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Forward commodity trading with private information

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Abstract

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1 Introduction

Forward contracting is a common arrangement between firms who seek to minimize the risk of price variation when trading commodities. A firm that wishes to sell a commodity that is delivered in the future might arrange a forward price with the buyer so that both have some certainty on what will be exchanged when the contract is settled. In this paper we are interested in such contracts that are negotiated over the counter by a seller and a buyer, as opposed to being traded in an exchange (like a futures contract, for example).

Such bilateral contracting is a common feature of many wholesale electricity markets that are characterized by occasional very high prices in the spot market caused by restrictions on the storage of electricity. It is no surprise that in such markets forward contracts are signed between retailers (buyers in the spot market) and generators (sellers in the spot market) in order to reduce the risk that arises from price spikes. The most common contracts take the form of financial instruments with payments depending on the price of electricity. Retailers sell power at a fixed price, but buy at the spot price, so a forward contract that fixes the price for the contract quantity helps to protect retailers from price spikes. Generators have an opposite set of incentives, with a forward contract guaranteeing their income against a situation where prices drop. Thus the appetite for forward contracts will depend on the degree of risk aversion of the participants.

A popular form of contract in electricity markets is called a *contract for differences* (CfD). If a contract trades at a price f then a firm i buying an amount Q of this contract from a firm j agrees to pay an amount $Q(f-p)$ to firm j (and receives this amount if it is negative) where p is the average price over a specified period of time. The average price p may be calculated over all hours, over peak hours only, or on the basis of a profile of average demand. The period of time in question is often of three months duration, or shorter. A CfD may be traded in a futures market (with daily mark-to-market payments being made), or it may be traded through a broker of some sort, or it may be an over -the-counter agreement between two firms. More details on forward market arrangements in different jurisdictions can be found in the literature (e.g. for Australia [2]; for the UK [12]; for the US [9], [14]; for Nordpool [6]).

In the electricity market literature there has been relatively little attention paid to the process by which contracts are negotiated, and the results of such negotiations. Although they can be traded in an exchange, CfD contracts are often bilateral, and involve a seller and a buyer settling a contract quantity Q and contract price f through some bargaining process. For example, this may happen when the energy manager of a large consumer contacts her counterpart in an electricity generator and seeks to settle a contract price and quantity over the telephone. In such a negotiation the participants may have different views of future price outcomes. If negotiators are risk neutral then different beliefs about $\mathbb{E}[p]$ would result in infinite contract quantities, so some form of risk aversion is needed to ensure finite outcomes. Contracting increases welfare by reducing risk, and by settling on Q and f, the agents arrive at a value for this surplus and how it should be distributed. The classical solution (based on an axiomatic approach) is the *Nash bargaining solution* [17].

Sometimes sellers of contracts wish to remain anonymous, for in competitive electricity markets the contract books of electricity generation companies are held in strict secrecy. One reason is that in imperfectly competitive markets, levels of contracting affect spot market offering behavior([1], [3]) and so generators are at a strategic disadvantage if their contract levels are known by competitors. In some circumstances buyers might also prefer to buy from a generator (to incentivize lower prices in the spot market) and so a speculator might prefer to be anonymous, so that this preference does not result in lower contract prices.

One way to preserve anonymity is for contracts to be arranged by a broker who mediates between the buyer and the seller. The simplest approach, which we call the *broker mechanism with simple offers,* involves no explicit negotiation. The seller constructs an increasing supply curve $Q(f)$ of contract quantities that maximize his expected utility, and the buyer constructs a decreasing demand curve $D(f)$ of contract purchases that maximizes her expected utility. They supply these curves to a broker, and agree with the broker to settle on a contract price f for which supply equals demand. This mechanism is a form of *take-it-or-leave-it* offer by each agent, where the supply and demand curves can be interpreted as a menu of offers to the other party from which it can select one option. Each pair of menus then determines an outcome for each agent.

In this paper we compare the broker mechanism with a *direct negotiation model* in which firms negotiate with each other, without using a broker, through discussion of potential price quantity pairs at which they may agree a contract. The model we have in mind involves repeated alternating offers made by the two parties until a price is agreed (with some probability of breakdown at any stage). Binmore et al. [5] show how this leads to a Nash bargaining solution in the limit as the breakdown probability approaches zero and the number of rounds of negotiation increases. In this context we need to know each player's expected utility in the case that there is no agreement. If we assume that there is ample opportunity for discussion between the two firms about to enter into a contract, then it is natural to assume that any agreed price quantity pair will be Pareto optimal, in the sense that there is no alternative contract that would produce improved expected utilities for both players. We show (for our limited model) that the direct negotiation model and the broker model yield very similar outcomes.

Given the similarity of the results, one might expect that a broker mechanism, with its anonymity advantages, would be preferred over direct negotiation (notwithstanding the broker's commission on the contract). There is however a catch. Sellers and buyers submitting supply and demand curves to the broker, might anticipate the other player's choice of curve, and alter their own. A seller will know that a buyer will buy less of a contract as the price goes up, and this information can be used to improve his outcome. If the function $D(f)$ offered by the buyer is known, then the seller will choose a point on this function that maximizes his own utility and then offer a supply function $Q(f)$ that goes through this point. Similarly the buyer will respond to her conjectures about the function $Q(f)$ offered by the seller. This strategic interaction gives rise to a game we call the *broker mechanism with strategic offers*. As in the simple broker mechanism, each player makes an offer that is a curve of acceptable Q and f pairs, and the broker determines the clearing price. The difference is that the curves offered by each player are optimal given the curves offered by the other.

This approach opens up a further possibility. Since the equilibrium assumes that each player makes optimal choices given a particular offer curve from the other, it makes sense to respond to the information embodied in the other offer curve. We can expect that combining the information held by the two firms will give a better estimate than either firm can achieve individually. Even when private information on a firm's expectation of the spot price is not shared explicitly, deductions can be made from the firm's readiness to buy or sell contracts at different prices. In simple terms we can say that if a seller is ready to sell significant numbers of contracts at a low price then the buyer can deduce that the seller anticipates low prices in the spot market. We will explore what happens to the equilibrium when players make use of these deductions.

Our work falls into the area of economics that relates to bargaining between parties with private information. Much has been written about this, with examples often drawn from the field of legal disputes or wage negotiations. The review by Kennan and Wilson [15] gives a summary of this work, and draws attention to the way that careful specification of the procedure used in reaching agreement is necessary to determine the equilibria that may occur. In this vein, we are interested in the particular case where agents with private information negotiate the forward trading of commodities in view of random future prices, and how the negotiation procedure affects the outcomes.

A standard approach in modeling forward contracting is to invoke an arbitrage argument so that forward prices will match actual prices in expectation. The problem then reduces to a careful consideration of the detailed stochastic behavior of spot prices, on which much has been written in the electricity market literature (see e.g. [16]). In practice, it has been observed that forward prices rarely match actual prices in expectation, and there is considerable discussion in this literature of the sign of the difference between forward and spot prices. This is an empirical question which is not straightforward to answer since the sign of the forward premium will depend on circumstances. An important paper in this area is Bessembinder and Lemmon [4] who analyze the premium as reflecting a supply and demand imbalance as risk averse players attempt to optimize their utilities using the contract market. Their empirical results are based on day ahead prices obtained from the PJM market. Recent work has identified similar premia in the day-ahead markets operated by the Midwest ISO [7], and the New England ISO [13].

Similar interest has emerged in Europe. Bunn and Chen [8] give a helpful discussion of the various factors that influence the British market looking at both the day ahead and month ahead data and using a vector autoregressive technique for estimating spot and premiums for both peak and non-peak prices. They show that premiums are positive for peak prices and negative for non-peak. There are significant behavioral effects where high peak premiums and high peak prices tend to induce higher premiums in the future. Related work can be found in Weron and Zator [20], who look at the Nord Pool market with weekly contracts between 1 and 6 weeks ahead.

In the absence of risk-neutral arbitrageurs, premia in bilateral forward contracts will depend on the risk attitudes of the agents as well as their private information. Indeed, hedging risks is a key driver for bilateral forward contract arrangements. The same argument has been made by Dong and Liu [10] who discuss a supply chain context for a non-storable commodity and use a Nash bargaining approach. They use a mean-variance utility function.

The use of Nash bargaining as a model for bilateral contract negotiation in an electricity market is found in Yu et al [22] who use a CVaR-based utility function to measure risk aversion, and Sreekumaran and Liu [19] who choose a CVaR type measure but based on cash flow. There are alternatives to Nash bargaining. For example Wu et al. [21] consider a contract quantity bid in a Cournot framework and use a mean-variance approach to allow for risk aversion. Similarly, Downward et al [11] consider a differentiated products model for fixed-price electricity contracting by retailers who use a risk measure that combines expectation and CVaR.

Our paper also sheds some light on a phenomenon that is important in practice. In wholesale electricity markets there are differences in the forecasts of average spot prices, so traders seeking to hedge their risk exposure will at the same time attempt to profit from their private information. Sanda, Olsen and Fleten [18] discuss the hedging behavior of hydro producers in the Nord Pool region. They find that large forward positions are taken with bilateral negotiation used for a significant fraction of these contracts (while the rest are traded in an exchange): some companies are able to use superior market forecasts to make considerable profits from these derivative contracts.

The paper is laid out as follows. In the next section we define a model of contracting under uncertainty, and then derive a simple broker mechanism for negotiating a contract. The outcomes of this mechanism are explored under various assumptions on the problem data. The following section then describes the Nash bargaining solution, and compares it with the simple broker mechanism. Section 4 introduces a model in which agents make conjectures on the beliefs of the counterparty and offer supply functions that respond to these. We conclude the paper with a general discussion in section 5. Proofs of all results are given in Appendix A.

2 The model

We consider a model in which there are two firms, a seller (firm 1) and a buyer (firm 2), who trade in a single divisible commodity. For simplicity we will assume that the buyer and seller have the same strictly concave, increasing utility function $U(z)$. Each firm views the spot price as a random variable W to which they assign (possibly) different probability distributions $\mathbb{P}_1$ and $\mathbb{P}_2$. We assume in the general case that W is distributed on $(-\infty, \infty)$. In each each price outcome w, firm i earns an operating profit $R_i(w)$. There may be different scenarios that result in the same spot price, but have different operating profits, and in this case $R_i(w)$ will be the expected operating profit given w.

The firms wish to arrange a forward contract quantity Q and contract price f. This is a purely financial contract under which the buyer (firm 1) buys this contract quantity and the seller (firm 2) sells this quantity. Under the contract terms a payment is made by the seller to the buyer of the difference between the spot price and strike price, f. Thus the total expected profit made by firm 1 in price outcome w is $R_1(w) + Q(w - f)$ and the total expected profit made by firm 2 in price outcome w is $R_2(w) + Q(f - w)$. The expected utilities for the two players if the contract quantity is Q and the contract price is f, are respectively

$$\Pi_1(Q, f) = \int_{-\infty}^{\infty} U\left(R_1(w) + Q(w - f)\right) d\mathbb{P}_1(w), \tag{1}$$

$$\Pi_2(Q, f) = \int_{-\infty}^{\infty} U\left(R_2(w) + Q(f - w)\right) d\mathbb{P}_2(w). \tag{2}$$

If f is below the range of possible prices estimated by the buyer then $\Pi_1(Q, f)$ increases with Q and so has no maximizer. If f is above the range of possible prices estimated by the buyer then she would sell contracts and $\Pi_1(Q, f)$ increases as $Q \to -\infty$ and so has no maximizer. Similarly if f is outside the range of possible prices estimated by the seller then $\Pi_2(Q, f)$ has no maximizer. To avoid this we impose the condition throughout the paper that $P_1((-\infty, f))$, $P_1((f, \infty))$, $P_2((-\infty, f))$, and $P_2((f, \infty,))$ are all strictly positive.

In what follows we will study the contracting outcomes (Q, f) that arise from a number of different negotiation procedures. The specific form of these outcomes will also depend on the problem data, so at different points we will make various assumptions on the problem data to simplify the analysis, without losing the essential structure of the negotiation process. The assumptions are listed as follows.

Assumption 1 *$U(z)$ is twice differentiable and strictly concave with $\lim_{z\to\infty} U'(z) = 0$ and $\lim_{z\to-\infty} U'(z) = \infty$.*

This assumption is satisfied by many utility functions including CARA utilities. Observe that it implies that $U'(z) > 0$. We shall make Assumption 1 throughout the paper.

Assumption 2 *There is some $\hat{f}$ such that for every w, $R_1(\hat{f}+w) = R_2(\hat{f}-w)$ (antisymmetry).*

A buyer and a seller are natural counterparties when their future risks are negatively correlated, so forward trading produces benefits for both players. This condition assumes that the rising profits of one player as prices increase are mirrored in the rising profits of the other as prices decrease. It will hold when the quantity bought and sold in the spot market by the two players is the same and independent of the price.

For an example in an electricity setting, consider a base load generator (seller) and a retailer (buyer). The buyer in the wholesale market buys an amount q and receives price p from retail sales (per unit of power used) and has fixed cost of operation K_1, giving a profit when the spot price is w of

$$R_1(w) = pq - wq - K_1.$$

The seller supplies an amount q and has fuel cost c and fixed cost of operation K_2. The seller's profit from spot market operations when the spot price is w is given by

$$R_2(w) = wq - cq - K_2.$$

Then we set

$$\hat{f} = \frac{p+c}{2} + \frac{K_2 - K_1}{2q},$$

and observe that

$$\begin{aligned} R_1(\hat{f} + w) &= pq - (\hat{f} + w)q - K_1 \\ &= q\frac{p-c}{2} - \frac{K_1 + K_2}{2} - wq \\ &= R_2(\hat{f} - w). \end{aligned}$$

The simplest form of this condition occurs when w has two outcomes, a high price w_H and a low price w_L. In this case, condition 2 implies that $\hat{f} = \frac{w_L + w_H}{2}$, $R_1(w_H) = R_2(w_L)$, and $R_2(w_H) = R_1(w_L)$.

Assumption 3 *$U''(\cdot)/U'(\cdot)$ is constant (CARA utility).*

The assumption of a CARA utility function enables us to prove that contract settlements are uniquely determined. Since it simplifies the analysis of contract outcomes, we will also resort to this case in the examples.

3 The broker mechanism with simple offers

We now consider the broker mechanism in which agents offer supply functions without anticipating their rival's choice. Given a fixed contract price f, the buyer (player 1) seeks an optimal contract quantity Q to buy. This gives the following first order condition.

$$\frac{\partial}{\partial Q}\Pi_1(Q, f) = \int_{-\infty}^{\infty} (w - f)U'\left(R_1(w) + Q(w - f)\right) d\mathbb{P}_1(w) = 0. \quad (3)$$

In the same way, we can find the first order condition for the seller determining the optimal contract quantity to sell:

$$\frac{\partial}{\partial Q}\Pi_2(Q, f) = \int_{-\infty}^{\infty} (f - w)U'\left(R_2(w) + Q(f - w)\right) d\mathbb{P}_2(w) = 0. \quad (4)$$

Proposition 1 *Under Assumption 1 the first order conditions define unique supply functions* $\widehat{Q}_1(f)$ *and* $\widehat{Q}_2(f)$*. If* $\mathbb{P}_1$ *and* $\mathbb{P}_2$ *have bounded support then the simple broker model has at least one solution* (Q^*, f^*) *where*

$$Q^* = \widehat{Q}_1(f^*) = \widehat{Q}_2(f^*).$$

This result leaves open the possibility of non-monotonic behavior of the optimal offer curves $\widehat{Q}_i(f)$ and hence more than one clearing price. However we can show that any solution obtained cannot be improved for one firm without making the other worse off.

Proposition 2 *Any solution to the simple broker model is Pareto optimal, i.e. it satisfies*

$$\Pi_1(Q, f) > \Pi_1(Q^*, f^*) \Rightarrow \Pi_2(Q, f) < \Pi_2(Q^*, f^*),$$
$$\Pi_2(Q, f) > \Pi_2(Q^*, f^*) \Rightarrow \Pi_1(Q, f) < \Pi_1(Q^*, f^*).$$

By restricting attention to the case of CARA utility functions (i.e. those satisfying Assumption 3) we can demonstrate that the offer curves are monotonic and hence the clearing price and quantity is unique.

Proposition 3 *Under Assumption 3 (CARA utility), the supply functions* $\widehat{Q}_1(f)$ *and* $\widehat{Q}_2(f)$ *are monotonic and if* $\mathbb{P}_1$ *and* $\mathbb{P}_2$ *have bounded support then there is a unique clearing price and quantity.*

3.1 Two price outcomes

In the remainder of this section we consider the special case when the spot price W has two outcomes w_L and w_H. To simplify notation in the two-outcome case we will henceforth write $\rho_1 = \mathbb{P}_1(w_H)$ and $\rho_2 = \mathbb{P}_2(w_H)$, so $\mathbb{P}_1(w_L) = 1 - \rho_1$ and $\mathbb{P}_2(w_L) = 1 - \rho_2$.

3.1.1 General utility functions

In the general case we obtain an expected utility for each player as follows.

$$\Pi_1(Q, f) = \rho_1 U\left(R_1(w_H) + Q(w_H - f)\right) + (1 - \rho_1) U\left(R_1(w_L) + Q(w_L - f)\right),$$
$$\Pi_2(Q, f) = \rho_2 U\left(R_2(w_H) + Q(f - w_H)\right) + (1 - \rho_2) U\left(R_2(w_L) + Q(f - w_L)\right).$$

The first order conditions (3) and (4) yield

$$(w_L - f)U'\left(R_1(w_L) + Q(w_L - f)\right)(1 - \rho_1) + (w_H - f)U'\left(R_1(w_H) + Q(w_H - f)\right)\rho_1 = 0, \quad (5)$$

$$(f - w_L)U'\left(R_2(w_L) + Q(f - w_L)\right)(1 - \rho_2) + (f - w_H)U'\left(R_2(w_H) + Q(f - w_H)\right)\rho_2 = 0. \quad (6)$$

Proposition 4 *If* $\rho_1 = \rho_2$ *then*

$$Q\left(w_H - w_L\right) \geq \min\{R_1(w_L) - R_1(w_H), R_2(w_H) - R_2(w_L)\},$$
$$Q\left(w_H - w_L\right) \leq \max\{R_1(w_L) - R_1(w_H), R_2(w_H) - R_2(w_L)\}.$$

Observe that if we also make Assumption 2 (antisymmetry) where $\hat{f} = \frac{w_L + w_H}{2}$, then we can set $R_2(w_H) = R_1(w_L) = r$ and $R_1(w_H) = R_2(w_L) = p$, so Proposition 4 gives

$$Q = \frac{r - p}{w_H - w_L}.$$

In fact, under Assumption 2 only, Proposition 4 can be generalized as follows.

Proposition 5 *Under Assumption 2 (antisymmetry), the quantity of the contract signed is greater than (less than) $Q^* = (r - p)/(w_H - w_L)$ when ρ_1 is greater than (less than) ρ_2 and is equal to Q^* when $\rho_1 = \rho_2$.*

3.1.2 CARA utility functions

Now we consider CARA utility functions with $U(0) = 0$ and $U(\infty) = 1$ which have the general form $U(x) = 1 - e^{-\alpha x}$. In this case the optimality conditions for the buyer are

$$\frac{\rho_1(w_H - f)}{(1 - \rho_1)(f - w_L)} = \frac{\exp(-\alpha R_1(w_L) - \alpha Q(w_L - f))}{\exp(-\alpha R_1(w_H) - \alpha Q(w_H - f))}.$$

Thus

$$\log\left(\frac{\rho_1(w_H - f)}{(1 - \rho_1)(f - w_L)}\right) - \alpha\left(R_1(w_H) + Q(w_H - f)\right) + \alpha\left(R_1(w_L) + Q(w_L - f)\right) = 0.$$

Hence the optimal contract quantity for the buyer is

$$Q = \frac{R_1(w_L) - R_1(w_H)}{w_H - w_L} + \frac{1}{\alpha(w_H - w_L)}\log\left(\frac{\rho_1(w_H - f)}{(1 - \rho_1)(f - w_L)}\right).$$

The seller's supply function (maximizing his expected utility) can be obtained similarly, and we get

$$Q = \frac{R_2(w_H) - R_2(w_L)}{w_H - w_L} - \frac{1}{\alpha(w_H - w_L)}\log\left(\frac{\rho_2(w_H - f)}{(1 - \rho_2)(f - w_L)}\right). \quad (7)$$

From these alternative expressions for Q we can find the value of f at which the market clears. This is the f value that satisfies

$$\frac{(w_H - f)^2}{(f - w_L)^2} = \frac{(1 - \rho_1)(1 - \rho_2)}{\rho_1\rho_2}\frac{e^{\alpha(R_1(w_H) + R_2(w_H))}}{e^{\alpha(R_1(w_L) + R_2(w_L))}}. \quad (8)$$

Under Assumption 2 (antisymmetry) we can solve (8) for Q and f explicitly. We have

$$\frac{e^{\alpha(R_1(w_H) + R_2(w_H))}}{e^{\alpha(R_1(w_L) + R_2(w_L))}} = 1$$

and so the market clears when

$$f = \frac{w_H + w_L\kappa}{1 + \kappa}$$

where $\kappa = \sqrt{\frac{(1-\rho_1)(1-\rho_2)}{\rho_1\rho_2}}$. Setting $\sigma = \sqrt{\frac{\rho_1(1-\rho_2)}{\rho_2(1-\rho_1)}}$ gives the clearing quantity at f to be

$$Q = \frac{r-p}{w_H - w_L} + \frac{\log \sigma}{\alpha(w_H - w_L)}, \tag{9}$$

where $r = R_2(w_H) = R_1(w_L)$ and $p = R_1(w_H) = R_2(w_L)$. Notice that the choice of f is independent of the coefficient of risk aversion α, but Q is not.

In the special case where $\rho_1 + \rho_2 = 1$ then $\kappa = 1$, so $f = (w_H + w_L)/2$. Another special case occurs when $\rho_1 = \rho_2 = \rho$ so $\sigma = 1$ and $\kappa = \frac{1-\rho}{\rho}$, giving $Q = \frac{r-p}{w_H - w_L}$and $f = \rho w_H + (1-\rho)w_L$, which is just the expected spot price. These will also be the values of f and Q if both players disclose their estimates of ρ (since by definition this implies both players have the same belief about ρ).

Example 1: (The broker mechanism in the antisymmetric case)

Let $w_H = 2$, $w_L = 1$, $r = 4$, $p = 1$. The buyer curve corresponding to ρ_1 is

$$Q = 3 + \frac{1}{\alpha} \log \left(\frac{\rho_1(2-f)}{(1-\rho_1)(f-1)} \right).$$

The seller curve corresponding to ρ_2 is

$$Q = 3 - \frac{1}{\alpha} \log \left(\frac{\rho_2(2-f)}{(1-\rho_2)(f-1)} \right).$$

For any ρ_1 and ρ_2 the market clears at

$$f = \frac{2+\kappa}{1+\kappa},$$
$$Q = 3 + \frac{1}{\alpha} \log \gamma.$$

This gives a grid of possible (Q, f) outcomes depending on the ρ_1, ρ_2 values of the two players. We show this in Figure 1 where we have allowed both ρ_1 and ρ_2 to vary between 0.4 and 0.6 in increments of 0.02. This is plotted for the case $\alpha = 0.2$.

4 Nash bargaining

We now consider a setting in which negotiation takes place directly between the two players. Our model for this is the Nash bargaining solution that occurs at the solution of

$$\max_{Q,f} \left(\Pi_1(Q,f) - \Pi_1(0,f) \right) \left(\Pi_2(Q,f) - \Pi_2(0,f) \right) \tag{10}$$

where $\Pi_1(Q, f)$ and $\Pi_2(Q, f)$ are defined by (1) and (2). As we mentioned in the introduction we can see this as the result of alternating offers when bargaining friction (associated with the possibility of a breakdown in negotiation) reduces to zero. The assumption here is that contract levels of zero will occur if there is no agreement.

We begin by showing that there is no difference between the Nash bargaining and simple broker offer solution in the antisymmetric case if in addition there is antisymmetry of the price distributions for the two players around the central price point $\widehat{f}$.

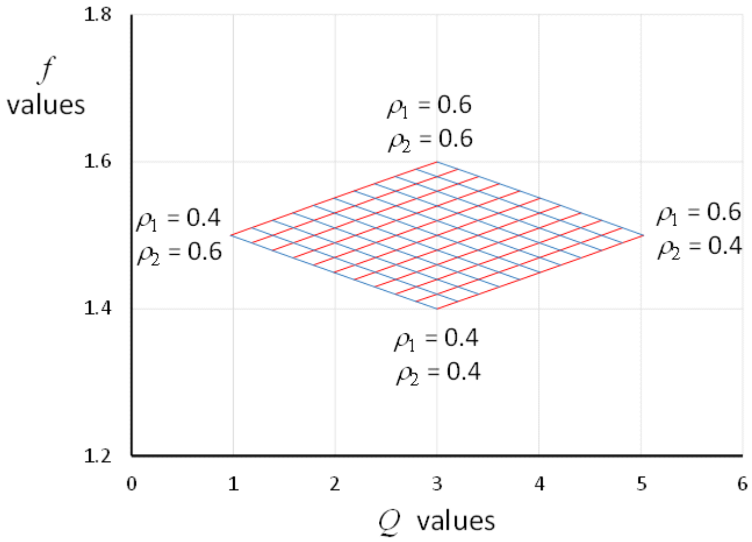

Figure 1: The grid of contract values for Example 1 with $\alpha = 0.2$ and ρ values that vary from 0.4 to 0.6. Seller offers are shown in red (ρ_2 decreases moving from left to right), purchaser offers in blue (ρ_1 increases moving from left to right).

Proposition 6 *Under Assumption 2 (antisymmetry), the Nash bargaining solution matches the simple-offer solution when for every y, $\mathbb{P}_1((-\infty, \widehat{f} + y)) = \mathbb{P}_2((\widehat{f} - y, \infty))$.*

4.1 Two price outcomes and CARA utility

Proposition 6 has a special case for a CARA utility function when the spot price W has two outcomes w_L and w_H, where $\rho_1 = P_1(w_H)$ and $\rho_2 = P_2(w_H)$, and $\hat{f} = \frac{w_L + w_H}{2}$. The assumption in the proposition then becomes $\rho_1 = 1 - \rho_2$. Since this gives $\kappa = 1$ the simple-offer solution yields $f = \hat{f}$ and

$$Q = \frac{(r-p)}{(w_H - w_L)} + \frac{\log \rho_1 - \log(1 - \rho_1)}{\alpha(w_H - w_L)}.$$

For any choice of ρ_1, this matches the Nash bargaining solution (at (ρ_1, ρ_2) values along the horizontal line $f = \hat{f}$ in Figure 1).

In general, the Nash bargaining solution does not quite match the simple-offer solution away from the centre line where $f = \widehat{f}$, but the two solutions are close to each other. We can illustrate this by considering a Nash bargaining model satisfying Assumption 2 (antisymmetry) and Assumption 3 (CARA utility). Recall $\sigma = \sqrt{\frac{\rho_1(1-\rho_2)}{\rho_2(1-\rho_1)}}$.

Proposition 7 *The solution to the Nash bargaining solution with two price outcomes, w_L and w_H, antisymmetry and CARA utilities is*

$$Q = \frac{(r-p)}{(w_H - w_L)} + \frac{\log \sigma}{\alpha(w_H - w_L)}, \tag{11}$$

$$f = w_L + \frac{1}{2\alpha Q}\log\left(\frac{e^{-\alpha p}\sigma\left(-\rho_2+\sigma\rho_2+1\right)\left(e^{-\alpha r}+e^{-\alpha p}\rho_1-e^{-\alpha r}\rho_1\right)}{e^{-\alpha r}\left(\sigma+\rho_1-\sigma\rho_1\right)\left(e^{-\alpha p}-e^{-\alpha p}\rho_2+e^{-\alpha r}\rho_2\right)}\right). \tag{12}$$

Observe that the contract quantity Q is the same as that obtained in the simple broker solution. It turns out that the value of f is also close to that arising from the simple broker solution. For example on the line where $\rho_1 = \rho_2 = \rho$, then $\sigma = 1$ and $Q = \frac{(r-p)}{(w_H-w_L)}$. This gives the contract price

$$f = \frac{(w_H+w_L)}{2} + \frac{(w_H-w_L)}{2\alpha(r-p)}\log\left(\frac{\rho+(1-\rho)e^{-\alpha(r-p)}}{\rho e^{-\alpha(r-p)}+(1-\rho)}\right), \tag{13}$$

which is close to the value $f = w_H\rho + w_L(1-\rho)$ obtained from the broker mechanism with simple offers (when both ρ values are equal). The difference between the solutions is illustrated by the following example.

Example 2: Suppose $\rho = 0.4$, $w_H = 2$, and $w_L = 1$, we have $f = 1.4$ for the broker case. When $\alpha = 0.2$, $r = 4$, and $p = 1$, the equation (13) gives the Nash bargaining solution as

$$f = \frac{3}{2} + \frac{1}{1.2}\log\left(\frac{0.4e^{-0.2}+0.6e^{-0.8}}{0.4e^{-0.8}+0.6e^{-0.2}}\right) = 1.4028.$$

The Nash bargaining contract prices are higher than those obtained from the broker when $\rho < 0.5$, and are lower when $\rho > 0.5$, and the differences increase with α. Figure 2 shows the size of these differences as ρ varies, for varying values of α.

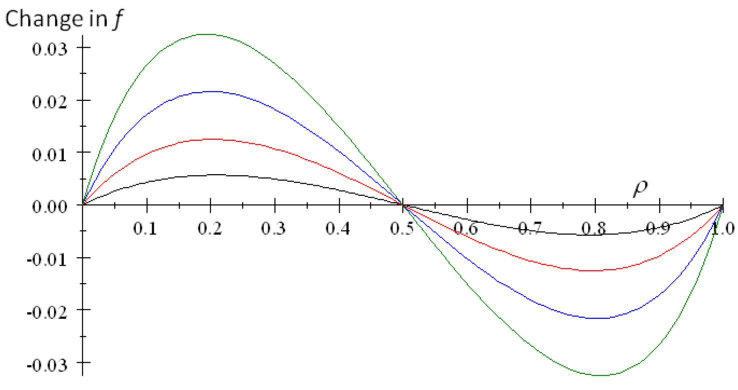

Figure 2: Increase in contract price f in Nash bargaining solution over broker solution for the example problem when both agents agree on ρ, the probability of a high price. Plots shown for $\alpha = 0.2, 0.3, 0.4, 0.5$.

It is interesting to reflect on why the broker mechanism with simple offers differs from Nash bargaining. The Nash bargaining solution is the only one that satisfies the bargaining axioms formulated by Nash: Pareto efficiency; symmetry; invariance to linear transformations of utility; and independence of irrelevant alternatives. We have already established Pareto efficiency for the simple broker mechanism and it is not difficult to check that linear transformations of the utilities will not change the result. Hence we may wonder which of the other axioms is not satisfied by the simple broker mechanism. However the Nash bargaining axioms apply to a bargaining solution that is a map from all convex compact sets of possible pairs of utility outcomes to a single choice. Our framework gives a mechanism based on the decision variables (Q, f) rather than the utility outcomes $\Pi_1(Q, f)$ and $\Pi_2(Q, f)$). There is no easy way to embed this mechanism within a bargaining solution that would apply to all bounded convex sets of utility pairs. In fact the set $U = (\Pi_1(Q, f), \Pi_2(Q, f))$ will in general not be convex. Moreover an embedding of the simple broker mechanism within a broader framework makes it very difficult to discuss the independence of irrelevant alternatives hypothesis.

5 Supply Function Equilibrium

The model described above simplifies the interaction we would expect between fully rational players in finding a clearing price and quantity. In the simple-offer model the supply function is chosen in a way that would be appropriate only if the other player was using an unknown fixed price and was prepared to supply (demand) any amount of contracts at that price. From a conjectural variations perspective this would be an extreme view for a firm to hold. In moving to a more realistic model it is appropriate to consider a supply function equilibrium, to which we now turn our attention.

We assume from now on that there are only two price outcomes w_L and w_H. The buyer (agent 1) bids an offer curve that depends on her private information ρ_1. We write this curve in parameterized form $Q_1(\rho_1, t)$, $f_1(\rho_1, t)$, where t is the parameter. Thus the buyer will purchase a quantity of contracts $Q_1(\rho_1, t)$ if the price is $f_1(\rho_1, t)$. Similarly the seller (agent 2) offers a supply curve that can be written in parameterized form as $Q_2(t, \rho_2)$, $f_2(t, \rho_2)$, where $Q_2(t, \rho_2)$ is the quantity of contracts that he wishes to sell at a price $f_2(t, \rho_2)$ when his private information is ρ_2. By defining $Q_2(t, \rho_2)$ as the sell amount (rather than the buy amount) we can say that the market clears at the price and quantity where these two curves intersect.

Each player takes the other player's supply function as fixed and optimizes their own offer against this. Each player anticipates that the other player's offer will depend on the other player's own information, but does not know what this information is. Hence player 1 (knowing ρ_1) is faced with a supply function in contracts being offered by player 2 that is determined by ρ_2. Player 1 would ideally make an offer that picks out the best point on the supply function of player 2. Linking these points together for different values of ρ_2, player 1 will then have an optimal supply function for any value of ρ_2.

We write $Q(\rho_1, \rho_2)$, and $f(\rho_1, \rho_2)$ as the quantity and price at the combination ρ_1, ρ_2. The expected utilities of each agent under information outcomes (ρ_1, ρ_2) are given by

$$\begin{aligned}
\Pi_1(\rho_1, \rho_2) &= \rho_1 U\left(R_1(w_H) + Q(\rho_1, \rho_2)(w_H - f(\rho_1, \rho_2))\right) \\
&\quad + (1 - \rho_1) U\left(R_1(w_L) + Q(\rho_1, \rho_2)(w_L - f(\rho_1, \rho_2))\right), \\
\Pi_2(\rho_1, \rho_2) &= \rho_2 U\left(R_2(w_H) + Q(\rho_1, \rho_2)(f(\rho_1, \rho_2) - w_H)\right) \\
&\quad + (1 - \rho_2) U\left(R_2(w_L) + Q(\rho_1, \rho_2)(f(\rho_1, \rho_2) - w_L)\right).
\end{aligned}$$

The offers made by the players are parameterized curves. So player 1 knowing its own ρ_1 and facing the other player's offer at ρ_2 given by $Q(t, \rho_2)$, and $f(t, \rho_2)$ indexed by the parameter t

would seek a point t that maximizes

$$\Pi_1(t,\rho_2) = \rho_1 U\left(R_1(w_H) + Q(t,\rho_2)(w_H - f(t,\rho_2))\right) + (1-\rho_1)U\left(R_1(w_L) + Q(t,\rho_2)(w_L - f(t,\rho_2))\right).$$

Hence the first order conditions for player 1 imply that

$$\begin{aligned}\rho_1 U'&\left(R_1(w_H) + Q(\rho_1,\rho_2)(w_H - f(\rho_1,\rho_2))\right)\\ &\times\left(Q_1'(\rho_1,\rho_2)(w_H - f(\rho_1,\rho_2)) - Q(\rho_1,\rho_2)f_1'(\rho_1,\rho_2)\right)\\ &+ (1-\rho_1)U'\left(R_1(w_L) + Q(\rho_1,\rho_2)(w_L - f(\rho_1,\rho_2))\right)\\ &\times\left(Q_1'(\rho_1,\rho_2)(w_L - f(\rho_1,\rho_2)) - Q(\rho_1,\rho_2)f_1'(\rho_1,\rho_2)\right) = 0,\end{aligned}$$

where we write Q_1' and f_1' for the derivatives with respect to the first component.

Similarly, from player 2's optimization, we have

$$\begin{aligned}\rho_2 U'&\left(R_2(w_H) + Q(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_H)\right)\\ &\times\left(Q_2'(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_H) + Q(\rho_1,\rho_2)f_2'(\rho_1,\rho_2)\right)\\ &+ (1-\rho_2)U'\left(R_2(w_L) + Q(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_L)\right)\\ &\times\left(Q_2'(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_L) + Q(\rho_1,\rho_2)f_2'(\rho_1,\rho_2)\right) = 0.\end{aligned}$$

Now we consider the *supply function model with information deduction* where the players each try to make use of the information from the other player that can be gleaned from their supply function. Each player chooses its supply function on the basis of its own information, but wants to do this in a way that takes advantage of the unknown offer of the other player, that in turn is influenced by the information that they have available. In this model player 1 will make calculations based on the private information of player 2, which is accessible to player 1 because of the use of a supply function that produces different results depending on the supply function chosen by player 2. When the information is of equal value the combined estimate, which is not available directly to either player, has $\rho = (\rho_1 + \rho_2)/2$.

The expected utilities of each agent under information outcomes (ρ_1,ρ_2) are now given by

$$\begin{aligned}\Pi_1(\rho_1,\rho_2) = &\frac{\rho_1+\rho_2}{2}U\left(R_1(w_H) + Q(\rho_1,\rho_2)(w_H - f(\rho_1,\rho_2))\right)\\ &+ (1 - \frac{\rho_1+\rho_2}{2})U\left(R_1(w_L) + Q(\rho_1,\rho_2)(w_L - f(\rho_1,\rho_2))\right),\end{aligned}$$

$$\begin{aligned}\Pi_2(\rho_1,\rho_2) = &\frac{\rho_1+\rho_2}{2}U\left(R_2(w_H) + Q(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_H)\right)\\ &+ (1 - \frac{\rho_1+\rho_2}{2})U\left(R_2(w_L) + Q(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_L)\right).\end{aligned}$$

The first order conditions for player 1are again derived from taking the supply function for player 2 as given in parameterized form by $Q(t,\rho_2)$, and $f(t,\rho_2)$. Player 1 can choose the value of t to optimize its own payoff, and we obtain the first order conditions:

$$\frac{\rho_1+\rho_2}{2}U'\left(R_1(w_H)+Q(\rho_1,\rho_2)(w_H-f(\rho_1,\rho_2))\right)\left(Q_1'(\rho_1,\rho_2)(w_H-f(\rho_1,\rho_2))-Q(\rho_1,\rho_2)f_1'(\rho_1,\rho_2)\right)$$
$$+(1-\frac{\rho_1+\rho_2}{2})U'\left(R_1(w_L)+Q(\rho_1,\rho_2)(w_L-f(\rho_1,\rho_2))\right)$$
$$\times\left(Q_1'(\rho_1,\rho_2)(w_L-f(\rho_1,\rho_2))-Q(\rho_1,\rho_2)f_1'(\rho_1,\rho_2)\right)=0.$$

and

$$\frac{\rho_1+\rho_2}{2}U'\left(R_2(w_H)+Q(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_H)\right)\left(Q_2'(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_H)+Q(\rho_1,\rho_2)f_2'(\rho_1,\rho_2)\right)$$
$$+(1-\frac{\rho_1+\rho_2}{2})U'\left(R_2(w_L)+Q(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_L)\right)$$
$$\times\left(Q_2'(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_L)+Q(\rho_1,\rho_2)f_2'(\rho_1,\rho_2)\right)=0.$$

Next we consider the supply function model with two outcomes under Assumption 2 (antisymmetry) (where as before $R_1(w_H)=R_2(w_L)=p$ and $R_2(w_H)=R_1(w_L)=r$ and $r>p$.)

Proposition 8 *With two outcomes and Assumption 2 (antisymmetry), for the supply function equilibrium both with and without information deduction, if the first order conditions for firm 1 hold for all (ρ_1,ρ_2) and the solution $Q(\rho_1,\rho_2)$, $f(\rho_1,\rho_2)$ satisfies the symmetry conditions:*

$$Q(\rho_1,\rho_2)=Q(1-\rho_2,1-\rho_1),\ \ f(\rho_1,\rho_2)=w_H+w_L-f(1-\rho_2,1-\rho_1), \tag{14}$$

then the first order conditions for firm 2 also hold.

We call an equilibrium solution satisfying (14) an anti-symmetric equilibrium (ASE). Observe that even under Assumption 2 there may also be other equilibria in this setting that are not antisymmetric.

If $\rho_1+\rho_2=1$ then note from (14) that $f(\rho_1,\rho_2)=w_H+w_L-f(\rho_1,\rho_2)$ and so $f(\rho_1,\rho_2)=\hat{f}=(w_H+w_L)/2$. In this case we can construct a differential equation for Q when restricted to the line $f=\hat{f}$.

Proposition 9 *Under Assumption 2 (antisymmetry), for a supply function equilibrium without information deduction, the function $h(\rho)=Q(\rho,1-\rho)$ satisfies the differential equation*

$$h'(\rho)=\frac{2h(\rho)}{\gamma}f_d\frac{\rho U'\left(p+\gamma h(\rho)\right)+(1-\rho)U'\left(r-\gamma h(\rho)\right)}{\rho U'\left(p+\gamma h(\rho)\right)-(1-\rho)U'\left(r-\gamma h(\rho)\right)}$$

for some positive constant f_d, and $\gamma=\frac{w_H-w_L}{2}$. With information deduction the differential equation is

$$h'(\rho)=\frac{2h(\rho)}{\gamma}f_d\frac{U'\left(p+\gamma h(\rho)\right)+U'\left(r-\gamma h(\rho)\right)}{U'\left(p+\gamma h(\rho)\right)-U'\left(r-\gamma h(\rho)\right)}$$

and all supply function equilibria have contract quantities bounded above by $Q^*=\frac{r-p}{w_H-w_L}$.

Observe that if $\rho_1 > \rho_2$ then Q^* is less than the contract quantity $Q(\hat{f})$ defined for the simple broker case by (9). Since $Q(\hat{f})$ is the optimal contract quantity at $\hat{f}$ for both buyer and seller, it must be a Pareto improvement on the equilibrium quantity with information deduction. This shows that the outcomes in a supply function equilibrium with information deduction need not be Pareto optimal.

The bound on Q values does not apply in the case without information deduction. In this case we need

$$\rho U'\left(p + \gamma h(\rho)\right) > (1-\rho)U'\left(r - \gamma h(\rho)\right) \tag{15}$$

to avoid h' being negative or infinite. Since high values of h occur at high values of ρ this is a weaker inequality than for the information deduction case. However it turns out to have implications at the other end of the h solution, when ρ is small. Consider the value

$$\rho^* = \frac{U'(r)}{U'(r) + U'(p)}.$$

For $\rho < \rho^*$ we have

$$\rho U'(p) < (1-\rho)U'(r)$$

and hence, when $h(\rho) \geq 0$ we find that the concavity of U implies that inequality (15) fails to hold. Thus for a solution which includes ρ values less than ρ^* we will need to allow h to be negative, corresponding to negative Q values. Negative contract quantities will be unusual in practice since they correspond to large differences in the private information held by the two firms (sufficient to make the generator buy contracts rather than sell them).

In the CARA case we get

$$\rho^* = \frac{\exp(-\alpha r)}{\exp(-\alpha r) + \exp(-\alpha p)}$$

and with the values from Example 2 of $\alpha = 0.2$, $p = 1$, $r = 4$ this becomes $\rho^* = 0.354\,34$. Hence if $\rho < 0.35434$ the derivative of h can only remain positive if h is negative, which shows that all solutions of interest to us go through zero at this ρ value.

In general the families of supply function equilibria are under-determined by the relationships that we have (this is reminiscent of other supply function equilibrium models). Determining the function h will be enough to determine the complete solution, since the Q values on the centre-line serve as a boundary condition to determine the rest of the (Q, f) solution.

Given that f_d is a second constant to be chosen as well as one of the $h(\rho)$ values as a starting point for the differential equation, we see that there are two degrees of freedom if we limit ourselves to antisymmetric solutions.

5.1 Example

To gain a better understanding of the character of supply function equilibria we will explore a particular example. We suppose that there is a CARA utility function with $\alpha = 0.2$. We take $R_1(w_H) = R_2(w_L) = 1$ and $R_2(w_H) = R_1(w_L) = 4$; a low price of $w_L = 1$ and a high price of $w_H = 2$. $U(x) = 1 - e^{-\alpha x}$, where α is the coefficient of absolute risk aversion.

In this case $U'(x) = \alpha e^{-\alpha x}$ and the first order conditions become (using S_1 and S_2 for the values of $\partial Q/df$ along the ρ_1 = constant and ρ_2 = constant lines respectively)

$$\rho_1 \exp\left(-\alpha p - \alpha Q(\rho_1,\rho_2)(w_H - f(\rho_1,\rho_2))\right)\left(S_2(\rho_1,\rho_2)(w_H - f(\rho_1,\rho_2)) - Q(\rho_1,\rho_2)\right)$$
$$+(1-\rho_1)\exp\left(-\alpha r - \alpha Q(\rho_1,\rho_2)(w_L - f(\rho_1,\rho_2))\right)\left(S_2(\rho_1,\rho_2)(w_L - f(\rho_1,\rho_2)) - Q(\rho_1,\rho_2)\right) = 0.$$

$$\rho_2 \exp\left(-\alpha r - \alpha Q(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_H)\right)\left(S_1(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_H) + Q(\rho_1,\rho_2)\right)$$
$$+(1-\rho_2)\exp\left(-\alpha p - \alpha Q(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_L)\right)\left(S_1(\rho_1,\rho_2)(f(\rho_1,\rho_2) - w_L) + Q(\rho_1,\rho_2)\right) = 0.$$

Thus

$$S_1 = Q\frac{\rho_2 \exp\left(-\alpha r - \alpha Q(f - w_H)\right) + (1-\rho_2)\exp\left(-\alpha p - \alpha Q(f - w_L)\right)}{\rho_2(w_H - f)\exp\left(-\alpha r - \alpha Q(f - w_H)\right) + (1-\rho_2)(w_L - f)\exp\left(-\alpha p - \alpha Q(f - w_L)\right)}, \quad (16)$$

$$S_2 = Q\frac{\rho_1 \exp\left(-\alpha p - \alpha Q(w_H - f)\right) + (1-\rho_1)\exp\left(-\alpha r - \alpha Q(w_L - f)\right)}{\rho_1(w_H - f)\exp\left(-\alpha p - \alpha Q(w_H - f)\right) + (1-\rho_1)(w_L - f)\exp\left(-\alpha r - \alpha Q(w_L - f)\right)}. \quad (17)$$

With information deduction the equations are the same with both ρ_1 and ρ_2 replaced with $(\rho_1 + \rho_2)/2$.

Next we discuss how these equations can be used to construct a numerical solution for all pairs (ρ_1, ρ_2) with $\rho_1, \rho_2 \in (\alpha, 1-\alpha)$. For the case without information deduction we begin by constructing a solution to the differential equation (34) for the Q value along line where $\rho_1 + \rho_2 = 1$, (on this line we have $f = (w_H + w_L)/2$). We will use an iterative process where at each stage we suppose we know the Q and f values for all ρ_1, ρ_2 values with $\rho_1 + \rho_2 = K$, then for a chosen small increment δ_ρ we construct the (Q, f) values along the line $\rho_1 + \rho_2 = K + \delta_\rho$. We do this by simply finding the crossing point when we extend the ρ_1 = constant, and ρ_2 = constant curves from the previous set of values. We already know the values (Q_A, f_A) at $(\rho_1, \rho_2 - \delta_\rho)$ and the values (Q_B, f_B) at $(\rho_1 - \delta_\rho, \rho_2)$. We writing S_{1A} for S_1 from (16) evaluated at (Q_A, f_A) with ρ_2 taking the value $\rho_2 - \delta_\rho$, and we write S_{2B} for S_2 from (17) evaluated at (Q_B, f_B) with ρ_1 taking the value $\rho_1 - \delta_\rho$. Then the crossing occurs when

$$Q_A + S_{1A}(f - f_A) = Q_B + S_{2B}(f - f_B),$$

so we can deduce

$$f = \frac{Q_A - Q_B + S_{2B} f_B - S_{1A} f_A}{S_{2B} - S_{1A}},$$
$$Q = \frac{S_{2B} Q_A - S_{1A} Q_B + S_{1A} S_{2B}(f_B - f_A)}{S_{2B} - S_{1A}}.$$

In this example we will choose to take a ρ range of 0.4 to 0.6. By doing this we stay above the value of $\rho^* = 0.354\,34$ that would lead to negative values of Q. It is very much harder to construct solutions that go across the $Q = 0$ boundary and, because this is unlikely in practice,

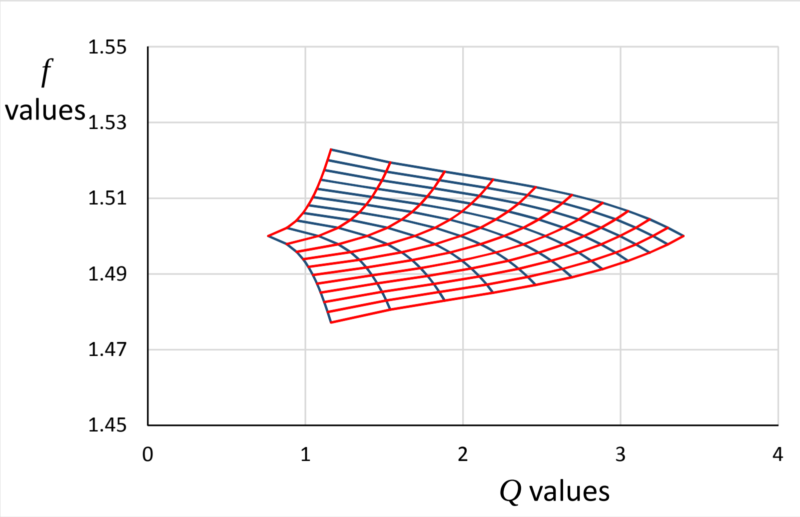

Figure 3: Symmetric supply function equilibrium without information deduction for $\rho = 0.4$ to $\rho = 0.6$.

it makes sense to restrict our attention to the $Q > 0$ case. Figure 3 below shows one solution possible in the case without information deduction.

We are interested in comparing the expected utility achieved for different equilibria. Because there is symmetry in outcomes the two players both have the same expected utility. This allows a natural coordination mechanism where both players select the equilibrium giving them the best outcome.

To find the expected utility we will consider the range of outcomes for different ρ_1 and ρ_2. But the values for the private information represented by ρ_1 and ρ_2 would not be expected to be independent, so evaluation of the expected utility requires us to be more explicit about the context for this game.

We suppose that the probability of a high price is unknown but is drawn from a prior on possible values. This prior is known to both players. The sequence is as follows. First a particular probability is selected from the prior by Nature. Then each player observes a number of price samples drawn with this probability (the number of high prices in this sample is the player's private information). Thus each sample is a single price, either high or low, and the player uses the complete set of samples to estimate its value of ρ which is then used in determining the optimal supply function to offer. The supply function offered thus depends on the observed sample. To evaluate the expected profit from a particular (ρ_1, ρ_2) pair we take expectations over the original prior (that is common knowledge). We will use a uniform distribution as the prior. In Appendix B we give more details of the way these calculations are carried out.

Assuming that each player observes 10 samples to determine its estimated probability of w_H, we can calculate that the solution shown in Figure 3 has expected utility of 0.38578, which is close to the maximum possible. Here we have varied the two parameters (f_d and the starting point $h(0.4)$) in order to achieve the best outcome.

In Figure 4 we show the behavior of a supply function equilibrium when both players make deductions about ρ from the supply function offered by the other. The method of construction is exactly analogous where we use the differential equation (35) and the appropriate forms for the definitions of S_1 and S_2. Again we have searched for good values of f_d and $h(0.4)$. This gives an expected utility of 0.38534 which is slightly worse than without information deduction. We notice that the variation in contract quantity sizes is substantially reduced in this case.

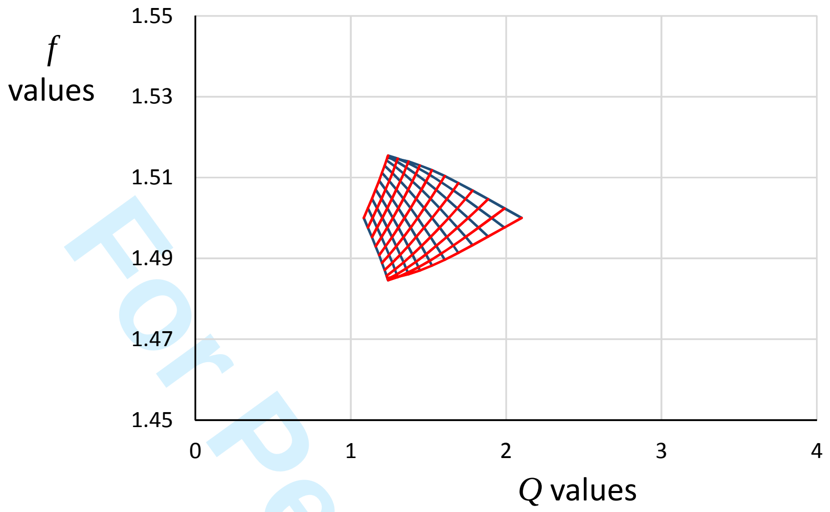

Figure 4: Symmetric supply function equilibrium with information deduction for $\rho = 0.4$ to $\rho = 0.6$.

We have computed the expected payoffs of symmetric equilibrium that maximizes expected utility in a large number of examples of this model for different choices of α, and p and r. The results of some of these experiments are reported in Table 1 below. The columns headed Π give the expected utility for each player under four assumptions, namely no contracting ($\Pi_{Q=0}$), simple broker (Π_{sb}), supply-function bidding with no information deduction (Π_{nid}), and supply-function bidding with information deduction (Π_{id}). In the last two cases we also provide the values of f_d and $Q_{\max}$ that maximize the expected utility.

It is difficult to be categorical about these results. In all cases reported the simple broker contracting solution improves the players' expected utility, as one would expect. Observe that for $\alpha = 0.25$ and $\alpha = 0.3$, we have $\Pi_{nid} < \Pi_{sb}$, but for $\alpha = 0.5$ this inequality reverses. So payoffs under supply-function bidding with no information deduction is typically worse than for simple broker contracting, but not always. Similarly, we typically have $\Pi_{id} < \Pi_{nid}$, but for $\alpha = 0.5$ $p = 1$, $r = 8$, this inequality reverses. So payoffs under supply-function bidding with information deduction are generally worse than those without infomation deduction, but not always. Observe (as shown in Figure 3 and Figure 4) that information deduction tends to reduce supply function equilibrium contract quantities (except for $\alpha = 0.5$ $p = 1$, $r = 8$), which are typically lower than the simple broker contract levels, so information deduction in this setting often moves us further from a more desirable outcome.

α	p	r	$\Pi_{Q=0}$	Π_{sb}	Π_{nid}	$Q_{\max}(nid)$	$f_d(nid)$	Π_{id}	$Q_{\max}(id)$	$f_d(id)$
0.25	1	3	0.3744	0.3888	0.3863	2.300	0.2025	0.3862	1.825	0.1975
0.30	1	3	0.4263	0.4469	0.4442	2.150	0.1050	0.4440	1.525	0.1325
0.20	1	4	0.3660	0.3887	0.3858	3.225	0.1050	0.3853	2.100	0.1200
0.25	1	4	0.4267	0.4603	0.4577	3.100	0.0500	0.4569	2.200	0.0850
0.30	1	4	0.4790	0.5235	0.5214	3.075	0.0325	0.5205	2.325	0.0675
0.25	1	5	0.4673	0.5233	0.5213	4.100	0.0275	0.5200	3.000	0.0500
0.30	1	5	0.5180	0.5893	0.5879	4.100	0.0250	0.5865	3.075	0.0350
0.25	1	6	0.4990	0.5788	0.5775	5.100	0.0250	0.5754	3.650	0.0200
0.30	1	6	0.5469	0.6459	0.6451	5.000	0.0250	0.6429	3.750	0.0100
0.50	1	6	0.6718	0.8227	0.8237	5.400	0.0100	0.8232	4.550	0.0200
0.50	1	8	0.6876	0.8911	0.8926	6.400	0.0070	0.8929	6.680	0.0165

Table 1: Expected utilities from different equilibria.

6 Discussion and conclusions

This paper has considered the problem faced by two players negotiating the terms of a forward (financial) contract when both are uncertain about the future spot price. Both price and quantity need to be determined. This problem only makes sense when the players are risk averse, since otherwise the different views they hold on the expected future price leads to an infinite contract quantity. In a simple model of this situation we compare the results of direct bilateral negotiation using a Nash bargaining concept, and the use of a broker who takes supply function offers from the two players. We show that these two methods produce very similar outcomes if utilities are CARA.

However the broker mechanism will lead to strategic action by the players, with each player anticipating the supply function offered by the other. This leads to a supply function equilibrium in contract offers. We show how these supply function equilibria can be calculated and demonstrate that they may well give worse expected utility for both players. Thus we have a type of prisoner's dilemma, where one player acting strategically improves their own utility, but when both of them do so there can be an overall loss of utility. Our numerical results suggest that this loss of expected utility is likely to occur unless there are high levels of risk aversion. In this context it is also possible to use the supply function offered by the other player to deduce the information they hold about the expected future spot market price. However this more sophisticated approach typically leads to smaller contract quantities and no overall improvement in expected utility.

Our results for supply function equilibrium have been restricted to the case of two outcomes with antisymmetric payoffs. In this setting we have constructed symmetric solutions, since it is reasonable to suppose that players will tend towards a symmetric solution, and attempt to coordinate on the equilibrium that gives them both the maximum expected utility. However it would be fair to say that the existence of non-symmetric solutions in which one player does better than the other will make this harder to achieve.

We may consider the implications for contract negotiation behavior in practice. Our results suggest that direct negotiation may have some advantages over dealing through a broker. This is particularly the case where players are relatively sophisticated and have a good knowledge of their counterparty's operating costs. In these cases strategic interactions should lead to a supply function equilibrium in offer curves. But the multiplicity of potential equilibria will make it hard to coordinate on a single equilibrium solution, and where equilibria are found they may have the effect of making both players worse off.

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A Proofs of Propositions

Proposition 1 *Under Assumption 1 the first order conditions define unique supply functions $\widehat{Q}_1(f)$ and $\widehat{Q}_2(f)$. If $\mathbb{P}_1$ and $\mathbb{P}_2$ have bounded support then the simple broker model has at least one solution (Q^*, f^*) where*

$$Q^* = \widehat{Q}_1(f^*) = \widehat{Q}_2(f^*).$$

Proof. Differentiating (3) with respect to Q gives

$$\int_{-\infty}^{\infty} (w-f)^2 U'' \left(R_1(w) + Q(w-f)\right) d\mathbb{P}_1(w) < 0, \tag{18}$$

so $\Pi_1(Q, f)$ is a strictly concave function of Q. The left-hand side of (3) can be written

$$-\int_{-\infty}^{f} (f-w) U' \left(R_1(w) - Q(f-w)\right) d\mathbb{P}_1(w) + \int_{f}^{\infty} (w-f) U' \left(R_1(w) + Q(w-f)\right) d\mathbb{P}_1(w)$$

where both integrands are strictly positive. Since $\lim_{z\to-\infty} U'(z) = \infty$ and $\lim_{z\to\infty} U'(z) = 0$, the left-hand side of (3) tends to $-\infty$ as $Q \to \infty$, and tends to $-\infty$ as $Q \to \infty$. Thus for any choice of f, (3) has a unique solution $\widehat{Q}_1$. Similarly (4) has a unique solution $\widehat{Q}_2$.

Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ have bounded support lying within the interval (a, b), say. This means that $w - a > 0$ and $b - w > 0$ for all $w \in \mathrm{supp}(\mathbb{P}_1)$. From (3), $\widehat{Q}_1(f)$ for the buyer satisfies

$$\int_a^b (w - f) U' \left(R_1(w) + \widehat{Q}_1(f)(w - f) \right) d\mathbb{P}_1(w) = 0$$

so if $f \to a$ then for all $w \in \mathrm{supp}(\mathbb{P}_1)$

$$\lim_{f\to a} U' \left(R_1(w) + \widehat{Q}_1(f)(w - f) \right) = 0,$$

implying $\lim_{f\to a} \widehat{Q}_1(f) = +\infty$. Similarly if $f \to b$ then for all $w \in \mathrm{supp}(\mathbb{P}_1)$

$$\lim_{f\to b} U' \left(R_1(w) + \widehat{Q}_1(f)(w - f) \right) = 0,$$

implying $\lim_{f\to b} \widehat{Q}_1(f) = -\infty$. In the same way, we can derive $\lim_{f\to a} \widehat{Q}_2(f) = -\infty$, $\lim_{f\to b} \widehat{Q}_1(f) = +\infty$.

The function $\widehat{Q}_1(f)$ (and similarly $\widehat{Q}_2(f)$) is continuous by virtue of (18) and the implicit function theorem, so the result follows by the intermediate value theorem. ■

Proposition 2 *Any solution to the simple broker model is Pareto-optimal, i.e. it satisfies*

$$\Pi_1(Q, f) > \Pi_1(Q^*, f^*) \Rightarrow \Pi_2(Q, f) < \Pi_2(Q^*, f^*),$$
$$\Pi_2(Q, f) > \Pi_2(Q^*, f^*) \Rightarrow \Pi_1(Q, f) < \Pi_1(Q^*, f^*).$$

Proof. Suppose $Q > 0$, and there is some $f \neq f^*$ for which the buyer has

$$\Pi_1(Q, f) > \Pi_1(Q^*, f^*) \tag{19}$$

By optimality of Q^* we have

$$\Pi_1(Q, f^*) \leq \Pi_1(Q^*, f^*). \tag{20}$$

If $f > f^*$ then

$$R_1(w) + Q(w - f) < R_1(w) + Q(w - f^*)$$

so the strict monotonicity of U gives

$$\Pi_1(Q, f) < \Pi_1(Q, f^*)$$

which contradicts (19) and (20). Thus $f < f^*$. This means

$$R_2(w) + Q(f - w) < R_2(w) + Q(f^* - w)$$

so

$$\Pi_2(Q, f) < \Pi_2(Q, f^*) \leq \Pi_2(Q^*, f^*).$$

The argument with $Q < 0$ is analogous. ■

Proposition 3 *Under Assumption 3 (CARA utility), the supply functions $\widehat{Q}_1(f)$ and $\widehat{Q}_2(f)$ are monotonic and if $\mathbb{P}_1$ and $\mathbb{P}_2$ have bounded support then there is a unique clearing price and quantity.*

Proof. Differentiating both sides of (3) implicitly with respect to f gives

$$\int_{-\infty}^{\infty}(w-f)U''\left(R_1(w)+\widehat{Q}_1(f)(w-f)\right)\left(\widehat{Q}_1'(f)(w-f)-\widehat{Q}_1(f)\right)d\mathbb{P}_1(w) = \int_{-\infty}^{\infty}U'\left(R_1(w)+\widehat{Q}_1(f)(w-f)\right)d\mathbb{P}_1(w).$$

Then we use Assumption 3 and write $\alpha = -U''(\cdot)/U'(\cdot) > 0$ for the coefficient of absolute risk aversion, to obtain

$$-\alpha\widehat{Q}_1'(f)\int_{-\infty}^{\infty}U'\left(R_1(w)+\widehat{Q}_1(f)(w-f)\right)(w-f)^2d\mathbb{P}_1(w) + \alpha\widehat{Q}_1(f)\int_{-\infty}^{\infty}(w-f)U'\left(R_1(w)+\widehat{Q}_1(f)(w-f)\right)d\mathbb{P}_1(w) = \int_{-\infty}^{\infty}U'\left(R_1(w)+\widehat{Q}_1(f)(w-f)\right)d\mathbb{P}_1(w).$$

Now the second term on the left hand side is zero from (3) and hence

$$\widehat{Q}_1'(f) = -\frac{\int_{-\infty}^{\infty}U'\left(R_1(w)+\widehat{Q}_1(f)(w-f)\right)d\mathbb{P}_1(w)}{\alpha\int_{-\infty}^{\infty}U'\left(R_1(w)+\widehat{Q}_1(f)(w-f)\right)(w-f)^2d\mathbb{P}_1(w)} < 0,$$

since U' is positive. Similarly we can show that $\widehat{Q}_2'(f) > 0$.

The uniqueness result follows immediately from the existence of a clearing price established in Proposition 1 once monotonicity is proved. ■

Proposition 4 *If $\rho_1 = \rho_2$ then*

$$Q\left(w_H - w_L\right) \geq \min\{R_1(w_L) - R_1(w_H), R_2(w_H) - R_2(w_L)\},$$
$$Q\left(w_H - w_L\right) \leq \max\{R_1(w_L) - R_1(w_H), R_2(w_H) - R_2(w_L)\}.$$

Proof. The first order conditions (5) and (6) yield

$$\frac{(w_H-f)}{(f-w_L)} = \frac{\rho_1 U'\left(R_1(w_L)-Q(f-w_L)\right)}{(1-\rho_1)U'\left(R_1(w_H)+Q(w_H-f)\right)} = \frac{\rho_2 U'\left(R_2(w_L)+Q(f-w_L)\right)}{(1-\rho_2)U'\left(R_2(w_H)-Q(w_H-f)\right)}$$

whence

$$U'\left(R_1(w_L)-Q(f-w_L)\right)U'\left(R_2(w_H)-Q(w_H-f)\right) = U'\left(R_2(w_L)+Q(f-w_L)\right)U'\left(R_1(w_H)+Q(w_H-f)\right).$$

Now setting $y = R_1(w_L) - R_2(w_H) + Q(w_L + w_H - 2f)$, $x = R_2(w_H) - Q(w_H - f)$ and $v = R_1(w_H) - R_2(w_L) + Q(w_L + w_H - 2f)$, $z = R_2(w_L) + Q(f - w_L)$ yields

$$U'(x+y)\,U'(x) = U'(z)\,U'(z+v)\,. \tag{21}$$

Now (21) and the strict concavity of U gives

$$x > z \iff U'(x) < U'(z) \iff U'(x+y) > U'(z+v) \iff x + y < z + v.$$

Substituting for x, y, z, and v gives

$$R_2(w_H) - R_2(w_L) > Q(w_H - w_L) \iff R_1(w_L) - R_1(w_H) < Q(w_H - w_L)\,.$$

Thus we obtain

$$Q(w_H - w_L) \geq \min\{R_1(w_L) - R_1(w_H), R_2(w_H) - R_2(w_L)\},$$
$$Q(w_H - w_L) \leq \max\{R_1(w_L) - R_1(w_H), R_2(w_H) - R_2(w_L)\},$$

as required. ■

Proposition 5 *Under Assumption 2 (antisymmetry), the quantity of the contract signed is greater than (less than) $Q^* = (r-p)/(w_H - w_L)$ when ρ_1 is greater than (less than) ρ_2 and is equal to Q^* when $\rho_1 = \rho_2$.*

Proof. The first order conditions (5) and (6) yield

$$\frac{(w_H - f)}{(f - w_L)} = \frac{\rho_1 U'(r - Q(f - w_L))}{(1-\rho_1)U'(p + Q(w_H - f))} = \frac{\rho_2 U'(p + Q(f - w_L))}{(1-\rho_2)U'(r - Q(w_H - f))},$$

whence

$$\frac{(1-\rho_1)}{\rho_1} U'(r - Q(f - w_L))\,U'(r - Q(w_H - f)) = \frac{(1-\rho_2)}{\rho_2} U'(p + Q(f - w_L))\,U'(p + Q(w_H - f))\,. \tag{22}$$

When $\rho_1 = \rho_2$ the result follows from Proposition 4. If $\rho_1 > \rho_2$ then (22) gives

$$U'(r - Q(f - w_L))\,U'(r - Q(w_H - f)) < U'(p + Q(f - w_L))\,U'(p + Q(w_H - f))\,,$$

and hence

$$r - Q(w_H - f) < p + Q(f - w_L)$$

from which we deduce $Q > Q^*$. (The argument in the opposite direction is similar.) ■

Proposition 6 *Under Assumption 2 (antisymmetry), the Nash bargaining solution matches the simple-offer solution when for every y, $\mathbb{P}_1((-\infty, \widehat{f} + y)) = \mathbb{P}_2((\widehat{f} - y, \infty))$.*

Proof. We use the expressions **(1)** and **(2)** for $\Pi_i(Q,f)$. Now assume $R_1(\widehat{f}+y) = R_2(\widehat{f}-y)$ and $\mathbb{P}_1((-\infty,\widehat{f}+y)) = \mathbb{P}_2((\widehat{f}-y,\infty))$. Then setting $w = \widehat{f}+y$ gives

$$\begin{aligned}
\Pi_1(Q,\widehat{f}) &= \int_{y=-\infty}^{y=\infty} U\left(R_1(\widehat{f}+y) + Q(\widehat{f}+y-\widehat{f})\right) d\mathbb{P}_1(\widehat{f}+y) \\
&= \int_{y=-\infty}^{y=\infty} U\left(R_2(\widehat{f}-y) + Qy\right) d\mathbb{P}_1(\widehat{f}+y) \\
&= -\int_{y=-\infty}^{y=\infty} U\left(R_2(\widehat{f}-y) + Qy\right) d\mathbb{P}_2(\widehat{f}-y) \\
&= -\int_{z=\infty}^{z=-\infty} U\left(R_2(z) + Q(\widehat{f}-z)\right) d\mathbb{P}_2(z), \text{ setting } z = \widehat{f}-y, \\
&= \int_{z=-\infty}^{z=\infty} U\left(R_2(z) + Q(\widehat{f}-z)\right) d\mathbb{P}_2(z) \\
&= \Pi_2(Q,\widehat{f})
\end{aligned} \tag{23}$$

Thus if $\mathbb{P}_1((-\infty,\widehat{f}+y)) = \mathbb{P}_2((\widehat{f}-y,\infty))$ then maximizing $\Pi_1(Q,\widehat{f})$ over Q has a solution, $\widehat{Q}$, which is the same as when we maximize $\Pi_2(Q,\widehat{f})$ over Q.

Also

$$\begin{aligned}
\left[\frac{\partial}{\partial f}\Pi_1(Q,f)\right]_{f=\widehat{f}} &= -Q\int_{y=-\infty}^{y=\infty} U'\left(R_1(\widehat{f}+y) + Q(\widehat{f}+y-\widehat{f})\right) d\mathbb{P}_1(\widehat{f}+y) \\
&= -Q\int_{-\infty}^{\infty} U'\left(R_2(\widehat{f}-y) + Qy\right) d\mathbb{P}_1(y) \\
&= Q\int_{y=-\infty}^{y=\infty} U'\left(R_2(\widehat{f}-y) + Qy\right) d\mathbb{P}_2(\widehat{f}-y) \\
&= Q\int_{z=\infty}^{z=-\infty} U'\left(R_2(z) + Q(\widehat{f}-z)\right) d\mathbb{P}_2(z) \\
&= -Q\int_{z=-\infty}^{z=\infty} U'\left(R_2(z) + Q(\widehat{f}-z)\right) d\mathbb{P}_2(z) \\
&= -\left[\frac{\partial}{\partial f}\Pi_2(Q,f)\right]_{f=\widehat{f}}.
\end{aligned} \tag{24}$$

Consider the first order conditions for the Nash bargaining solution to (10). We need

$$(\Pi_2(Q,f) - \Pi_2(0,f))\frac{\partial}{\partial Q}\Pi_1(Q,f) + (\Pi_1(Q,f) - \Pi_1(0,f))\frac{\partial}{\partial Q}\Pi_2(Q,f) = 0, \tag{25}$$

$$\begin{aligned}
&(\Pi_2(Q,f) - \Pi_2(0,f))\left(\frac{\partial}{\partial f}\Pi_1(Q,f) - \frac{\partial}{\partial f}\Pi_1(0,f)\right) \\
&+ (\Pi_1(Q,f) - \Pi_1(0,f))\left(\frac{\partial}{\partial f}\Pi_2(Q,f) - \frac{\partial}{\partial f}\Pi_2(0,f)\right) \\
&= 0.
\end{aligned} \tag{26}$$

Since we know that $\frac{\partial}{\partial Q}\Pi_i(Q,f)=0$ when $f=\widehat{f}$ and $Q=\widehat{Q}$, (25) is satisfied immediately at $(\widehat{Q},\widehat{f})$.

From (23) we know that

$$\left(\Pi_1(\widehat{Q},\widehat{f})-\Pi_1(0,\widehat{f})\right)=\left(\Pi_2(\widehat{Q},\widehat{f})-\Pi_2(0,\widehat{f})\right).$$

From (24) we have

$$\frac{\partial}{\partial f}\Pi_1(Q,f)-\frac{\partial}{\partial f}\Pi_1(0,f)=-\left(\frac{\partial}{\partial f}\Pi_2(Q,f)-\frac{\partial}{\partial f}\Pi_2(0,f)\right),$$

and so (26) is satisfied at $(\widehat{Q},\widehat{f})$. Hence we have established the result we need. ■

Proposition 7 *The solution to the Nash bargaining solution with two price outcomes, w_L and w_H, antisymmetry and CARA utilities is*

$$Q=\frac{(r-p)}{(w_H-w_L)}+\frac{\log\sigma}{\alpha(w_H-w_L)},$$
$$f=w_L+\frac{1}{2\alpha Q}\log\left(\frac{e^{-\alpha p}\sigma\left(-\rho_2+\sigma\rho_2+1\right)\left(e^{-\alpha r}+e^{-\alpha p}\rho_1-e^{-\alpha r}\rho_1\right)}{e^{-\alpha r}\left(\sigma+\rho_1-\sigma\rho_1\right)\left(e^{-\alpha p}-e^{-\alpha p}\rho_2+e^{-\alpha r}\rho_2\right)}\right).$$

Proof. In Nash bargaining, we seek the maximum over Q and f of

$$\begin{aligned}&\left(\rho_1U\left(p+Q(w_H-f)\right)+(1-\rho_1)U\left(r+Q(w_L-f)\right)-\rho_1U\left(p\right)-(1-\rho_1)U\left(r\right)\right)\\&\times\left(\rho_2U\left(r+Q(f-w_H)\right)+(1-\rho_2)U\left(p+Q(f-w_L)\right)-\rho_2U\left(r\right)-(1-\rho_2)U\left(p\right)\right).\end{aligned}$$

The first order conditions for the Nash bargaining solution are (differentiating with respect to Q)

$$\begin{aligned}&\rho_1(w_H-f)U'\left(p+Q(w_H-f)\right)+(1-\rho_1)(w_L-f)U'\left(r+Q(w_L-f)\right)\\&\times\left(\rho_2U\left(r+Q(f-w_H)\right)+(1-\rho_2)U\left(p+Q(f-w_L)\right)-\rho_2U\left(r\right)-(1-\rho_2)U\left(p\right)\right)\\&+\rho_2(f-w_H)U'\left(r+Q(f-w_H)\right)+(1-\rho_2)(f-w_L)U'\left(p+Q(f-w_L)\right)\\&\times\left(\rho_1U\left(p+Q(w_H-f)\right)+(1-\rho_1)U\left(r+Q(w_L-f)\right)-\rho_1U\left(p\right)-(1-\rho_1)U\left(r\right)\right)=0\end{aligned}\tag{27}$$

and (differentiating with respect to f)

$$\begin{aligned}&-Q\rho_1U'\left(p+Q(w_H-f)\right)-Q(1-\rho_1)U'\left(r+Q(w_L-f)\right)\\&\times\left(\rho_2U\left(r+Q(f-w_H)\right)+(1-\rho_2)U\left(p+Q(f-w_L)\right)-\rho_2U\left(r\right)-(1-\rho_2)U\left(p\right)\right)\\&+\rho_2QU'\left(r+Q(f-w_H)\right)+(1-\rho_2)Q(f-w_L)U'\left(p+Q(f-w_L)\right)\\&\times\left(\rho_1U\left(p+Q(w_H-f)\right)+(1-\rho_1)U\left(r+Q(w_L-f)\right)-\rho_1U\left(p\right)-(1-\rho_1)U\left(r\right)\right)=0.\end{aligned}\tag{28}$$

With CARA utility functions these equations become

$$\begin{aligned}&\left(\rho_1(w_H-f)e^{-\alpha p-\alpha Q(w_H-f)}+(1-\rho_1)(w_L-f)e^{-\alpha r-\alpha Q(w_L-f)}\right)\\&\quad\times\left(\rho_2e^{-\alpha r}(1-e^{-\alpha Q(f-w_H)})+(1-\rho_2)e^{-\alpha p}(1-e^{-\alpha Q(f-w_L)})\right)\\&\quad+\left(\rho_2(f-w_H)e^{-\alpha r-\alpha Q(f-w_H)}+(1-\rho_2)(f-w_L)e^{-\alpha p-\alpha Q(f-w_L)}\right)\\&\quad\quad\times\left(\rho_1e^{-\alpha p}(1-e^{-\alpha Q(w_H-f)})+(1-\rho_1)e^{-\alpha r}(1-e^{-\alpha Q(w_L-f)})\right)=0,\end{aligned}$$

and

$$\left(-\rho_1 e^{-\alpha p-\alpha Q(w_H-f)} - (1-\rho_1)e^{-\alpha r-\alpha Q(w_L-f)}\right) \times \left(\rho_2 e^{-\alpha r}(1-e^{-\alpha Q(f-w_H)}) + (1-\rho_2)e^{-\alpha p}(1-e^{-\alpha Q(f-w_L)})\right) + \left(\rho_2 e^{-\alpha r-\alpha Q(f-w_H)} + (1-\rho_2)e^{-\alpha p-\alpha Q(f-w_L)}\right) \times \left(\rho_1 e^{-\alpha p}(1-e^{-\alpha Q(w_H-f)}) + (1-\rho_1)e^{-\alpha r}(1-e^{-\alpha Q(w_L-f)})\right) = 0.$$

These equations can be rearranged to give

$$\frac{\left(-\rho_1 e^{-\alpha p-\alpha Q(w_H-f)} - (1-\rho_1)e^{-\alpha r-\alpha Q(w_L-f)}\right)}{\left(\rho_2 e^{-\alpha r-\alpha Q(f-w_H)} + (1-\rho_2)e^{-\alpha p-\alpha Q(f-w_L)}\right)} = -\frac{\left(\rho_1 e^{-\alpha p}(1-e^{-\alpha Q(w_H-f)}) + (1-\rho_1)e^{-\alpha r}(1-e^{-\alpha Q(w_L-f)})\right)}{\left(\rho_2 e^{-\alpha r}(1-e^{-\alpha Q(f-w_H)}) + (1-\rho_2)e^{-\alpha p}(1-e^{-\alpha Q(f-w_L)})\right)} = \frac{\left(\rho_1(w_H-f)e^{-\alpha p-\alpha Q(w_H-f)} + (1-\rho_1)(w_L-f)e^{-\alpha r-\alpha Q(w_L-f)}\right)}{\left(\rho_2(f-w_H)e^{-\alpha r-\alpha Q(f-w_H)} + (1-\rho_2)(f-w_L)e^{-\alpha p-\alpha Q(f-w_L)}\right)}.$$

Thus

$$\left(\rho_1 e^{-\alpha p-\alpha Q(w_H-f)} + (1-\rho_1)e^{-\alpha r-\alpha Q(w_L-f)}\right) \times \left(\rho_2(f-w_H)e^{-\alpha r-\alpha Q(f-w_H)} + (1-\rho_2)(f-w_L)e^{-\alpha p-\alpha Q(f-w_L)}\right) + \left(\rho_2 e^{-\alpha r-\alpha Q(f-w_H)} + (1-\rho_2)e^{-\alpha p-\alpha Q(f-w_L)}\right) \times \left(\rho_1(w_H-f)e^{-\alpha p-\alpha Q(w_H-f)} + (1-\rho_1)(w_L-f)e^{-\alpha r-\alpha Q(w_L-f)}\right) = 0,$$

and this simplifies to

$$(w_H-w_L)\,e^{-2\alpha(p+r)}e^{-Q\alpha(w_H+w_L)}\left(\rho_1(1-\rho_2)e^{2r\alpha}e^{2(Q\alpha w_L)} - (1-\rho_1)\rho_2 e^{2p\alpha}e^{2(Q\alpha w_H)}\right) = 0.$$

Solving for Q gives

$$Q = \frac{(r-p)}{(w_H-w_L)} + \frac{1}{2\alpha(w_H-w_L)}\log\left(\frac{(1-\rho_2)\rho_1}{\rho_2(1-\rho_1)}\right)$$

so

$$Q = \frac{(r-p)}{(w_H-w_L)} + \frac{\log\sigma}{\alpha(w_H-w_L)}$$

where we recall $\sigma = \sqrt{\frac{\rho_1(1-\rho_2)}{(1-\rho_1)\rho_2}}$.

Now we turn to the contract prices. From (28) we have

$$\left(-\rho_1 e^{-\alpha p-\alpha Q(w_H)} - (1-\rho_1)e^{-\alpha r-\alpha Q(w_L)}\right) \times \left(\rho_2 e^{-\alpha r}(e^{\alpha Qf} - e^{-\alpha Q(-w_H)}) + (1-\rho_2)e^{-\alpha p}(e^{\alpha Qf} - e^{-\alpha Q(-w_L)})\right) + \left(\rho_2 e^{-\alpha r-\alpha Q(-w_H)} + (1-\rho_2)e^{-\alpha p-\alpha Q(-w_L)}\right) \times \left(\rho_1 e^{-\alpha p}(e^{-\alpha Qf} - e^{-\alpha Q(w_H)}) + (1-\rho_1)e^{-\alpha r}(e^{-\alpha Qf} - e^{-\alpha Q(w_L)})\right) = 0,$$

and from the solution for Q we know

$$\rho_1(1-\rho_2)e^{2r\alpha}e^{2(Q\alpha w_L)} = (1-\rho_1)\rho_2 e^{2p\alpha}e^{2(Q\alpha w_H)},$$

so $e^{Q\alpha w_H} = e^{\alpha(r-p)}e^{Q\alpha w_L}\sigma$. Thus

$$\begin{aligned}&\left(-\rho_1\gamma^{-1}e^{-\alpha r-\alpha Q w_L} - (1-\rho_1)e^{-\alpha r-\alpha Q w_L}\right)\\ &\quad\times\left(\rho_2 e^{-\alpha r}(e^{\alpha Q f} - e^{\alpha(r-p)}e^{Q\alpha w_L}\sigma) + (1-\rho_2)e^{-\alpha p}(e^{\alpha Q f} - e^{\alpha Q w_L})\right)\\ &\quad+\left(\rho_2 e^{-\alpha p+Q\alpha w_L}\sigma + (1-\rho_2)e^{-\alpha p+\alpha Q w_L}\right)\\ &\quad\times\left(\rho_1 e^{-\alpha p}(e^{-\alpha Q f} - e^{-\alpha(r-p)}e^{-Q\alpha w_L}\sigma^{-1}) + (1-\rho_1)e^{-\alpha r}(e^{-\alpha Q f} - e^{-\alpha Q w_L})\right) = 0.\end{aligned}$$

This is a quadratic in $e^{\alpha Q f}$ which has a single positive root

$$e^{\alpha Q f} = e^{Q\alpha w_L}\sqrt{\left(\frac{e^{-\alpha p}\sigma\left(-\rho_2+\sigma\rho_2+1\right)\left(e^{-\alpha r}+e^{-\alpha p}\rho_1-e^{-\alpha r}\rho_1\right)}{e^{-\alpha r}\left(\sigma+\rho_1-\sigma\rho_1\right)\left(e^{-\alpha p}-e^{-\alpha p}\rho_2+e^{-\alpha r}\rho_2\right)}\right)}$$

from which the contract price f can be derived in terms of Q, namely

$$f = w_L + \frac{1}{2\alpha Q}\log\left(\frac{e^{-\alpha p}\sigma\left(-\rho_2+\sigma\rho_2+1\right)\left(e^{-\alpha r}+e^{-\alpha p}\rho_1-e^{-\alpha r}\rho_1\right)}{e^{-\alpha r}\left(\sigma+\rho_1-\sigma\rho_1\right)\left(e^{-\alpha p}-e^{-\alpha p}\rho_2+e^{-\alpha r}\rho_2\right)}\right).$$

■

Proposition 8 *With two outcomes and Assumption 2 (antisymmetry), for the supply function equilibrium both with and without information deduction, if the first order conditions for firm 1 holds for all* (ρ_1,ρ_2) *and the solution* $Q(\rho_1,\rho_2)$, $f(\rho_1,\rho_2)$ *satisfies the symmetry conditions:*

$$Q(\rho_1,\rho_2) = Q(1-\rho_2,1-\rho_1),\quad f(\rho_1,\rho_2) = w_H + w_L - f(1-\rho_2,1-\rho_1),$$

then the first order conditions for firm 2 also holds.

Proof. We write $\eta_i = 1-\rho_i$, $i=1,2$. Then $Q_2'(\rho_1,\rho_2) = -Q_1'(\eta_2,\eta_1)$ and $f_2'(\rho_1,\rho_2) = f_1'(\eta_2,\eta_1)$. Begin with the case without information deduction. The first-order conditions are

$$\begin{aligned}&\rho_1 U'\left(p+Q(\rho_1,\rho_2)(w_H-f(\rho_1,\rho_2))\right)\left(Q_1'(\rho_1,\rho_2)(w_H-f(\rho_1,\rho_2)) - Q(\rho_1,\rho_2)f_1'(\rho_1,\rho_2)\right)\\ &\quad+(1-\rho_1)U'\left(r+Q(\rho_1,\rho_2)(w_L-f(\rho_1,\rho_2))\right)\\ &\quad\times(Q_1'(\rho_1,\rho_2)(w_L-f(\rho_1,\rho_2)) - Q(\rho_1,\rho_2)f_1'(\rho_1,\rho_2)) = 0.\end{aligned}\tag{29}$$

$$\begin{aligned}&\rho_2 U'\left(r+Q(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_H)\right)\left(Q_2'(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_H) + Q(\rho_1,\rho_2)f_2'(\rho_1,\rho_2)\right)\\ &\quad+(1-\rho_2)U'\left(p+Q(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_L)\right)\\ &\quad\times(Q_2'(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_L) + Q(\rho_1,\rho_2)f_2'(\rho_1,\rho_2)) = 0.\end{aligned}\tag{30}$$

Thus we can rewrite (30) as

$$\begin{aligned}&(1-\eta_2)U'\left(r+Q(\eta_2,\eta_1)(w_L-f(\eta_2,\eta_1))\right)\left(-Q_1'(\eta_2,\eta_1)(w_L-f(\eta_2,\eta_1)) + Q(\eta_2,\eta_1)f_1'(\eta_2,\eta_1)\right)\\ &+\eta_2 U'\left(p+Q(\eta_2,\eta_1)(w_H-f(\eta_2,\eta_1))\right)\left(-Q_1'(\eta_2,\eta_1)(w_H-f(\eta_2,\eta_1)) + Q(\eta_2,\eta_1)f_1'(\eta_2,\eta_1)\right) = 0.\end{aligned}$$

which is simply (29) multiplied through by -1 and with ρ_1 replaced with η_2, and ρ_2 replaced with η_1. Thus if (14) holds, then it is enough that (30) is true everywhere to deduce that both sets of first order conditions are satisfied.

Now we consider the supply function equilibrium with information deduction. The first-order conditions become

$$\frac{\rho_1+\rho_2}{2}U'\left(p+Q(\rho_1,\rho_2)(w_H-f(\rho_1,\rho_2))\right)\left(Q_1'(\rho_1,\rho_2)(w_H-f(\rho_1,\rho_2))-Q(\rho_1,\rho_2)f_1'(\rho_1,\rho_2)\right)$$
$$+(1-\frac{\rho_1+\rho_2}{2})U'\left(r+Q(\rho_1,\rho_2)(w_L-f(\rho_1,\rho_2))\right)$$
$$\times(Q_1'(\rho_1,\rho_2)(w_L-f(\rho_1,\rho_2))-Q(\rho_1,\rho_2)f_1'(\rho_1,\rho_2))=0. \quad (31)$$

$$\frac{\rho_1+\rho_2}{2}U'\left(r+Q(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_H)\right)\left(Q_2'(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_H)+Q(\rho_1,\rho_2)f_2'(\rho_1,\rho_2)\right)$$
$$+(1-\frac{\rho_1+\rho_2}{2})U'\left(p+Q(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_L)\right)$$
$$\times(Q_2'(\rho_1,\rho_2)(f(\rho_1,\rho_2)-w_L)+Q(\rho_1,\rho_2)f_2'(\rho_1,\rho_2))=0. \quad (32)$$

In the same way as before, when (14) holds, we can rewrite (32) as

$$(1-\frac{\eta_1+\eta_2}{2})U'\left(r+Q(\eta_2,\eta_1)(w_L-f(\eta_2,\eta_1))\right)\left(-Q_1'(\eta_2,\eta_1)(w_L-f(\eta_2,\eta_1))+Q(\eta_2,\eta_1)f_1'(\eta_2,\eta_1)\right)$$
$$+(\frac{\eta_1+\eta_2}{2})U'\left(p+Q(\eta_2,\eta_1)(w_H-f(\eta_2,\eta_1))\right)\left(-Q_1'(\eta_2,\eta_1)(w_H-f(\eta_2,\eta_1))+Q(\eta_2,\eta_1)f_1'(\eta_2,\eta_1)\right)=0,$$

which is (31) multiplied through by -1 and with ρ_1 replaced with η_2, and ρ_2 replaced with η_1. Thus when (31) is true everywhere we can deduce that both sets of first order conditions are satisfied. ■

Proposition 9 *Under Assumption 2 (antisymmetry), for a supply function equilibrium without information deduction, the function $h(\rho)=Q(\rho,1-\rho)$ satisfies the differential equation*

$$h'(\rho)=\frac{2h(\rho)}{\gamma}f_d\frac{\rho U'\left(p+\gamma h(\rho)\right)+(1-\rho)U'\left(r-\gamma h(\rho)\right)}{\rho U'\left(p+\gamma h(\rho)\right)-(1-\rho)U'\left(r-\gamma h(\rho)\right)}$$

for some positive constant f_d, and $\gamma=\frac{w_H-w_L}{2}$. With information deduction the differential equation is

$$h'(\rho)=\frac{2h(\rho)}{\gamma}f_d\frac{U'\left(p+\gamma h(\rho)\right)+U'\left(r-\gamma h(\rho)\right)}{U'\left(p+\gamma h(\rho)\right)-U'\left(r-\gamma h(\rho)\right)}$$

and all supply function equilibria have contract quantities bounded above by $Q^*=\frac{r-p}{w_H-w_L}$.

Proof. We start with the case without information deduction. Then from (29) we have

$$\rho U'\left(p+h(\rho)\gamma\right)\left(Q_1'(\rho,1-\rho)\gamma-h(\rho)f_1'(\rho,1-\rho)\right)$$
$$+(1-\rho)U'\left(r-\gamma h(\rho)\right)\left(-\gamma Q_1'(\rho,1-\rho)-h(\rho)f_1'(\rho,1-\rho)\right)$$
$$=0,$$

where $\gamma = (w_H - w_L)/2$. Thus

$$Q_1'(\rho, 1-\rho) = \frac{h(\rho)}{\gamma} f_1'(\rho, 1-\rho) \frac{\rho U'(p + h(\rho)\gamma) + (1-\rho)U'(r - \gamma h(\rho))}{\rho U'(p + h(\rho)\gamma) - (1-\rho)U'(r - \gamma h(\rho))} \tag{33}$$

Suppose we start at a point on the central line $(\rho, 1-\rho)$ and move to the point $(\rho + \delta, 1-\rho)$. Then

$$Q(\rho + \delta, 1-\rho) = h(\rho) + \delta Q_1'(\rho, 1-\rho) + O(\delta^2),$$
$$f(\rho + \delta, 1-\rho) = (w_H + w_L)/2 + \delta f_1'(\rho, 1-\rho) + O(\delta^2).$$

But we can also consider starting at the point $(\rho + \delta, 1 - \rho - \delta)$ and moving to the point $(\rho + \delta, 1 - \rho)$. This gives

$$Q(\rho + \delta, 1-\rho) = h(\rho + \delta) + \delta Q_2'(\rho + \delta, 1 - \rho - \delta) + O(\delta^2),$$
$$f(\rho + \delta, 1-\rho) = (w_H + w_L)/2 + \delta f_2'(\rho + \delta, 1 - \rho - \delta) + O(\delta^2).$$

Equating these expressions and observing that in this antisymmetric case we have $Q_2'(\rho+\delta, 1-\rho-\delta) = -Q_1'(\rho + \delta, 1 - \rho - \delta)$ and $f_2'(\rho + \delta, 1 - \rho - \delta) = f_1'(\rho + \delta, 1 - \rho - \delta)$, shows:

$$f_1'(\rho, 1-\rho) = f_1'(\rho + \delta, 1 - \rho - \delta) + O(\delta^2),$$
$$h(\rho + \delta) - \delta Q_1'(\rho + \delta, 1 - \rho - \delta) = h(\rho) + \delta Q_1'(\rho, 1-\rho) + O(\delta^2).$$

Thus we can demonstrate (considering δ small) that f_1' takes the same value on $(\rho, 1-\rho)$ for all values of ρ. In other words it is a constant say f_d .

Then

$$h(\rho + \delta) - h(\rho) = \delta \left(Q_1'(\rho, 1-\rho) + Q_1'(\rho + \delta, 1 - \rho - \delta)\right) + O(\delta^2).$$

Thus letting $\delta \to 0$ and using continuity of $Q_1'(\rho, 1-\rho)$ we obtain from (33)

$$h'(\rho) = \frac{2h(\rho)}{\gamma} f_d \frac{\rho U'(p + \gamma h(\rho)) + (1-\rho)U'(r - \gamma h(\rho))}{\rho U'(p + \gamma h(\rho)) - (1-\rho)U'(r - \gamma h(\rho))}. \tag{34}$$

In the case with information deduction we use (31) and this equation becomes

$$h'(\rho) = \frac{2h(\rho)}{\gamma} f_d \frac{U'(p + \gamma h(\rho)) + U'(r - \gamma h(\rho))}{U'(p + \gamma h(\rho)) - U'(r - \gamma h(\rho))}. \tag{35}$$

Now notice that with information deduction, to avoid $h'(\rho)$ becoming infinite or negative we require

$$U'(p + \gamma h(\rho)) > U'(r - \gamma h(\rho))$$

which implies (since U' is decreasing) that $p + \gamma h(\rho) < r - \gamma h(\rho)$, i.e. $h(\rho) < (r-p)/(2\gamma) = Q^*$. This is the highest value of Q possible when $\rho_1 + \rho_2 = 1$ and occurs when ρ_1 takes its highest value. However this is also the ρ combination that leads to the highest possible contract quantity, and hence we have Q^* as an overall bound on Q values. ■

B Calculation of expected utilities

We give some additional detail on the way that we calculate the expected utility given a particular equilibrium solution, which specifies values for contract quantities $Q(\rho_1, \rho_2)$ and contract prices $f(\rho_1, \rho_2)$ as functions of ρ_1 and ρ_2, the estimated probabilities for the two players.

All our calculations are based on a grid of possible values $\rho^{(1)} < \rho^{(2)} < ... < \rho^{(M)}$ where $(\rho^{(1)}, \rho^{(M)})$ is the range of possible values ($(0.4, 0.6)$ in our examples). We will assume that the true value is equally likely to be any of these M possibilities - this is Nature's prior and is known to both players.

Suppose that Nature selects a value $\rho^{(m)}$ then firm 1 samples N_1 outcomes (each either high or low) using this value of ρ. Firm 2 samples N_2 outcomes. Each firm then makes a maximum likelihood estimate of Nature's choice of ρ on the basis of the sample they have observed. In other words the firms choose ρ values as the most likely grid value for ρ on the basis of the sample.

For each of the M possible values of ρ we can calculate the probability of observing any combination of high and low outcomes. Let S_1 be the sample for firm 1 containing n_H high outcomes and n_L low outcomes. Similarly let S_2 be the sample of m_H high outcomes and m_L low outcomes. With these results firm 1 selects $\rho_1 = \rho_0(S_1)$ as the closest grid point to n_H/N_1 and firm 2 selects $\rho_2 = \rho_0(S_2)$ as the closest grid point to m_H/N_2. Using this we can calculate the contract quantity and price associated with the pair $(\rho_0(S_1), \rho_0(S_2))$ in this equilibrium. The expected utility is calculated on the basis of the actual value of ρ. For firm 1 this is

$$\begin{aligned}\Pi_1(S_1, S_2, \rho) = \rho U\left(R_H^{(1)} + Q(\rho_0(S_1), \rho_0(S_2))(w_H - f(\rho_0(S_1), \rho_0(S_2)))\right) \\ + (1-\rho)U\left(R_L^{(1)} + Q(\rho_0(S_1), \rho_0(S_2))(w_L - f(\rho_0(S_1), \rho_0(S_2)))\right).\end{aligned}$$

and a similar expression for firm 2.

The probability of seeing a pair of samples (S_1, S_2) given ρ is

$$p_{S_1,S_2}(\rho) = \frac{N_1!}{n_H! n_L!}\rho^{n_H}(1-\rho)^{n_L}\frac{N_2!}{m_H! m_L!}\rho^{m_H}(1-\rho)^{m_L}.$$

Hence given a fixed ρ from Nature the expected utility for firm 1 is the sum over all possible samples of $p_{S_1,S_2}(\rho^{(m)})\Pi_1(S_1, S_2, \rho^{(m)})$, and the final expected utility given a uniform distribution for Nature's choice of ρ is

$$(1/M)\sum_{m=1}^{M}\sum_{S_1,S_2} p_{S_1,S_2}(\rho^{(m)})\Pi_1(S_1, S_2, \rho^{(m)}),$$

with a similar expression for firm 2.