directly between the two firms. Our model for this is the Nash bargaining solution that occurs at the solution of

$$\max_{Q,f} \left(\Pi_1(Q,f) - \Pi_1(0,f)\right)\left(\Pi_2(Q,f) - \Pi_2(0,f)\right) \tag{14}$$

where $\Pi_1(Q,f)$ and $\Pi_2(Q,f)$ are defined by (1) and (2). As we mentioned in the introduction we can see this as the result of alternating offers when bargaining friction (associated with the possibility of a breakdown in negotiation) reduces to zero. The assumption here is that contract levels of zero will occur if there is no agreement.

We begin by showing that there is no difference between the Nash bargaining solution and the broker solution with non-strategic offers if there is antisymmetry of both the operating profits and the price distributions for the two firms around a central price point $\widehat{f} = \frac{a+b}{2}$. With these conditions, both Nash bargaining and non-strategic offer solution are guaranteed to end up at the central price $\widehat{f}$. Moreover the profit to player 1 from agreeing a contract level Q at price $\widehat{f}$ is the same as the profit to player 2. So both players agree on the best contract quantity and this then emerges as the choice from both mechanisms.

Proposition 4. *If $U_1 = U_2$, and for every y, $R_1(\widehat{f}+y) = R_2(\widehat{f}-y)$ and $\mathbb{P}_1([a,\widehat{f}+y)) = \mathbb{P}_2((\widehat{f}-y),b]$, where $\widehat{f} = \frac{a+b}{2}$, then the Nash bargaining solution matches the broker solution with non-strategic offers.*

When the spot price W has two outcomes w_L and w_H, the antisymmetry conditions in Proposition 4 imply $\rho_1 = 1-\rho_2$, where $\rho_1 = \mathbb{P}_1(w_H)$ and $\rho_2 = \mathbb{P}_2(w_H)$. If $R_2(w_H) = R_1(w_L)$ and $R_2(w_L) = R_1(w_H)$, and both agents have the same utility function, then the Nash bargaining solution matches the broker solution with non-strategic offers and $f = (w_L + w_H)/2$. For example, with CARA utilities as in Example 1 in the previous section, this shows that the Nash bargaining solution for $(\rho_1, 1-\rho_1)$ is $(Q, (w_L + w_H)/2)$ where

$$Q = \frac{\Delta}{(w_H - w_L)} + \frac{\log\rho_1 - \log(1-\rho_1)}{\alpha(w_H - w_L)}.$$

More generally, in the case of CARA utilities we have a striking result, showing that the contract quantities in the Nash bargaining solution exactly match the contract quantities in the broker solution with non-strategic offers. This result derives from properties of the derivatives of CARA utilities, and holds without the restriction of two price outcomes. Thus our previous observation that differences in risk aversion will not change the contract quantities for the broker solution with non-strategic offers, will also apply to the Nash-bargaining solution as long as R_1 and R_2 are linear functions with opposite slopes, and $\mathbb{E}_{\mathbb{P}_1}[w] = \mathbb{E}_{\mathbb{P}_2}[w]$.

PROPOSITION 5. *The contract quantity in the Nash bargaining solution will match the broker solution with non-strategic offers when there are CARA utilities.*

In general, the contract price in the Nash bargaining solution does not quite match the broker solution away from the centre line where $f = \widehat{f}$, but the two solutions are close to each other.

Example 1 continued

To illustrate Proposition 5 we consider the case with two price outcomes w_L and w_H, and CARA utilities. We extend Example 1 by adding Assumption 2, which enables a simplification of notation to r and s where $R_2(w_H) = R_1(w_L) = r$, $R_1(w_H) = R_2(w_L) = s$, and $r > s$. It follows from Proposition 5 that the contract quantity for the Nash bargaining solution is the same as the broker solution with non-strategic offers defined by (12) and (13). We can also show (as we establish in Lemma 4 in Appendix A) that when $\alpha_1 = \alpha_2 = \alpha$ we have

$$f = w_L + \frac{1}{2\alpha Q} \log\left(\frac{e^{-\alpha s}\sigma\left(-\rho_2 + \sigma\rho_2 + 1\right)\left(e^{-\alpha r} + e^{-\alpha s}\rho_1 - e^{-\alpha r}\rho_1\right)}{e^{-\alpha r}\left(\sigma + \rho_1 - \sigma\rho_1\right)\left(e^{-\alpha s} - e^{-\alpha s}\rho_2 + e^{-\alpha r}\rho_2\right)} \right), \tag{15}$$

where we write $\sigma = \sqrt{\frac{\rho_1(1-\rho_2)}{\rho_2(1-\rho_1)}}$.

It turns out that, as well as the contract quantity Q being the same as that obtained in the broker solution with non-strategic offers, the value of f is also close to that arising from the broker solution. For example on the line where $\rho_1 = \rho_2 = \rho$, then $\sigma = 1$ and $Q = \frac{\Delta}{(w_H - w_L)}$. This gives the contract price

$$f = \frac{(w_H + w_L)}{2} + \frac{(w_H - w_L)}{2\alpha(r-s)} \log\left(\frac{\rho + (1-\rho)e^{-\alpha(r-s)}}{\rho e^{-\alpha(r-s)} + (1-\rho)} \right), \tag{16}$$

which is close to the value $f = w_H\rho + w_L(1-\rho)$ obtained from the broker mechanism with non-strategic offers (when both ρ values are equal). For example, with the parameters shown in Figure 1, the maximum difference is about 0.005.

5. Supply-Function Equilibrium

We now consider a broker mechanism with strategic offers. In this case we need to consider an equilibrium between policies that map private information to supply-function offers, so that each agent's policy comprises a family of supply functions. When private information becomes known, each agent's policy determines their supply function that is then offered to the broker.

We suppose that the players have a type that is determined by a single one-dimensional parameter ρ. We will be interested in the case that ρ represents a player's private information, or beliefs, about the distribution of future prices. However we could use a similar framework to deal with types of player distinguished by having different costs, or by having different levels of risk aversion.

As in the broker mechanism with non-strategic offers, and in contrast to the Nash bargaining framework, we will assume that the private information is not shared with the other player, so that each player makes a supply function offer to the broker having no knowledge of the other player's type and therefore its supply function. Using a broker means that offers do not need to be made at exactly the same time. The key property of a supply function equilibrium is that a player has no incentive to change their supply function even if they know the type (and hence the offer) of the other player. This property occurs when the clearing price-quantity pair is ex-post optimal for each of the possible offers of the other player. We may think of player 1 observing the supply-function offer from player 2 and finding the best point (Q, f) on that curve to maximize $\Pi_1(Q, f)$. By joining up these best points for each of the different supply functions that might be offered by player 2 into a single supply-function offer, player 1 is guaranteed the best possible result no matter what type player 2 turns out to be.

If this ideal arrangement is possible we automatically achieve incentive compatibility - the truthful bid already achieves the best possible outcome. It is not clear however that such a supply-function equilibrium will exist. In this section we will demonstrate the existence of supply-function equilibria and show how they can be calculated in the special case where there are only two price outcomes w_L and w_H, Assumption 2 holds, and both players use the same CARA utility function.

5.1. Equilibrium with no information deduction

We begin with the case where the agents do not adjust their beliefs based on deductions made from the other agent's behavior. The buyer (firm 1) bids an offer curve that depends on her private information ρ_1. We write this curve in parameterized form $Q_1(\rho_1, t)$, $f_1(\rho_1, t)$, where t is the parameter. Thus the buyer will purchase a quantity of contracts $Q_1(\rho_1, t)$ if the price is $f_1(\rho_1, t)$. Similarly the seller (firm 2) offers a supply curve that can be written in parameterized form as $Q_2(t, \rho_2)$, $f_2(t, \rho_2)$, where $Q_2(t, \rho_2)$ is the quantity of contracts that he wishes to sell at a price $f_2(t, \rho_2)$ when his private information is ρ_2. By defining $Q_2(t, \rho_2)$ as the sell amount (rather than the buy amount) we can say that the market clears at the price and quantity where these two curves intersect.

Each player takes the other player's supply function as fixed and optimizes their own offer against this. Each player anticipates that the other player's offer will depend on the other player's own information, but does not know what this information is. Hence firm 1 (knowing ρ_1) is faced with a supply function in contracts being offered by firm 2 that is determined by ρ_2. Firm 1 would ideally make an offer that picks out the best point on the supply function of firm 2. Linking these points together for different values of ρ_2, firm 1 will then have an optimal supply function for any value of ρ_2.

To construct a Nash equilibrium, we suppose that firm 1 knows the complete set of supply-function offers to be made by firm 2 depending on its private information ρ_2. Thus firm 1 knows firm 2's supply function $(Q_2(t, \rho_2), f_2(t, \rho_2))$ for each possible value of ρ_2. We seek an optimal response to this set of supply functions.

Given a single supply-function offer by firm 2, corresponding to ρ_2, firm 1 (with private information ρ_1) seeks a value t that maximizes

$$\Pi_1(t, \rho_2) = \int_a^b U_1\left(R_1(w) + Q_2(t, \rho_2)\left(w - f_2(t, \rho_2)\right)\right) d\mathbb{P}_1(\rho_1, w), \tag{17}$$

so that firm 1 picks the best point t on firm 2's supply function $(Q_2(t, \rho_2), f_2(t, \rho_2))$, using firm 1's belief about price distribution $\mathbb{P}_1$. We write this price distribution as $\mathbb{P}_1(\rho_1, w)$ to reflect the fact that it depends on firm 1's private information ρ_1. Suppose that the optimal value of t is denoted by $t^*(\rho_2)$. Player 1 thus wishes to use a supply function that passes through the point

$(Q_2(t^*(\rho_2), \rho_2), f_2(t^*(\rho_2), \rho_2))$. By considering all possible values of ρ_2, firm 1 can construct a supply function (based on its private information ρ_1) that we can write as $(Q_1(\rho_1, t'), f_1(\rho_1, t'))$. In fact if we use ρ_2 as the parameter t' we have $Q_1(\rho_1, \rho_2) = Q_2(t^*(\rho_2), \rho_2)$ and $f_1(\rho_1, \rho_2) = f_2(t^*(\rho_2), \rho_2)$. This process can be repeated for different values of firm 1's private information ρ_1 to produce a complete set of supply-function offers for firm 1 that is an optimal response to the set of supply-function offers by firm 2.

In the same way firm 2 can pick out its optimal supply-function offers by finding values of t (which we can write as $t^*(\rho_1)$) to maximize

$$\Pi_2(\rho_1, t) = \int_a^b U_2\left(R_2(w) + Q_1(\rho_1, t)\left(f_1(\rho_1, t) - w\right)\right) d\mathbb{P}_2(\rho_2, w), \tag{18}$$

and then constructing a supply function that passes through the points $(Q_1(\rho_1, t^*(\rho_1)), f_1(\rho_1, t^*(\rho_1)))$. A Nash equilibrium then occurs if each player's complete set of supply functions is an optimal response to that of the other player.

Suppose that we wish to use this framework for a case where the univariate type for firm 1 conveyed by private information ρ_1 relates to costs, then the formulation is the same except that the ρ_1 dependence in $\Pi_1(t, \rho_2)$ occurs in the function $R_1(\rho_1, w)$ rather than in $\mathbb{P}_1$. Similarly the ρ_2 dependence in $\Pi_2(\rho_1, t)$ occurs in the function $R_2(\rho_2, w)$.

Once a complete set of supply functions is determined for each player we can write $Q(\rho_1, \rho_2)$ and $f(\rho_1, \rho_2)$ as the quantity and price at the combination ρ_1, ρ_2. The expected utilities of each agent under information outcomes (ρ_1, ρ_2) are given by

$$\Pi_1(\rho_1, \rho_2) = \int_a^b U_1\left(R_1(w) + Q(\rho_1, \rho_2)\left(w - f(\rho_1, \rho_2)\right)\right) d\mathbb{P}_1(\rho_1, w),$$
$$\Pi_2(\rho_1, \rho_2) = \int_a^b U_2\left(R_2(w) + Q(\rho_1, \rho_2)\left(f(\rho_1, \rho_2) - w\right)\right) d\mathbb{P}_2(\rho_2, w).$$

We will henceforth restrict attention to the setting of two price outcomes $\{w_L, w_H\}$, so the type ρ_i can be interpreted as the probability that firm i assigns to a high spot price w_H. We also assume $U_1 = U_2$. The expected utilities of each firm under outcomes (ρ_1, ρ_2) are then given by

$$\Pi_1(\rho_1, \rho_2) = \rho_1 U\left(R_1(w_H) + Q(\rho_1, \rho_2)(w_H - f(\rho_1, \rho_2))\right)$$
$$+ (1 - \rho_1) U\left(R_1(w_L) + Q(\rho_1, \rho_2)(w_L - f(\rho_1, \rho_2))\right),$$

$$\Pi_2(\rho_1,\rho_2) = \rho_2 U\left(R_2(w_H) + Q(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_H)\right) \\ + (1-\rho_2)U\left(R_2(w_L) + Q(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_L)\right).$$

The first order conditions for firm 1 are derived from taking the supply function for firm 2 as given in parameterized form by $Q(t,\rho_2)$, and $f(t,\rho_2)$. Firm 1 can choose the value of t (and hence the (Q,f) pair) to optimize its own payoff, and we obtain the first order conditions:

$$\rho_1 U'\left(R_1(w_H) + Q(\rho_1,\rho_2)(w_H - f(\rho_1,\rho_2))\right)\left(Q_1'(\rho_1,\rho_2)(w_H - f(\rho_1,\rho_2)) - Q(\rho_1,\rho_2)f_1'(\rho_1,\rho_2)\right) \\ + (1-\rho_1)U'\left(R_1(w_L) + Q(\rho_1,\rho_2)(w_L - f(\rho_1,\rho_2))\right) \\ \times \left(Q_1'(\rho_1,\rho_2)(w_L - f(\rho_1,\rho_2)) - Q(\rho_1,\rho_2)f_1'(\rho_1,\rho_2)\right) = 0,$$

and

$$\rho_2 U'\left(R_2(w_H) + Q(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_H)\right)\left(Q_2'(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_H) + Q(\rho_1,\rho_2)f_2'(\rho_1,\rho_2)\right) \\ + (1-\rho_2)U'\left(R_2(w_L) + Q(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_L)\right) \\ \times \left(Q_2'(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_L) + Q(\rho_1,\rho_2)f_2'(\rho_1,\rho_2)\right) = 0.$$

Given Assumption 2 we show in Appendix A (Proposition 6) that there is a set of potential supply-function equilibria that satisfy

$$Q(\rho_1,\rho_2) = Q(1-\rho_2, 1-\rho_1), \quad f(\rho_1,\rho_2) = w_H + w_L - f(1-\rho_2, 1-\rho_1). \tag{19}$$

We call an equilibrium solution satisfying (19) an anti-symmetric equilibrium (ASE). Observe that even under Assumption 2 there may also be other equilibria in this setting that are not antisymmetric. If $\rho_1 + \rho_2 = 1$ then note from (19) that $f(\rho_1,\rho_2) = w_H + w_L - f(\rho_1,\rho_2)$ and so $f(\rho_1,\rho_2) = \hat{f} = (w_H + w_L)/2$. We will write $h(\rho)$ for the value $Q(\rho, 1-\rho)$, that is the Q value on the line $f = \hat{f}$. As we establish in Appendix B (Proposition 7), the function $h(\rho)$ satisfies the differential equation

$$h'(\rho) = \frac{2h(\rho)}{\gamma} f_d \frac{\rho U'(s + \gamma h(\rho)) + (1-\rho)U'(r - \gamma h(\rho))}{\rho U'(s + \gamma h(\rho)) - (1-\rho)U'(r - \gamma h(\rho))} \tag{20}$$

for some positive constant f_d, and $\gamma = (w_H - w_L)/2$.

In general the families of supply-function equilibria are under-determined by the relationships that we have (this is reminiscent of other supply function equilibrium models). Determining the function h will be enough to determine the complete solution, since the Q values on the centre-line serve as a boundary condition to determine the rest of the (Q, f) solution. Given that f_d is a second constant to be chosen as well as one of the $h(\rho)$ values as a starting point for the differential equation, we see that there are two degrees of freedom if we limit ourselves to antisymmetric solutions.

Observe that we will require

$$\rho U'(s+\gamma h(\rho)) > (1-\rho)U'(r-\gamma h(\rho)) \tag{21}$$

to avoid h' being negative or infinite. However if ρ is less than

$$\rho^* = \frac{U'(r)}{U'(r)+U'(s)}$$

then

$$\rho U'(s) < (1-\rho)U'(r)$$

and the concavity of U implies that inequality (21) fails to hold when $h(\rho) \geq 0$. Thus for a solution which includes ρ values less than ρ^* we will need to allow h to be negative, corresponding to negative Q values. Negative contract quantities will be unusual in practice since they correspond to large differences in the private information held by the two firms (sufficient to make the generator buy contracts rather than sell them).

In the CARA case we get

$$\rho^* = \frac{\exp(-\alpha r)}{\exp(-\alpha r)+\exp(-\alpha s)}$$

and with the values from Example 1 of $\alpha = 0.2$, $s = 1$, $r = 4$ this becomes $\rho^* = 0.354\,34$. Hence if $\rho < 0.35434$ the derivative of h can only remain positive if h is negative, which shows that all solutions of interest to us go through zero at this ρ value. By setting the lowest ρ value to be 0.4 in our numerical examples we avoid this difficulty.

5.2. Equilibrium with information deduction

We now consider the supply-function model with information deduction. In evaluating $\Pi_1(\rho_1, \rho_2)$ firm 1 uses the probability distribution $\mathbb{P}_1(\rho_1, w)$. But the information outcome for the other player is implicitly available for firm 1 if she observes (after submitting the supply-function bid) the crossover point Q, f from which can be deduced the private information of the other player that is captured in their type. Thus we consider a function $\widehat{\mathbb{P}}_1(\rho_1, \rho_2)$ that gives firm 1's deduction of the distribution of outcomes given firm 1's initial estimate for type ρ_1, and firm 2's estimate ρ_2.

Up to now we have assumed $\widehat{\mathbb{P}}_1(\rho_1, \rho_2) = \mathbb{P}_1(\rho_1)$ so that the distribution of outcomes assumed by firm 1 remains unchanged no matter what the type of the other firm. Hence the expected utility for firm 1 is computed using the probability distribution corresponding to ρ_1, and similarly for firm 2. This would be appropriate if for some reason each player believes that the information being used by the other is completely unreliable.

With information deduction each possible type ρ_2 of the counterparty defines a probability distribution of the spot price that captures the private information of firm 2, and firm 1 will account for this in her calculation of expected utility. One way to specify this information deduction is to use a Bayesian approach. Under this arrangement there is a prior on the set of price distributions, each player observes some private information and on this basis makes their own assessment of the posterior on the set of distributions. These posterior distributions are dependent on the private information ρ_1 and ρ_2. If firm 1 is aware of both pieces of private information, ρ_1 and ρ_2, then she can deduce a new posterior on the set of distributions. This then gives rise to the final distribution $\widehat{\mathbb{P}}_1(\rho_1, \rho_2)$. Under this model firms make equal use of ρ_1 and ρ_2 and we will have $\widehat{\mathbb{P}}_1(\rho_1, \rho_2) = \widehat{\mathbb{P}}_2(\rho_1, \rho_2)$. In this framework the expected utility for firm 1, $\Pi_1(t, \rho_2)$, has exactly the same formulation as (17) with $\mathbb{P}_1(\rho_1, w)$ replaced with $\widehat{\mathbb{P}}_1(\rho_1, \rho_2, w)$, and similarly $\Pi_2(\rho_1, t)$ has the same formulation as (18) with $\mathbb{P}_2(\rho_2, w)$ replaced with $\widehat{\mathbb{P}}_2(\rho_1, \rho_2, w)$. The process for selecting the maximizing choice of t and then constructing an optimal supply-function response is the same as before.

With information deduction, the private information of one player as revealed by their choice of supply function has an impact on the calculations carried out by the other player. However there can be no advantage for a player in choosing a different supply function (pretending to have different private information than actually is the case) in order to change the behavior of the other player. Because there is no exchange of information prior to the decision on the supply function to be offered, and each player chooses a supply function that is optimal against any of the set of possible supply functions of the other, it follows that firm 1 knowing ρ_1 and pretending to be some other type ρ_1' does not cause any change to the supply function chosen by firm 2.

As for the previous case we will illustrate the supply-function model with two outcomes $\{w_L, w_H\}$, so the type ρ_i can be interpreted as the probability that firm i assigns to a high spot price w_H. When information is considered of equal value, then it is natural to set $\widehat{\mathbb{P}}_1(\rho_1,\rho_2) = \frac{\rho_1+\rho_2}{2}$. For example this would be the correct outcome when the private information is derived from a sample taken from the actual distribution, and both firms use samples of the same size. The expected utilities of each firm using information deduction under outcomes (ρ_1, ρ_2) are now given by

$$\begin{aligned}\Pi_1(\rho_1,\rho_2) = &\frac{\rho_1+\rho_2}{2} U\left(R_1(w_H) + Q(\rho_1,\rho_2)(w_H - f(\rho_1,\rho_2))\right)\\ &+ (1-\frac{\rho_1+\rho_2}{2}) U\left(R_1(w_L) + Q(\rho_1,\rho_2)(w_L - f(\rho_1,\rho_2))\right),\end{aligned}$$

$$\begin{aligned}\Pi_2(\rho_1,\rho_2) = &\frac{\rho_1+\rho_2}{2} U\left(R_2(w_H) + Q(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_H)\right)\\ &+ (1-\frac{\rho_1+\rho_2}{2}) U\left(R_2(w_L) + Q(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_L)\right),\end{aligned}$$

and we obtain the first order conditions:

$$\begin{aligned}&\frac{\rho_1+\rho_2}{2} U'\left(R_1(w_H) + Q(\rho_1,\rho_2)(w_H - f(\rho_1,\rho_2))\right)\left(Q_1'(\rho_1,\rho_2)(w_H - f(\rho_1,\rho_2)) - Q(\rho_1,\rho_2) f_1'(\rho_1,\rho_2)\right)\\ &\quad + (1-\frac{\rho_1+\rho_2}{2}) U'\left(R_1(w_L) + Q(\rho_1,\rho_2)(w_L - f(\rho_1,\rho_2))\right)\\ &\qquad \times \left(Q_1'(\rho_1,\rho_2)(w_L - f(\rho_1,\rho_2)) - Q(\rho_1,\rho_2) f_1'(\rho_1,\rho_2)\right) = 0,\end{aligned}$$

and

$$\begin{aligned}&\frac{\rho_1+\rho_2}{2} U'\left(R_2(w_H) + Q(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_H)\right)\left(Q_2'(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_H) + Q(\rho_1,\rho_2) f_2'(\rho_1,\rho_2)\right)\\ &\quad + (1-\frac{\rho_1+\rho_2}{2}) U'\left(R_2(w_L) + Q(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_L)\right)\\ &\qquad \times \left(Q_2'(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_L) + Q(\rho_1,\rho_2) f_2'(\rho_1,\rho_2)\right) = 0.\end{aligned}$$

Under Assumption 2, we show in Appendix A (Proposition 7) that if $(Q(\rho_1,\rho_2), f(\rho_1,\rho_2))$ is a supply-function equilibrium with information deduction then the function $h(\rho) = Q(\rho, 1-\rho)$ satisfies the differential equation

$$h'(\rho) = \frac{2h(\rho)}{\gamma} f_d \frac{U'(s+\gamma h(\rho)) + U'(r-\gamma h(\rho))}{U'(s+\gamma h(\rho)) - U'(r-\gamma h(\rho))} \tag{22}$$

for some positive constant f_d and $\gamma = \frac{w_H - w_L}{2}$, and all supply-function equilibria have contract quantities bounded above by $Q^* = \frac{r-s}{w_H - w_L}$.

Observe by (8) that if $\rho_1 > \rho_2$ then Q^* is less than the contract quantity $Q(\hat{f})$ defined for the broker case with non-strategic offers. Since $Q(\hat{f})$ is the optimal contract quantity at $\hat{f}$ for both buyer and seller, it must be a Pareto improvement on the equilibrium quantity with information deduction. This shows that the outcomes in a supply-function equilibrium with information deduction need not be Pareto optimal.

5.3. Numerical examples

To gain a better understanding of the character of supply-function equilibria we will explore some numerical examples. We suppose that there is a CARA utility function, $U(x) = 1 - e^{-\alpha x}$. We take $R_1(w_H) = R_2(w_L)$ and $R_2(w_H) = R_1(w_L)$; a low price of $w_L = 1$ and a high price of $w_H = 2$. In Appendix B we describe how we use (20) and (22) to construct an ASE for the no information deduction case and the information deduction case respectively.

We consider values for ρ ranging from 0.4 to 0.6. By doing this we stay above the value of $\rho^* = 0.354\,34$ that would lead to negative values of Q in the case of no information deduction. It is very much harder to construct solutions that go across the $Q = 0$ boundary and, because this is unlikely in practice, it makes sense to restrict our attention to the $Q > 0$ case. Figure 2 below shows one solution possible in the case without information deduction.

We are interested in comparing the expected utility achieved for different equilibria. Because there is symmetry in outcomes the two players both have the same expected utility. This allows a natural coordination mechanism where both players select the equilibrium giving them the best outcome. To find the expected utility we will consider the range of outcomes for different ρ_1 and ρ_2.

But the values for the private information represented by ρ_1 and ρ_2 would not be expected to be independent, so evaluation of the expected utility requires us to be more explicit about the context for this game.

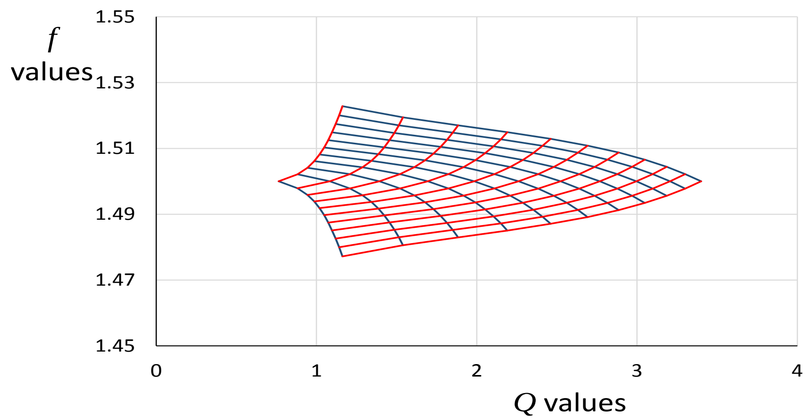

Figure 2 Symmetric supply-function equilibrium without information deduction for $\rho = 0.4$ to $\rho = 0.6$.

We suppose that the actual probability distribution for price w is unknown but is drawn from a prior on possible distributions. This prior is known to both players. The sequence is as follows. First a particular probability distribution for w is selected from the prior by Nature. Then each player observes a number of samples from this distribution (this is the player's private information). Each sample is just one price outcome and the player uses the complete set of samples to estimate a probability distribution on price. It is these probabilities that are used to determine the optimal supply function to offer. The supply function offered thus depends on the observed sample. The supply-function equilibrium reflects supply functions offered for all the different observations that are possible.

To evaluate the expected profit from a particular equilibrium we take expectations over the original prior (that is common knowledge). In the case of just two possible prices w_L and w_H, the prior on the distribution of w becomes a prior on the set of all possible probabilities for w_H. We will use a uniform distribution as the prior. In Appendix C we give more details of the way these calculations are carried out.

Assuming that each player observes 10 samples to determine its estimated probability of w_H, we can calculate that the solution shown in Figure 2 has expected utility of 0.38578, which is close to the maximum possible. Here we have varied the two parameters (f_d and the maximum value for Q which is $Q_{max} = h(0.6)$) in order to achieve the best outcome.

In Figure 3 we show the behavior of a supply-function equilibrium when both players make deductions about ρ from the supply function offered by the other. The method of construction is exactly analogous where we use the differential equation (41) and the appropriate forms for the definitions of S_1 and S_2. Again we have searched for good values of f_d and $h(0.6)$. This gives an expected utility of 0.38534 which is slightly worse than without information deduction. We notice that the variation in contract quantity sizes is substantially reduced in this case.

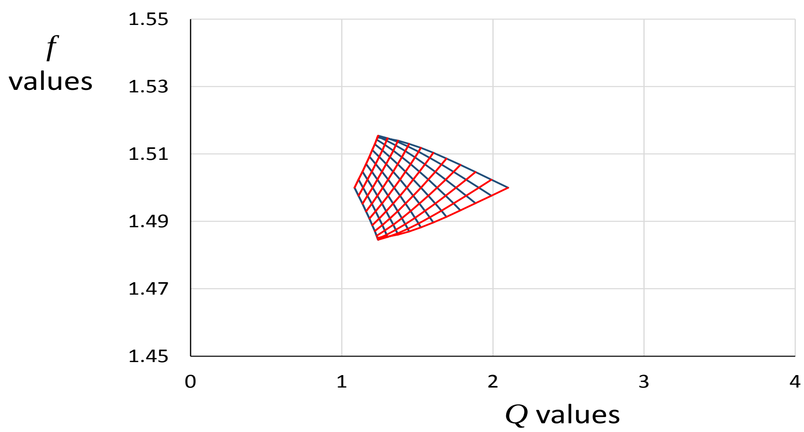

Figure 3 Symmetric supply-function equilibrium with information deduction for $\rho = 0.4$ to $\rho = 0.6$.

We have computed the expected payoffs of a symmetric equilibrium that maximizes expected utility in a large number of examples of this model for different choices of α, and s and r. The results of some of these experiments are reported in Table 1 below. The columns headed Π give the expected utility for each player under four assumptions, namely no contracting ($\Pi_{Q=0}$), broker with non-strategic offers (Π_{sb}), supply-function bidding with no information deduction (Π_{nid}), and supply-function bidding with information deduction (Π_{id}). In the last two cases we also provide the values of f_d and $Q_{\max}$ that maximize the expected utility.

It is difficult to be categorical about these results. In all cases reported the broker solution with non-strategic offers improves the players' expected utility, as one would expect. Observe that for $\alpha = 0.25$ and $\alpha = 0.3$, we have $\Pi_{nid} < \Pi_{sb}$, but for $\alpha = 0.5$ this inequality reverses. So payoffs under supply-function bidding with no information deduction are typically worse than those for payoffs from non-strategic offers, but not always. Similarly, we typically have $\Pi_{id} < \Pi_{nid}$, but for $\alpha = 0.5$, $s = 1$, $r = 8$, this inequality reverses. So payoffs under supply-function bidding with information deduction are generally worse than those without information deduction, but not always. Observe (as shown in Figure 2 and Figure 3) that information deduction tends to reduce supply-function equilibrium contract quantities (except for $\alpha = 0.5$, $s = 1$, $r = 8$), which are typically lower than non-strategic contract quantities, so information deduction in this setting often moves us further from a more desirable outcome.

α	s	r	$\Pi_{Q=0}$	Π_{sb}	Π_{nid}	$Q_{\max}(nid)$	$f_d(nid)$	Π_{id}	$Q_{\max}(id)$	$f_d(id)$
0.25	1	3	0.3744	0.3888	0.3863	2.300	0.2025	0.3862	1.825	0.1975
0.30	1	3	0.4263	0.4469	0.4442	2.150	0.1050	0.4440	1.525	0.1325
0.20	1	4	0.3660	0.3887	0.3858	3.225	0.1050	0.3853	2.100	0.1200
0.25	1	4	0.4267	0.4603	0.4577	3.100	0.0500	0.4569	2.200	0.0850
0.30	1	4	0.4790	0.5235	0.5214	3.075	0.0325	0.5205	2.325	0.0675
0.25	1	5	0.4673	0.5233	0.5213	4.100	0.0275	0.5200	3.000	0.0500
0.30	1	5	0.5180	0.5893	0.5879	4.100	0.0250	0.5865	3.075	0.0350
0.25	1	6	0.4990	0.5788	0.5775	5.100	0.0250	0.5754	3.650	0.0200
0.30	1	6	0.5469	0.6459	0.6451	5.000	0.0250	0.6429	3.750	0.0100
0.50	1	6	0.6718	0.8227	0.8237	5.400	0.0100	0.8232	4.550	0.0200
0.50	1	8	0.6876	0.8911	0.8926	6.400	0.0070	0.8929	6.680	0.0165

Table 1 Expected utilities from different equilibria.

6. Discussion and conclusions

This paper has considered the problem faced by two players negotiating the terms of a forward (financial) contract when both are uncertain about the future spot price, each with their own estimate of its probability distribution. Both price and quantity need to be determined. This problem only makes sense when the players are risk averse, since otherwise the different views they hold on the expected future price leads to an infinite contract quantity. In a simple model of this situation we compare the results of direct bilateral negotiation using a Nash bargaining concept, and the use of a broker who takes supply-function offers from the two players. We show that these two methods produce similar outcomes if utilities are CARA, with an exact match of contract quantities.

Our results show that the contract quantities are related to the differences in price expectation between the two players. When firm profits are linear in prices with slopes $-\lambda$ and λ, so that total profit is constant, then the contract quantity will be greater (or less) than λ if player 1 (the purchaser) has the higher expected forecast for spot price. Differences in risk aversion between the players will not affect these inequalities.

The broker mechanism can be expected to result in strategic behavior by the players, with each player anticipating the supply function offered by the other. This leads to a supply-function equilibrium in contract offers. We show how these supply function equilibria can be calculated and demonstrate that they may well give worse expected utility for both players. Thus we have a type of prisoner's dilemma, where one player acting strategically improves their own utility, but when both of them do so, there can be an overall loss of utility. Our numerical results suggest that this loss of expected utility is likely to occur unless there are high levels of risk aversion. In this context it is also possible to use the supply function offered by the other player to deduce the information they hold about the expected future spot market price. However this more sophisticated approach typically leads to smaller contract quantities and no overall improvement in expected utility.

Large contract quantities occur when the two players have very different views on the probability of a high price. It is thus inevitable that an approach which uses a combination of the two ρ estimates will reduce the difference between the final estimates and hence reduce contract sizes.

We have constructed supply-function equilibria for examples with two price outcomes w_L and w_H and common CARA utility function. Our numerical technique is applied to instances with $R_1(w_H) = R_2(w_L)$ and $R_1(w_L) = R_2(w_H)$, and yields antisymmetric equilibria. It is reasonable to suppose that players will tend towards such a solution, and attempt to coordinate on the equilibrium that gives them both the maximum expected utility. However the existence of alternative equilbria, in which one player does better than the other, will make this coordination harder to achieve.

We may consider the implications for contract negotiation behavior in practice. Sometimes sellers of contracts wish to remain anonymous, for in competitive electricity markets the contract books of electricity generation companies are held in strict secrecy. One reason is that in imperfectly competitive markets, levels of contracting affect spot market offering behavior([1], [3]) and so generators are at a strategic disadvantage if their contract levels are known by competitors. In some circumstances purchasers might also prefer to buy from a generator (to incentivize lower prices in the spot market) and so a speculator might prefer to be anonymous, so that this preference does not result in lower contract prices.

However our results suggest that direct negotiation may have some advantages over dealing through a broker. This is particularly the case where players are relatively sophisticated and have a good knowledge of their counterparty's operating costs. In these cases strategic interactions when dealing through a broker should lead to a supply-function equilibrium in offer curves. But the multiplicity of potential equilibria will make it hard to coordinate on a single equilibrium solution, and where equilibria are found they often have the effect of making both players worse off.

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Appendix A: Proofs of Propositions

PROPOSITION 1. *Under Assumption 1 the first order conditions define unique supply functions* $\widehat{Q}_1(f)$ *and* $\widehat{Q}_2(f)$ *and the broker model with non-strategic offers has at least one solution* (Q^*, f^*) *where*

$$Q^* = \widehat{Q}_1(f^*) = \widehat{Q}_2(f^*).$$

Proof Differentiating (3) with respect to Q gives

$$\int_a^b (w-f)^2 U_1''\left(R_1(w) + Q(w-f)\right) d\mathbb{P}_1(w) < 0, \tag{23}$$

so $\Pi_1(Q,f)$ is a strictly concave function of Q. The left-hand side of (3) can be written

$$-\int_a^f (f-w) U_1'\left(R_1(w) - Q(f-w)\right) d\mathbb{P}_1(w) + \int_f^b (w-f) U_1'\left(R_1(w) + Q(w-f)\right) d\mathbb{P}_1(w)$$

where both integrands are strictly positive. Since $\lim_{z\to-\infty} U_1'(z) = \infty$ and $\lim_{z\to\infty} U_1'(z) = 0$, the left-hand side of (3) tends to $-\infty$ as $Q \to \infty$, and tends to ∞ as $Q \to -\infty$. Thus for any choice of f, (3) has a unique solution $\widehat{Q}_1$. Similarly (4) has a unique solution $\widehat{Q}_2$.

Since $\mathbb{P}_1(w)$ has bounded support, $w - a > 0$ and $b - w > 0$ for all $w \in \operatorname{supp}(\mathbb{P}_1)$. From (3), $\widehat{Q}_1(f)$ for the buyer satisfies

$$\int_a^b (w-f) U_1'\left(R_1(w) + \widehat{Q}_1(f)(w-f)\right) d\mathbb{P}_1(w) = 0,$$

so if $f \to a$ then for all $w \in \operatorname{supp}(\mathbb{P}_1)$

$$\lim_{f\to a} U_1'\left(R_1(w) + \widehat{Q}_1(f)(w-f)\right) = 0,$$

implying $\lim_{f\to a} \widehat{Q}_1(f) = +\infty$. Similarly if $f \to b$ then for all $w \in \operatorname{supp}(\mathbb{P}_1)$

$$\lim_{f\to b} U_1'\left(R_1(w) + \widehat{Q}_1(f)(w-f)\right) = 0,$$

implying $\lim_{f\to b} \widehat{Q}_1(f) = -\infty$. In the same way, we can derive $\lim_{f\to a} \widehat{Q}_2(f) = -\infty$, $\lim_{f\to b} \widehat{Q}_1(f) = +\infty$.

The function $\widehat{Q}_1(f)$ (and similarly $\widehat{Q}_2(f)$) is continuous by virtue of (23) and the implicit function theorem, so the result follows by the intermediate value theorem. □

Proposition 2. *Any solution to the broker model with non-strategic offers is Pareto-optimal, i.e. it satisfies*

$$\Pi_1(Q,f) > \Pi_1(Q^*,f^*) \Rightarrow \Pi_2(Q,f) < \Pi_2(Q^*,f^*),$$

$$\Pi_2(Q,f) > \Pi_2(Q^*,f^*) \Rightarrow \Pi_1(Q,f) < \Pi_1(Q^*,f^*).$$

Proof Suppose $Q > 0$, and there is some $f \neq f^*$ for which the buyer has

$$\Pi_1(Q,f) > \Pi_1(Q^*,f^*). \tag{24}$$

By optimality of Q^* we have

$$\Pi_1(Q,f^*) \leq \Pi_1(Q^*,f^*). \tag{25}$$

If $f > f^*$ then

$$R_1(w) + Q(w-f) < R_1(w) + Q(w-f^*),$$

so the strict monotonicity of U_1 gives

$$\Pi_1(Q,f) < \Pi_1(Q,f^*)$$

which contradicts (24) and (25). Thus $f < f^*$. This means

$$R_2(w) + Q(f-w) < R_2(w) + Q(f^*-w),$$

so

$$\Pi_2(Q,f) < \Pi_2(Q,f^*) \leq \Pi_2(Q^*,f^*).$$

The argument with $Q < 0$ is analogous. □

Proposition 3. *Under Assumption 3 (CARA utility), the supply functions $\widehat{Q}_1(f)$ and $\widehat{Q}_2(f)$ are monotonic and there is a unique clearing price and quantity.*

Proof Differentiating both sides of (3) implicitly with respect to f gives

$$\int_a^b (w-f)U_1''\left(R_1(w) + \widehat{Q}_1(f)(w-f)\right)\left(\widehat{Q}_1'(f)(w-f) - \widehat{Q}_1(f)\right)d\mathbb{P}_1(w)$$
$$= \int_a^b U_1'\left(R_1(w) + \widehat{Q}_1(f)(w-f)\right)d\mathbb{P}_1(w).$$

Then we use Assumption 3 and write $\alpha = -U_1''(\cdot)/U_1'(\cdot) > 0$ for the coefficient of absolute risk aversion, to obtain

$$-\alpha\widehat{Q}_1'(f)\int_a^b U_1'\left(R_1(w)+\widehat{Q}_1(f)(w-f)\right)(w-f)^2 d\mathbb{P}_1(w) \\ +\alpha\widehat{Q}_1(f)\int_a^b (w-f)U_1'\left(R_1(w)+\widehat{Q}_1(f)(w-f)\right)d\mathbb{P}_1(w) \\ =\int_a^b U_1'\left(R_1(w)+\widehat{Q}_1(f)(w-f)\right)d\mathbb{P}_1(w).$$

Now the second term on the left hand side is zero from (3) and hence

$$\widehat{Q}_1'(f) = -\frac{\int_a^b U_1'\left(R_1(w)+\widehat{Q}_1(f)(w-f)\right)d\mathbb{P}_1(w)}{\alpha\int_a^b U_1'\left(R_1(w)+\widehat{Q}_1(f)(w-f)\right)(w-f)^2 d\mathbb{P}_1(w)} < 0,$$

since U_1' is positive. Similarly we can show that $\widehat{Q}_2'(f) > 0$.

The uniqueness result follows immediately from the existence of a clearing price established in Proposition 1 once monotonicity is proved. □

LEMMA 1. *Suppose $R_1(w)$ is nonincreasing in w and $R_2(w)$ is nondecreasing in w and $\mathbb{P}_1 = \mathbb{P}_2$. Then the solution to the broker model with non-strategic offers solution satisfies*

$$\min\{\lambda_1,\lambda_2\} \le Q \le \max\{\mu_1,\mu_2\}.$$

Proof Denote $\mathbb{P}_1$ and $\mathbb{P}_2$ by $\mathbb{P}$. Now observe that the values of

$$\lambda_i = \inf_{w_1\neq w_2}\left\{\frac{R_i(w_2)-R_i(w_1)}{w_1-w_2} : w_1,w_2\in \operatorname{supp}(\mathbb{P})\right\}$$

and

$$\mu_i = \sup_{w_1\neq w_2}\left\{\frac{R_i(w_2)-R_i(w_1)}{w_1-w_2} : w_1,w_2\in \operatorname{supp}(\mathbb{P})\right\}$$

remain unchanged if we define $R_i(w)$ for all $w\in[a,b]$ by linear interpolation between points in $\operatorname{supp}(\mathbb{P})$, and set

$$\lambda_i = \inf_{w_1\neq w_2}\left\{\frac{R_i(w_2)-R_i(w_1)}{w_1-w_2} : w_1,w_2\in [a,b]\right\}$$

and

$$\mu_i = \sup_{w_1\neq w_2}\left\{\frac{R_i(w_2)-R_i(w_1)}{w_1-w_2} : w_1,w_2\in [a,b]\right\}.$$

So we assume without loss of generality that $R_i(w)$ is defined for all $w \in [a,b]$.

Given f, the optimality conditions for the broker model with non-strategic offers are

$$\int_a^b (f-w)U_1'\left(R_1(w) - Q(f-w)\right) d\mathbb{P}(w) = 0, \tag{26}$$

$$\int_a^b (f-w)U_2'\left(R_2(w) + Q(f-w)\right) d\mathbb{P}(w) = 0. \tag{27}$$

Suppose $Q > \mu_1$. Then for all $w \neq f$,

$$Q > \frac{R_1(w) - R_1(f)}{f-w}$$

which gives

$$R_1(w) - Q(f-w) \begin{cases} < R_1(f), \text{ if } (f-w) > 0 \\ > R_1(f), \text{ if } (f-w) < 0 \end{cases},$$

thus giving by strict concavity of U_1,

$$(f-w)U_1'\left(R_1(w) - Q(f-w)\right) > (f-w)U_1'\left(R_1(f)\right).$$

Thus

$$\begin{aligned} U_1'\left(R_1(f)\right)\int_a^b (f-w)d\mathbb{P}(w) &< \int_a^b (f-w)U_1'\left(R_1(w) - Q(f-w)\right) d\mathbb{P}(w) \\ &= 0 \end{aligned}$$

by (26). It follows that

$$\int_a^b (f-w)d\mathbb{P}(w) < 0. \tag{28}$$

Now suppose $Q > \mu_2$. By a similar argument to the above we can show that for all w,

$$Q > \frac{R_2(f) - R_2(w)}{f-w}$$

so

$$R_2(w) + Q(f-w) \begin{cases} > R_2(f), \text{ if } (f-w) > 0 \\ < R_2(f), \text{ if } (f-w) < 0 \end{cases}$$

giving by strict concavity of U_2,

$$(f-w)U_2'\left(R_2(w) + Q(f-w)\right) < (f-w)U_2'\left(R_2(f)\right).$$

Thus

$$U_2'(R_2(f))\int_a^b (f-w)d\mathbb{P}(w) > \int_a^b (f-w)U_2'(R_2(w)+Q(f-w))\,d\mathbb{P}(w)$$
$$= 0,$$

by (27). Thus

$$\int_a^b (f-w)d\mathbb{P}(w) > 0$$

contradicting (28). It follows that $Q \le \max\{\mu_1,\mu_2\}$. By a similar argument $Q \ge \min\{\lambda_1,\lambda_2\}$ where

$$\lambda_i = \inf_{w_1\neq w_2}\{\left|\frac{R_i(w_1)-R_i(w_2)}{w_1-w_2}\right| : w_1,w_2 \in \mathrm{supp}(\mathbb{P})\}.$$

□

LEMMA 2. *Suppose* $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = \lambda$. *Then*

$$Q\begin{cases} > \\ = \lambda \\ < \end{cases} \iff \begin{cases} \mathbb{E}_{\mathbb{P}_1}[w] > \mathbb{E}_{\mathbb{P}_2}[w] \\ \mathbb{E}_{\mathbb{P}_1}[w] = \mathbb{E}_{\mathbb{P}_2}[w] \\ \mathbb{E}_{\mathbb{P}_1}[w] < \mathbb{E}_{\mathbb{P}_2}[w] \end{cases}$$

Proof ($\Longrightarrow$) If $Q > \lambda$ then $\lambda = \mu_1$ and the proof of Lemma 1 shows

$$f < \int_a^b w d\mathbb{P}_1(w).$$

Similarly from $\lambda = \mu_2$ we have

$$f > \int_a^b w d\mathbb{P}_2(w)$$

so $\mathbb{E}_{\mathbb{P}_1}[w] > \mathbb{E}_{\mathbb{P}_2}[w]$. Similarly If $Q < \lambda$, then we have $\mathbb{E}_{\mathbb{P}_1}[w] < \mathbb{E}_{\mathbb{P}_2}[w]$.

If $Q = \lambda$ then

$$R_1(w) - Q(f-w) = R_1(f).$$

so

$$\int_a^f (f-w)U_1'(R_1(w)-Q(f-w))\,d\mathbb{P}(w) = U_1'(R_1(f))\int_a^f (f-w)d\mathbb{P}_1(w).$$

Similarly

$$\int_f^b (w-f)U_1'(R_1(w)+Q(w-f))\,d\mathbb{P}(w) = U_1'(R_1(f))\int_f^b (f-w)d\mathbb{P}_1(w).$$

It follows from (26) that

$$f = \int_a^b w d\mathbb{P}_1(w).$$

Similarly from (27) we have $f = \int_a^b w d\mathbb{P}_2(w)$.

($\Longleftarrow$) $\mathbb{E}_{\mathbb{P}_1}[w] = \mathbb{E}_{\mathbb{P}_2}[w] \implies Q = \lambda$ follows directly from contraposing the above. If $\mathbb{E}_{\mathbb{P}_1}[w] > \mathbb{E}_{\mathbb{P}_2}[w]$ then we have $Q \geq \lambda$ and $Q \neq \lambda$, so $Q > \lambda$. Similarly if $\mathbb{E}_{\mathbb{P}_1}[w] < \mathbb{E}_{\mathbb{P}_2}[w]$ then we have $Q < \lambda$. $\square$

Lemma 3. *Suppose*

$$R_2(w_H) - R_2(w_L) = R_1(w_L) - R_1(w_H) = \Delta. \tag{29}$$

The broker solution with non-strategic offers with two price outcomes, w_L and w_H, and CARA utilities is

$$Q = \frac{\Delta}{w_H - w_L} + \frac{1}{(\alpha_1 + \alpha_2)(w_H - w_L)} \log\left(\frac{\rho_1(1-\rho_2)}{(1-\rho_1)\rho_2}\right),$$
$$f = \frac{w_H + w_L \kappa}{1+\kappa},$$

where $\kappa = \left(\frac{(1-\rho_1)}{\rho_1}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} \left(\frac{(1-\rho_2)}{\rho_2}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}}$.

Proof In this case the optimality conditions for the buyer are

$$\frac{\rho_1(w_H - f)}{(1-\rho_1)(f - w_L)} = \frac{\exp(-\alpha_1 R_1(w_L) - \alpha Q(w_L - f))}{\exp(-\alpha_1 R_1(w_H) - \alpha Q(w_H - f))}.$$

Thus

$$\log\left(\frac{\rho_1(w_H - f)}{(1-\rho_1)(f - w_L)}\right) - \alpha_1 \left(R_1(w_H) + Q(w_H - f)\right) + \alpha_1 \left(R_1(w_L) + Q(w_L - f)\right) = 0.$$

Hence the optimal contract quantity for the buyer is

$$Q = \frac{R_1(w_L) - R_1(w_H)}{w_H - w_L} + \frac{1}{\alpha_1(w_H - w_L)} \log\left(\frac{\rho_1(w_H - f)}{(1-\rho_1)(f - w_L)}\right).$$

The seller's supply function (maximizing his expected utility) can be obtained similarly, and we get

$$Q = \frac{R_2(w_H) - R_2(w_L)}{w_H - w_L} - \frac{1}{\alpha_2(w_H - w_L)} \log\left(\frac{\rho_2(w_H - f)}{(1-\rho_2)(f - w_L)}\right).$$

From these alternative expressions for Q we can find the value of f at which the market clears. We obtain

$$\frac{1}{\alpha_1} \log\left(\frac{\rho_1(w_H - f)}{(1-\rho_1)(f - w_L)}\right) + \frac{1}{\alpha_2} \log\left(\frac{\rho_2(w_H - f)}{(1-\rho_2)(f - w_L)}\right) = R_2(w_H) - R_2(w_L) - R_1(w_L) + R_1(w_H).$$

With (29) the right hand side is zero, and so

$$\left(\frac{\rho_1(w_H-f)}{(1-\rho_1)(f-w_L)}\right)^{\alpha_2} = \left(\frac{\rho_2(w_H-f)}{(1-\rho_2)(f-w_L)}\right)^{-\alpha_1}.$$

This simplifies to

$$f = \frac{w_H + w_L\kappa}{1+\kappa}$$

as required.

Now we calculate the clearing quantity as

$$\begin{aligned} Q &= \frac{\Delta}{w_H - w_L} + \frac{1}{\alpha_1(w_H-w_L)}\log\left(\frac{\rho_1}{(1-\rho_1)}\left(\frac{(1-\rho_1)}{\rho_1}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}\left(\frac{(1-\rho_2)}{\rho_2}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\right) \\ &= \frac{\Delta}{w_H - w_L} + \frac{1}{(\alpha_1+\alpha_2)(w_H-w_L)}\log\left(\frac{\rho_1(1-\rho_2)}{(1-\rho_1)\rho_2}\right). \end{aligned}$$

□

Proposition 4. *If $U_1 = U_2$, and for every y, $R_1(\widehat{f}+y) = R_2(\widehat{f}-y)$, and $\mathbb{P}_1([a,\widehat{f}+y)) = \mathbb{P}_2((\widehat{f}-y),b]$, where $\widehat{f} = \frac{a+b}{2}$, then the Nash bargaining solution matches the non-strategic offer solution.*

Proof We use the expressions (1) and (2) for $\Pi_i(Q,f)$. Now assume $R_1(\widehat{f}+y) = R_2(\widehat{f}-y)$ and $\mathbb{P}_1([a,\widehat{f}+y)) = \mathbb{P}_2((\widehat{f}-y,b])$. Let $\delta = \frac{b-a}{2}$. Then setting $w = \widehat{f}+y$ gives

$$\begin{aligned} \Pi_1(Q,\widehat{f}) &= \int_{y=-\delta}^{y=\delta} U\left(R_1(\widehat{f}+y) + Q(\widehat{f}+y-\widehat{f})\right) d\mathbb{P}_1(\widehat{f}+y) \\ &= \int_{y=-\delta}^{y=\delta} U\left(R_2(\widehat{f}-y) + Qy\right) d\mathbb{P}_1(\widehat{f}+y) \\ &= -\int_{y=-\delta}^{y=\delta} U\left(R_2(\widehat{f}-y) + Qy\right) d\mathbb{P}_2(\widehat{f}-y) \\ &= -\int_{z=b}^{z=a} U\left(R_2(z) + Q(\widehat{f}-z)\right) d\mathbb{P}_2(z), \text{ setting } z = \widehat{f}-y, \\ &= \int_{z=a}^{z=b} U\left(R_2(z) + Q(\widehat{f}-z)\right) d\mathbb{P}_2(z) \\ &= \Pi_2(Q,\widehat{f}). \end{aligned} \tag{30}$$

Thus if $\mathbb{P}_1([a,\widehat{f}+y)) = \mathbb{P}_2((\widehat{f}-y,b])$ then maximizing $\Pi_1(Q,\widehat{f})$ over Q has a solution, $\widehat{Q}$, which is the same as when we maximize $\Pi_2(Q,\widehat{f})$ over Q.

Also

$$\begin{aligned}\left[\frac{\partial}{\partial f}\Pi_1(Q,f)\right]_{f=\widehat{f}} &= -Q\int_{y=-\delta}^{y=\delta} U'\left(R_1(\widehat{f}+y)+Q(\widehat{f}+y-\widehat{f})\right)d\mathbb{P}_1(\widehat{f}+y)\\ &= -Q\int_{y=-\delta}^{y=\delta} U'\left(R_2(\widehat{f}-y)+Qy\right)d\mathbb{P}_1(\widehat{f}+y)\\ &= Q\int_{y=-\delta}^{y=\delta} U'\left(R_2(\widehat{f}-y)+Qy\right)d\mathbb{P}_2(\widehat{f}-y)\\ &= Q\int_{z=b}^{z=a} U'\left(R_2(z)+Q(\widehat{f}-z)\right)d\mathbb{P}_2(z)\\ &= -Q\int_{z=a}^{z=b} U'\left(R_2(z)+Q(\widehat{f}-z)\right)d\mathbb{P}_2(z)\\ &= -\left[\frac{\partial}{\partial f}\Pi_2(Q,f)\right]_{f=\widehat{f}}. \end{aligned} \tag{31}$$

Consider the first order conditions for the Nash bargaining solution to (14). We need

$$(\Pi_2(Q,f)-\Pi_2(0,f))\frac{\partial}{\partial Q}\Pi_1(Q,f)+(\Pi_1(Q,f)-\Pi_1(0,f))\frac{\partial}{\partial Q}\Pi_2(Q,f)=0, \tag{32}$$

$$\begin{aligned}&(\Pi_2(Q,f)-\Pi_2(0,f))\left(\frac{\partial}{\partial f}\Pi_1(Q,f)-\frac{\partial}{\partial f}\Pi_1(0,f)\right)\\ &\qquad+(\Pi_1(Q,f)-\Pi_1(0,f))\left(\frac{\partial}{\partial f}\Pi_2(Q,f)-\frac{\partial}{\partial f}\Pi_2(0,f)\right)=0. \end{aligned} \tag{33}$$

Since we know that $\frac{\partial}{\partial Q}\Pi_i(Q,f)=0$ when $f=\widehat{f}$ and $Q=\widehat{Q}$, (32) is satisfied immediately at $(\widehat{Q},\widehat{f})$.

From (30) we know that

$$\left(\Pi_1(\widehat{Q},\widehat{f})-\Pi_1(0,\widehat{f})\right)=\left(\Pi_2(\widehat{Q},\widehat{f})-\Pi_2(0,\widehat{f})\right).$$

From (31) we have

$$\frac{\partial}{\partial f}\Pi_1(Q,f)-\frac{\partial}{\partial f}\Pi_1(0,f)=-\left(\frac{\partial}{\partial f}\Pi_2(Q,f)-\frac{\partial}{\partial f}\Pi_2(0,f)\right),$$

and so (33) is satisfied at $(\widehat{Q},\widehat{f})$. Hence we have established the result we need. □

PROPOSITION 5. *The contract quantity in the Nash bargaining solution will match the broker solution with non-strategic offers when there are CARA utilities.*

Proof In Nash bargaining, we seek the maximum over Q and f of

$$\left(\int_a^b (U_1(Q,f)-U_1(0,0))\,d\mathbb{P}_1\right)\left(\int_a^b (U_2(Q,f)-U_2(0,0))\,d\mathbb{P}_2\right).$$

With CARA utilities we have

$$U_1(Q,f) = 1 - \exp(-\alpha_1 R_1(w) - \alpha_1 Q(w-f)),$$

$$U_2(Q,f) = 1 - \exp(-\alpha_2 R_2(w) - \alpha_2 Q(f-w)).$$

The first order conditions defining the non-strategic offer optimum (Q^*, f^*) are

$$\int_a^b (w - f^*)\alpha_1 \exp(-\alpha_1 (R_1(w) + Q^*(w - f^*)))d\mathbb{P}_1(w) = 0,$$

$$\int_a^b (f^* - w)\alpha_2 \exp(-\alpha_2 (R_2(w) + Q^*(f^* - w)))d\mathbb{P}_2(w) = 0,$$

giving

$$\int_a^b (w - f^*) \exp(-\alpha_1 R_1(w) - \alpha_1 Q^* w)d\mathbb{P}_1 = 0,$$
$$\int_a^b (f^* - w) \exp(-\alpha_2 R_2(w) + \alpha_2 Q^* w))d\mathbb{P}_2 = 0,$$

after cancelling term $\alpha_1 \exp(\alpha_1 Q^* f^*)$ in the first equation and $\alpha_2 \exp(-\alpha_2 Q^* f^*)$ in the second. We will write

$$\begin{aligned}
X_1(Q^*) &= \int_a^b \exp(-\alpha_1 R_1(w) - \alpha_1 Q^* w)\ d\mathbb{P}_1, \\
Y_1(Q^*) &= \int_a^b w \exp(-\alpha_1 R_1(w) - \alpha_1 Q^* w)\ d\mathbb{P}_1, \\
X_2(Q^*) &= \int_a^b \exp(-\alpha_2 R_2(w) + \alpha_2 Q^* w))\ d\mathbb{P}_2, \\
Y_2(Q^*) &= \int_a^b w \exp(-\alpha_2 R_2(w) + \alpha_2 Q^* w))\ d\mathbb{P}_2.
\end{aligned}$$

Thus $Y_1(Q^*) = f^* X_1(Q^*)$ and $Y_2(Q^*) = f^* X_2(Q^*)$. We can eliminate f^* and obtain

$$Y_1(Q^*)X_2(Q^*) = Y_2(Q^*)X_1(Q^*).$$

Now we turn to the first-order conditions for the Nash bargaining problem

$$\int_a^b \frac{\partial U_1(Q,f)}{\partial Q}\ d\mathbb{P}_1 \left(\int_a^b (U_2(Q,f) - U_2(0))\, d\mathbb{P}_2\right) + \int_a^b \frac{\partial U_2(Q,f)}{\partial Q}\ d\mathbb{P}_2 \left(\int_a^b (U_1(Q,f) - U_1(0))\, d\mathbb{P}_1\right) = 0,$$
$$\int_a^b \frac{\partial U_1(Q,f)}{\partial f}\ d\mathbb{P}_1 \left(\int_a^b (U_2(Q,f) - U_2(0))\, d\mathbb{P}_2\right) + \int_a^b \frac{\partial U_2(Q,f)}{\partial f}\ d\mathbb{P}_2 \left(\int_a^b (U_1(Q,f) - U_1(0))\, d\mathbb{P}_1\right) = 0.$$

From the Q derivative

$$\alpha_1 \left(Y_1(Q) - fX_1(Q)\right) \exp(\alpha_1 Qf) \left(\int_a^b \exp(-\alpha_2 R_2(w) d\mathbb{P}_2 - X_2(Q) \exp(-\alpha_2 Qf) \right)$$
$$= \alpha_2 (Y_2(Q) - fX_2(Q)) \exp(-\alpha_2 Qf) \left(\int_a^b \exp(-\alpha_1 R_1(w) d\mathbb{P}_1 - X_1(Q) \exp(\alpha_1 Qf) \right).$$

From the f derivative

$$-\alpha_1 QX_1(Q) \exp(\alpha_1 Qf) \left(\int_a^b \exp(-\alpha_2 R_2(w) d\mathbb{P}_2 - X_2(Q) \exp(-\alpha_2 Qf) \right)$$
$$+ \alpha_2 QX_2(Q) \exp(-\alpha_2 Qf) \left(\int_a^b \exp(-\alpha_1 R_1(w) d\mathbb{P}_1 - X_1(Q) \exp(\alpha_1 Qf) \right) = 0.$$

Thus

$$\frac{\alpha_1 X_1(Q)}{\alpha_2 X_2(Q)} = \frac{\exp(-\alpha_2 Qf) \left(\int_a^b \exp(-\alpha_1 R_1(w) d\mathbb{P}_1 - X_1(Q) \exp(\alpha_1 Qf) \right)}{\exp(\alpha_1 Qf) \left(\int_a^b \exp(-\alpha_2 R_2(w) d\mathbb{P}_2 - X_2(Q) \exp(-\alpha_2 Qf) \right)}.$$

Hence

$$\alpha_1 \left(Y_1(Q) - fX_1(Q)\right) = \alpha_2 (Y_2(Q) - fX_2(Q)) \frac{\alpha_1 X_1(Q)}{\alpha_2 X_2(Q)},$$

and so

$$Y_1(Q) X_2(Q) = Y_2(Q) X_1(Q).$$

This has a solution Q^* given by the broker solution with non-strategic offers. $\square$

Lemma 4. *Under Assumption 2, the solution (Q, f) to the Nash bargaining solution with two price outcomes, w_L and w_H, and CARA utilities with $\alpha_1 = \alpha_2 = \alpha$ has*

$$f = w_L + \frac{1}{2\alpha Q} \log \left(\frac{e^{-\alpha s} \sigma \left(-\rho_2 + \sigma \rho_2 + 1\right) \left(e^{-\alpha r} + e^{-\alpha s} \rho_1 - e^{-\alpha r} \rho_1\right)}{e^{-\alpha r} \left(\sigma + \rho_1 - \sigma \rho_1\right) \left(e^{-\alpha s} - e^{-\alpha s} \rho_2 + e^{-\alpha r} \rho_2\right)} \right).$$

Proof The first order conditions for the Nash bargaining solution are (differentiating with respect to f)

$$-Q\rho_1 U_1' \left(s + Q(w_H - f)\right) - Q(1 - \rho_1) U_1' \left(r + Q(w_L - f)\right)$$
$$\times \left(\rho_2 U_2 \left(r + Q(f - w_H)\right) + (1 - \rho_2) U_2 \left(s + Q(f - w_L)\right) - \rho_2 U_2 (r) - (1 - \rho_2) U_2 (s)\right)$$
$$+\rho_2 Q U_2' \left(r + Q(f - w_H)\right) + (1 - \rho_2) Q(f - w_L) U_2' \left(s + Q(f - w_L)\right)$$
$$\times \left(\rho_1 U_1 \left(s + Q(w_H - f)\right) + (1 - \rho_1) U_1 \left(r + Q(w_L - f)\right) - \rho_1 U_1 (s) - (1 - \rho_1) U_1 (r)\right) = 0. \quad (34)$$

From (34) we have when $\alpha_1 = \alpha_2 = \alpha$

$$\begin{aligned}
&\left(-\rho_1 e^{-\alpha s-\alpha Q(w_H)} - (1-\rho_1)e^{-\alpha r-\alpha Q(w_L)}\right) \\
&\qquad \times \left(\rho_2 e^{-\alpha r}(e^{\alpha Qf} - e^{-\alpha Q(-w_H)}) + (1-\rho_2)e^{-\alpha s}(e^{\alpha Qf} - e^{-\alpha Q(-w_L)})\right) \\
&\qquad\qquad + \left(\rho_2 e^{-\alpha r-\alpha Q(-w_H)} + (1-\rho_2)e^{-\alpha s-\alpha Q(-w_L)}\right) \\
&\qquad\qquad \times \left(\rho_1 e^{-\alpha s}(e^{-\alpha Qf} - e^{-\alpha Q(w_H)}) + (1-\rho_1)e^{-\alpha r}(e^{-\alpha Qf} - e^{-\alpha Q(w_L)})\right) = 0,
\end{aligned}$$

and from the solution for Q we know

$$\rho_1(1-\rho_2)e^{2r\alpha}e^{2(Q\alpha w_L)} = (1-\rho_1)\rho_2 e^{2s\alpha}e^{2(Q\alpha w_H)},$$

so $e^{Q\alpha w_H} = e^{\alpha(r-s)}e^{Q\alpha w_L}\sigma$. Thus

$$\begin{aligned}
&\left(-\rho_1\gamma^{-1} e^{-\alpha r-\alpha Q w_L} - (1-\rho_1)e^{-\alpha r-\alpha Q w_L}\right) \\
&\qquad \times \left(\rho_2 e^{-\alpha r}(e^{\alpha Qf} - e^{\alpha(r-s)}e^{Q\alpha w_L}\sigma) + (1-\rho_2)e^{-\alpha s}(e^{\alpha Qf} - e^{\alpha Q w_L})\right) \\
&\qquad\qquad + \left(\rho_2 e^{-\alpha s+Q\alpha w_L}\sigma + (1-\rho_2)e^{-\alpha s+\alpha Q w_L}\right) \\
&\qquad \times \left(\rho_1 e^{-\alpha s}(e^{-\alpha Qf} - e^{-\alpha(r-s)}e^{-Q\alpha w_L}\sigma^{-1}) + (1-\rho_1)e^{-\alpha r}(e^{-\alpha Qf} - e^{-\alpha Q w_L})\right) = 0.
\end{aligned}$$

This is a quadratic in $e^{\alpha Qf}$ which has a single positive root

$$e^{\alpha Qf} = e^{Q\alpha w_L}\sqrt{\left(\frac{e^{-\alpha s}\sigma\left(-\rho_2+\sigma\rho_2+1\right)\left(e^{-\alpha r}+e^{-\alpha s}\rho_1-e^{-\alpha r}\rho_1\right)}{e^{-\alpha r}\left(\sigma+\rho_1-\sigma\rho_1\right)\left(e^{-\alpha s}-e^{-\alpha s}\rho_2+e^{-\alpha r}\rho_2\right)}\right)}$$

from which the contract price f can be derived in terms of Q, namely

$$f = w_L + \frac{1}{2\alpha Q}\log\left(\frac{e^{-\alpha s}\sigma\left(-\rho_2+\sigma\rho_2+1\right)\left(e^{-\alpha r}+e^{-\alpha s}\rho_1-e^{-\alpha r}\rho_1\right)}{e^{-\alpha r}\left(\sigma+\rho_1-\sigma\rho_1\right)\left(e^{-\alpha s}-e^{-\alpha s}\rho_2+e^{-\alpha r}\rho_2\right)}\right).$$

□

Proposition 6. *With two outcomes and Assumption 2, for the supply-function equilibrium both with and without information deduction, if the first order conditions for firm 1 holds for all* (ρ_1, ρ_2) *and the solution* $Q(\rho_1,\rho_2)$, $f(\rho_1,\rho_2)$ *satisfies the symmetry conditions:*

$$Q(\rho_1,\rho_2) = Q(1-\rho_2, 1-\rho_1), \quad f(\rho_1,\rho_2) = w_H + w_L - f(1-\rho_2, 1-\rho_1),$$

then the first order conditions for firm 2 also holds.

Proof We write $\eta_i = 1-\rho_i$, $i=1,2$. Then $Q_2'(\rho_1,\rho_2) = -Q_1'(\eta_2,\eta_1)$ and $f_2'(\rho_1,\rho_2) = f_1'(\eta_2,\eta_1)$. Begin with the case without information deduction. The first-order conditions are

$$\begin{aligned}\rho_1 U'\left(s+Q(\rho_1,\rho_2)(w_H - f(\rho_1,\rho_2))\right)\left(Q_1'(\rho_1,\rho_2)(w_H-f(\rho_1,\rho_2)) - Q(\rho_1,\rho_2)f_1'(\rho_1,\rho_2)\right) \\ +(1-\rho_1)U'\left(r+Q(\rho_1,\rho_2)(w_L-f(\rho_1,\rho_2))\right) \\ \times(Q_1'(\rho_1,\rho_2)(w_L-f(\rho_1,\rho_2)) - Q(\rho_1,\rho_2)f_1'(\rho_1,\rho_2)) = 0,\end{aligned} \tag{35}$$

$$\begin{aligned}\rho_2 U'\left(r+Q(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_H)\right)\left(Q_2'(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_H) + Q(\rho_1,\rho_2)f_2'(\rho_1,\rho_2)\right) \\ +(1-\rho_2)U'\left(s+Q(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_L)\right) \\ \times(Q_2'(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_L) + Q(\rho_1,\rho_2)f_2'(\rho_1,\rho_2)) = 0.\end{aligned} \tag{36}$$

Thus we can rewrite (36) as

$$\begin{aligned}(1-\eta_2)U'\left(r+Q(\eta_2,\eta_1)(w_L-f(\eta_2,\eta_1))\right)\left(-Q_1'(\eta_2,\eta_1)(w_L-f(\eta_2,\eta_1)) + Q(\eta_2,\eta_1)f_1'(\eta_2,\eta_1)\right) \\ +\eta_2 U'\left(s+Q(\eta_2,\eta_1)(w_H-f(\eta_2,\eta_1))\right)\left(-Q_1'(\eta_2,\eta_1)(w_H-f(\eta_2,\eta_1)) + Q(\eta_2,\eta_1)f_1'(\eta_2,\eta_1)\right) = 0,\end{aligned}$$

which is simply (35) multiplied through by -1 and with ρ_1 replaced with η_2, and ρ_2 replaced with η_1. Thus if (19) holds, then it is enough that (36) is true everywhere to deduce that both sets of first order conditions are satisfied.

Now we consider the supply-function equilibrium with information deduction. The first-order conditions become

$$\begin{aligned}\frac{\rho_1+\rho_2}{2}U'\left(s+Q(\rho_1,\rho_2)(w_H - f(\rho_1,\rho_2))\right)\left(Q_1'(\rho_1,\rho_2)(w_H-f(\rho_1,\rho_2)) - Q(\rho_1,\rho_2)f_1'(\rho_1,\rho_2)\right) \\ +(1-\frac{\rho_1+\rho_2}{2})U'\left(r+Q(\rho_1,\rho_2)(w_L-f(\rho_1,\rho_2))\right) \\ \times(Q_1'(\rho_1,\rho_2)(w_L-f(\rho_1,\rho_2)) - Q(\rho_1,\rho_2)f_1'(\rho_1,\rho_2)) = 0,\end{aligned} \tag{37}$$

$$\begin{aligned}\frac{\rho_1+\rho_2}{2}U'\left(r+Q(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_H)\right)\left(Q_2'(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_H) + Q(\rho_1,\rho_2)f_2'(\rho_1,\rho_2)\right) \\ +(1-\frac{\rho_1+\rho_2}{2})U'\left(s+Q(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_L)\right) \\ \times(Q_2'(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_L) + Q(\rho_1,\rho_2)f_2'(\rho_1,\rho_2)) = 0.\end{aligned} \tag{38}$$

In the same way as before, when (19) holds, we can rewrite (38) as

$$(1-\frac{\eta_1+\eta_2}{2})U'\left(r+Q(\eta_2,\eta_1)(w_L-f(\eta_2,\eta_1))\right)\left(-Q_1'(\eta_2,\eta_1)(w_L-f(\eta_2,\eta_1))+Q(\eta_2,\eta_1)f_1'(\eta_2,\eta_1)\right)$$
$$+(\frac{\eta_1+\eta_2}{2})U'\left(s+Q(\eta_2,\eta_1)(w_H-f(\eta_2,\eta_1))\right)\left(-Q_1'(\eta_2,\eta_1)(w_H-f(\eta_2,\eta_1))+Q(\eta_2,\eta_1)f_1'(\eta_2,\eta_1)\right)=0,$$

which is (37) multiplied through by -1 and with ρ_1 replaced with η_2, and ρ_2 replaced with η_1. Thus when (37) is true everywhere we can deduce that both sets of first order conditions are satisfied.

□

Appendix B: Numerical solution of anti-symmetric equilibrium

We derive here some results that are used to compute the numerical solution of the supply-function equilibria.

PROPOSITION 7. *Under Assumption 2, for a supply-function equilibrium without information deduction, the function $h(\rho)=Q(\rho,1-\rho)$ satisfies the differential equation*

$$h'(\rho)=\frac{2h(\rho)}{\gamma}f_d\frac{\rho U'\left(s+\gamma h(\rho)\right)+(1-\rho)U'\left(r-\gamma h(\rho)\right)}{\rho U'\left(s+\gamma h(\rho)\right)-(1-\rho)U'\left(r-\gamma h(\rho)\right)}$$

for some positive constant f_d, and $\gamma=\frac{w_H-w_L}{2}$. With information deduction the differential equation is

$$h'(\rho)=\frac{2h(\rho)}{\gamma}f_d\frac{U'\left(s+\gamma h(\rho)\right)+U'\left(r-\gamma h(\rho)\right)}{U'\left(s+\gamma h(\rho)\right)-U'\left(r-\gamma h(\rho)\right)}$$

and all supply-function equilibria have contract quantities bounded above by $Q^*=\frac{r-s}{w_H-w_L}$.

Proof We start with the case without information deduction. Then from (35) we have

$$\rho U'\left(s+h(\rho)\gamma\right)\left(Q_1'(\rho,1-\rho)\gamma-h(\rho)f_1'(\rho,1-\rho)\right)$$
$$+(1-\rho)U'\left(r-\gamma h(\rho)\right)\left(-\gamma Q_1'(\rho,1-\rho)-h(\rho)f_1'(\rho,1-\rho)\right)=0,$$

where $\gamma=(w_H-w_L)/2$. Thus

$$Q_1'(\rho,1-\rho)=\frac{h(\rho)}{\gamma}f_1'(\rho,1-\rho)\frac{\rho U'\left(s+h(\rho)\gamma\right)+(1-\rho)U'\left(r-\gamma h(\rho)\right)}{\rho U'\left(s+h(\rho)\gamma\right)-(1-\rho)U'\left(r-\gamma h(\rho)\right)} \tag{39}$$

Suppose we start at a point on the central line $(\rho,1-\rho)$ and move to the point $(\rho+\delta,1-\rho)$. Then

$$Q(\rho+\delta,1-\rho)=h(\rho)+\delta Q_1'(\rho,1-\rho)+O(\delta^2),$$
$$f(\rho+\delta,1-\rho)=(w_H+w_L)/2+\delta f_1'(\rho,1-\rho)+O(\delta^2).$$

But we can also consider starting at the point $(\rho+\delta, 1-\rho-\delta)$ and moving to the point $(\rho+\delta, 1-\rho)$. This gives

$$Q(\rho+\delta, 1-\rho) = h(\rho+\delta) + \delta Q_2'(\rho+\delta, 1-\rho-\delta) + O(\delta^2),$$

$$f(\rho+\delta, 1-\rho) = (w_H + w_L)/2 + \delta f_2'(\rho+\delta, 1-\rho-\delta) + O(\delta^2).$$

Equating these expressions and observing that in this case we have $Q_2'(\rho+\delta, 1-\rho-\delta) = -Q_1'(\rho+\delta, 1-\rho-\delta)$ and $f_2'(\rho+\delta, 1-\rho-\delta) = f_1'(\rho+\delta, 1-\rho-\delta)$, shows:

$$f_1'(\rho, 1-\rho) = f_1'(\rho+\delta, 1-\rho-\delta) + O(\delta^2),$$

$$h(\rho+\delta) - \delta Q_1'(\rho+\delta, 1-\rho-\delta) = h(\rho) + \delta Q_1'(\rho, 1-\rho) + O(\delta^2).$$

Thus we can demonstrate (considering δ small) that f_1' takes the same value on $(\rho, 1-\rho)$ for all values of ρ. In other words it is a constant say f_d .

Then

$$h(\rho+\delta) - h(\rho) = \delta\left(Q_1'(\rho, 1-\rho) + Q_1'(\rho+\delta, 1-\rho-\delta)\right) + O(\delta^2).$$

Thus letting $\delta \to 0$ and using continuity of $Q_1'(\rho, 1-\rho)$ we obtain from (39)

$$h'(\rho) = \frac{2h(\rho)}{\gamma} f_d \frac{\rho U'(s+\gamma h(\rho)) + (1-\rho)U'(r-\gamma h(\rho))}{\rho U'(s+\gamma h(\rho)) - (1-\rho)U'(r-\gamma h(\rho))}. \tag{40}$$

In the case with information deduction we use (37) and this equation becomes

$$h'(\rho) = \frac{2h(\rho)}{\gamma} f_d \frac{U'(s+\gamma h(\rho)) + U'(r-\gamma h(\rho))}{U'(s+\gamma h(\rho)) - U'(r-\gamma h(\rho))}. \tag{41}$$

Now notice that with information deduction, to avoid $h'(\rho)$ becoming infinite or negative we require

$$U'(s+\gamma h(\rho)) > U'(r-\gamma h(\rho))$$

which implies (since U' is decreasing) that $s+\gamma h(\rho) < r-\gamma h(\rho)$, i.e. $h(\rho) < (r-s)/(2\gamma) = Q^*$. This is the highest value of Q possible when $\rho_1 + \rho_2 = 1$ and occurs when ρ_1 takes its highest value. However this is also the ρ combination that leads to the highest possible contract quantity, and hence we have Q^* as an overall bound on Q values. $\square$

When $U(x)=1-e^{-\alpha x}$, $U'(x)=\alpha e^{-\alpha x}$ and the first order conditions become (using S_1 and S_2 for the values of $\partial Q/df$ along the $\rho_1=$ constant and $\rho_2=$ constant lines respectively):

$$\begin{aligned}&\rho_1 \exp\left(-\alpha s-\alpha Q(\rho_1,\rho_2)(w_H-f(\rho_1,\rho_2))\right)\left(S_2(\rho_1,\rho_2)(w_H-f(\rho_1,\rho_2))-Q(\rho_1,\rho_2)\right)\\ &\quad+(1-\rho_1)\exp\left(-\alpha r-\alpha Q(\rho_1,\rho_2)(w_L-f(\rho_1,\rho_2))\right)\left(S_2(\rho_1,\rho_2)(w_L-f(\rho_1,\rho_2))-Q(\rho_1,\rho_2)\right)=0,\end{aligned}$$

$$\begin{aligned}&\rho_2 \exp\left(-\alpha r-\alpha Q(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_H)\right)\left(S_1(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_H)+Q(\rho_1,\rho_2)\right)\\ &\quad+(1-\rho_2)\exp\left(-\alpha s-\alpha Q(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_L)\right)\left(S_1(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_L)+Q(\rho_1,\rho_2)\right)=0.\end{aligned}$$

Thus

$$S_1=Q\frac{\rho_2\exp\left(-\alpha r-\alpha Q(f-w_H)\right)+(1-\rho_2)\exp\left(-\alpha s-\alpha Q(f-w_L)\right)}{\rho_2(w_H-f)\exp\left(-\alpha r-\alpha Q(f-w_H)\right)+(1-\rho_2)(w_L-f)\exp\left(-\alpha s-\alpha Q(f-w_L)\right)},\tag{42}$$

$$S_2=Q\frac{\rho_1\exp\left(-\alpha s-\alpha Q(w_H-f)\right)+(1-\rho_1)\exp\left(-\alpha r-\alpha Q(w_L-f)\right)}{\rho_1(w_H-f)\exp\left(-\alpha s-\alpha Q(w_H-f)\right)+(1-\rho_1)(w_L-f)\exp\left(-\alpha r-\alpha Q(w_L-f)\right)}.\tag{43}$$

With information deduction the equations are the same with both ρ_1 and ρ_2 replaced with $(\rho_1+\rho_2)/2$.

Next we discuss how these equations can be used to construct a numerical solution for all pairs (ρ_1,ρ_2) in a given range. For the case without information deduction we begin by constructing a solution to the differential equation (40) for the Q value along the line where $\rho_1+\rho_2=1$, (on this line we have $f=(w_H+w_L)/2$). We will use an iterative process where at each stage we suppose we know the Q and f values for all ρ_1, ρ_2 values with $\rho_1+\rho_2=K$, then for a chosen small increment δ_ρ we construct the (Q,f) values along the line $\rho_1+\rho_2=K+\delta_\rho$. We do this by simply finding the crossing point when we extend the $\rho_1=$ constant, and $\rho_2=$ constant curves from the previous set of values. We already know the values (Q_A,f_A) at $(\rho_{1,}\rho_2-\delta_\rho)$ and the values (Q_B,f_B) at $(\rho_1-\delta_\rho,\rho_2)$. We writing S_{1A} for S_1 from (42) evaluated at (Q_A,f_A) with ρ_2 taking the value $\rho_2-\delta_\rho$, and we write S_{2B} for S_2 from (43) evaluated at (Q_B,f_B) with ρ_1 taking the value $\rho_1-\delta_\rho$. Then the crossing occurs when

$$Q_A+S_{1A}(f-f_A)=Q_B+S_{2B}(f-f_B),$$

so we can deduce

$$f=\frac{Q_A-Q_B+S_{2B}f_B-S_{1A}f_A}{S_{2B}-S_{1A}},$$
$$Q=\frac{S_{2B}Q_A-S_{1A}Q_B+S_{1A}S_{2B}(f_B-f_A)}{S_{2B}-S_{1A}}.$$

Appendix C: Calculation of expected utilities

We give some additional detail on the way that we calculate the expected utility given a particular equilibrium solution, which specifies values for contract quantities $Q(\rho_1, \rho_2)$ and contract prices $f(\rho_1, \rho_2)$ as functions of ρ_1 and ρ_2, the estimated probabilities for the two players.

All our calculations are based on a grid of possible values $\rho^{(1)} < \rho^{(2)} < ... < \rho^{(M)}$ where $(\rho^{(1)}, \rho^{(M)})$ is the range of possible values ($(0.4, 0.6)$ in our examples). We will assume that the true value is equally likely to be any of these M possibilities - this is Nature's prior and is known to both players.

Suppose that Nature selects a value $\rho^{(m)}$ then firm 1 samples N_1 outcomes (each either high or low) using this value of ρ. Firm 2 samples N_2 outcomes. Each firm then makes a maximum likelihood estimate of Nature's choice of ρ on the basis of the sample they have observed. In other words the firms choose ρ values as the most likely grid value for ρ on the basis of the sample.

For each of the M possible values of ρ we can calculate the probability of observing any combination of high and low outcomes. Let S_1 be the sample for firm 1 containing n_H high outcomes and n_L low outcomes. Similarly let S_2 be the sample of m_H high outcomes and m_L low outcomes. With these results firm 1 selects $\rho_1 = \rho_0(S_1)$ as the closest grid point to n_H/N_1 and firm 2 selects $\rho_2 = \rho_0(S_2)$ as the closest grid point to m_H/N_2. Using this we can calculate the contract quantity and price associated with the pair $(\rho_0(S_1), \rho_0(S_2))$ in this equilibrium. The expected utility is calculated on the basis of the actual value of ρ. For firm 1 this is

$$\begin{aligned}\Pi_1(S_1, S_2, \rho) = \rho U\left(R_H^{(1)} + Q(\rho_0(S_1), \rho_0(S_2))(w_H - f(\rho_0(S_1), \rho_0(S_2)))\right) \\ + (1-\rho) U\left(R_L^{(1)} + Q(\rho_0(S_1), \rho_0(S_2))(w_L - f(\rho_0(S_1), \rho_0(S_2)))\right),\end{aligned}$$

and a similar expression for firm 2.

The probability of seeing a pair of samples (S_1, S_2) given ρ is

$$p_{S_1,S_2}(\rho) = \frac{N_1!}{n_H! n_L!} \rho^{n_H} (1-\rho)^{n_L} \frac{N_2!}{m_H! m_L!} \rho^{m_H} (1-\rho)^{m_L}.$$

Hence given a fixed ρ from Nature the expected utility for firm 1 is the sum over all possible samples of $p_{S_1,S_2}(\rho^{(m)}) \Pi_1(S_1, S_2, \rho^{(m)})$, and the final expected utility given a uniform distribution for Nature's choice of ρ is

$$(1/M) \sum_{m=1}^{M} \sum_{S_1,S_2} p_{S_1,S_2}(\rho^{(m)}) \Pi_1(S_1, S_2, \rho^{(m)}),$$

with a similar expression for firm 2.