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# On Cournot Equilibria in Electricity Transmission Networks

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We consider electricity pool markets in radial transmission networks in which the lines have capacities. At each node there is a strategic generator injecting generation quantities into the pool. Prices are determined by a linear competitive fringe at each node (or equivalently a linear demand function) through a convex dispatch optimization. We derive a set of linear inequalities satisfied by the line capacities that gives necessary and sufficient conditions for the unconstrained one-shot Cournot equilibrium to remain an equilibrium in the constrained network. We discuss the extension of this model to general networks and to lines with transmission losses, and conclude by discussing the application of this methodology to the New Zealand electricity transmission network.

*Key words*: electricity markets, transmission, game theory, Cournot

## 1. Introduction

There has been much interest in recent years in extending Nash equilibrium models for oligopolistic competition to the setting of nodal electricity pool markets (see Cardell et al. 1997, Metzler et al. 2003, Neuhoff et al. 2005). In nodal electricity pool markets, generators offer quantities of power at the nodes of a transmission network, and inform the system operator of the price they wish to be paid for this power. The system operator solves a convex optimization model that schedules the generation to meet the load at least cost, and then *dispatches* the generation that is to be used. This economic dispatch model is commonly based on the linear equations determining DC load flow (or quadratic equations if line losses are included). The shadow prices of the flow-conservation constraints define an electricity price at each node of the transmission network. This is the additional cost to the system of meeting one more unit of demand at a particular node; this pricing scheme is referred to as nodal pricing or locational marginal pricing. There are a number of references available on this model and its properties, see e.g. Metzler et al. (2003), Philpott and Pritchard (2004), Schweppe et al. (1988).

When generators offer their power at marginal cost, the economic dispatch model gives nodal prices at which the total welfare of the market participants is maximized. Each generator's utility is measured by its revenue (nodal price times dispatched quantity) minus its cost, and consumer welfare is measured by the area between the demand curve and the nodal price. Because of economies of scale, electricity markets typically have small numbers of generating companies with incentives to offer in their power strategically. The study of market outcomes from such strategic behaviour requires equilibrium models from game theory. The most popular model for studying strategic behaviour in electricity markets follows the Cournot paradigm.

In the Nash-Cournot equilibrium model, all strategic generators simultaneously choose a fixed quantity of power to inject at their own location, each aiming to maximize its own welfare. The price at each node is determined by transmitting the injected power to meet elastic demand at the nodes. In most wholesale electricity markets, demand is inelastic in the short term, at least for residential consumers, and so demand elasticity is interpreted liberally in this setting. It can correspond to industrial load shedding or fixed demand and a *competitive fringe* who are assumed to offer fixed increasing supply curves at some nodes. Although there are some important assumptions in the Cournot model that make it less attractive than the more realistic supply-function equilibrium models (see e.g. Green and Newbery 1992, Klemperer and Meyer 1989) it has proved to be a popular modelling tool because of its tractability.

As outlined by Yao, Oren and Adler (2008), the different classes of Cournot equilibrium models one can derive depend on the assumptions made about the rationality of the generators. The most comprehensive and arguably realistic is the *full-rationality* assumption. In the transmission congestion setting (where lines may become saturated) this assumes that generators anticipate the effect of their generation decisions on the congestion in the network, and therefore on the payments that they receive from potentially higher prices. In a full-rationality model, the generators act as leaders, each choosing a generation quantity simultaneously, assuming that other generation quantities are fixed, but anticipating a follower stage in which the system operator computes clearing prices using the economic dispatch model.

An early contribution in the understanding of this model was provided by the paper of Borenstein, Bushnell and Stoft (2000) that showed, in a two-node symmetric setting, that a Nash-Cournot equilibrium in pure strategies might not exist if the transmission capacity of the line joining the nodes was not sufficiently large. In asymmetric situations with congestion, full-rationality models may result in either no equilibrium, a unique equilibrium, or more than one equilibrium. This has been illustrated in models of the European market by Neuhoff (2003), and in a model of the New Zealand electricity market by Downward (2006).

The development of models for computing Nash-Cournot equilibria in congested electricity transmission networks is an active area of research. These equilibria are given by the solutions to equilibrium problems with equilibrium constraints (EPECs), as modelled by Hu and Ralph (2007). Here each generator maximizes its profit by solving a mathematical program with equilibrium constraints (MPEC); these problems are non-convex due to the ability of generators to congest lines. The resulting EPEC is inherently difficult to solve, because the first order conditions are insufficient to guarantee global optimality for each MPEC. The solutions that are computed are referred to as local Nash equilibria, but are not guaranteed to be full-rationality equilibria as defined above.

Since the lack of existence or non-uniqueness of Nash equilibria is a serious impediment to the economic analysis of electricity markets with transmission systems, most authors seeking to quantify the effects of market power in transmission networks have chosen to relax the full-rationality assumption to so-called *bounded rationality.* Here it is assumed that the system operator is itself a player in the game who makes decisions simultaneously with the generators choosing their levels of generation. Observe that this is in contrast to the more realistic full-rationality model, in which the system operator determines the optimal transmission flows after the generators have chosen their generation levels in a Stackelberg-type game where the system operator's response is taken account of in the generators' choices of strategies.

One may interpret bounded rationality as an assumption that generators will act as price-takers with respect to transmission, but price-setters at their own node. Within this framework, there are several common variations, which are explored by Yao et al. (2008) for example. In one variation, generators assume that competitors' injections and all transmission flows are fixed, and then optimize their injection. Simultaneous optimization of injections gives an equilibrium. Unfortunately, in some circumstances, this approach can give solutions that are not intuitively reasonable. For example, when applied to the Borenstein *et al.* example of symmetric generators at opposite ends of an uncongested line, the bounded rationality approach would yield two local monopolies as opposed to the symmetric Cournot duopoly solution. Alternative approaches (in which generators assume nodal price premiums are fixed rather than transmission quantities) will overcome this problem, but have other limitations, such as yielding the symmetric Cournot duopoly solution for the example by Borenstein et al. (2000) when the line capacity is small.

As mentioned above, Borenstein *et al.* analyse an electricity market with a two-node grid, and compute the minimum line capacity required to support the unconstrained Nash-Cournot equilibrium, under the assumption that generators behave with full rationality. It was found that if the line were too small there would be an incentive for some generator to withhold supply, allowing the single line to congest toward their node; the generator would in essence act as a local monopolist over the remaining demand. In this paper, we apply this concept to more general networks, and seek to derive conditions which ensure that the unconstrained Nash-Cournot equilibrium remains a Nash equilibrium in the presence of (lossless) line capacities. The difficulty in deriving such conditions is due to the combinatorial nature of problems involving congested networks; here the number of ways that a generator can congest the grid increases exponentially with the number of lines. However by restricting ourselves to radial networks we show for linear demand curves and constant marginal generation costs that the problem simplifies in such a way that the set of arc capacities which are necessary and sufficient for the existence of an unconstrained Nash-Cournot equilibrium is a convex polyhedral set, which we call the *competitive capacity set.*

This result is significant for several reasons. In the first instance, our model allows planners to assume participants have full rationality. We feel that a major drawback of bounded-rationality models is that while the assumptions made about the rationality of the participants can materially affect the results of the model, these assumptions appear to be impossible to test a priori. Our results can be used to characterize the circumstances in which the full-rationality model yields the unconstrained Nash equilibrium.

Throughout this paper we assume linear demand functions (or competitive fringe), which is a common approach in electricity market modelling, see e.g. Yao et al. (2008). In practice, the supply functions making up a competitive fringe are typically piecewise constant, and so a linear function is an approximation. However, this conveniently enables the competitive capacity set for radial networks to be described explicitly as a set of linear inequalities. The convexity of this set is useful when planning transmission grid expansions using optimization models.

Although the owner of the transmission network is primarily concerned with providing a reliable transmission service at low cost, regulators in nodal electricity markets are concerned with providing enough grid capacity to enable competition between geographically separate agents. Exploring the tradeoffs between these is important as the transmission network is expanded to meet growing electricity demand and generation.

One could argue that a network with capacities ensuring an unconstrained Cournot equilibrium might be too expensive. Indeed in the symmetric two-node example of Borenstein et al. (2000) the line carries no flow in equilibrium and so appears to be redundant. However in many cases the capacity needed to avoid gaming of congestion is lower than what might be thought necessary. In any case, knowing this limit is important to evaluate the tradeoffs between the competition (and other) benefits of building a line and its capital cost. We illustrate the use of this methodology in practice by applying our model to the New Zealand electricity transmission network.

The paper is laid out as follows. In the next section, we formulate a version of the electricity economic dispatch problem for our setting with radial networks and give its optimality conditions. We then use these conditions in section 2.2 to establish a key property of the residual demand curve faced by a generator at a given node. In section 2.3, this property is used to derive conditions on the line capacities that guarantee an unconstrained Nash-Cournot equilibrium. In section 3, we consider what may happen if the underlying network includes any loops. In a DC load flow model the presence of loops adds constraints (from Kirchhoff's laws) to the dispatch problem that invalidates the analysis in section 2. We also discuss the effect of transmission losses, and show that a Nash equilibrium always exists for the two-node symmetric case considered in Borenstein et al. (2000). We conclude with an application of the model to the New Zealand electricity transmission network.

## 2. Radial transmission networks

In this section we consider a radial (tree) network, $(\mathcal{N}, \mathcal{A})$, of nodes $i \in \mathcal{N}$, and directed lossless arcs denoted $ij \in \mathcal{A}$, where $i$ is the tail node and $j$ is the head node. The flow on arc $ij$ is denoted $f_{ij}$ and the capacity of arc $ij$ is denoted $K_{ij}$. At each node $i$ there is a known demand $d_i$, such that $\sum_{i \in \mathcal{N}} d_i > 0$, and a competitive fringe defined by a linear supply function $S_i(p) = a_i p$, where $a_i > 0$ for all $i$. Observe that this is equivalent to assuming a linear demand function of the form

$$D_i(p) = d_i - a_i p.$$

We assume at each node, $i$, that there is a single generator with constant marginal cost $c_i \geq 0$, which injects power $q_i$ (this assumption is not essential and can easily be relaxed to allow for multiple generators at a node). We denote by $x_i$ the dispatch of the competitive fringe. Occasionally we refer to the above variables in vector form, for instance $x$ stands for the vector with components $x_i$. The generators at each node are assumed to behave strategically; we model this behaviour by way of a Cournot game, whereby each generator, $i$, chooses its $q_i$ so as to maximize its profit.

### 2.1. The economic dispatch problem

In nodal pool markets, given an injection $q_i$ for each generator, nodal prices are determined by a system operator whose objective is to minimize the cost of meeting demand. This is achieved by solving the following *economic dispatch problem.*

$$\begin{aligned}
\mathrm{P}(q) := \min \; & \sum_{i \in \mathcal{N}} \tfrac{1}{2a_i} x_i{}^2 \\
\text{s.t.} \; & -x_i + \sum_{j, ij \in \mathcal{A}} f_{ij} - \sum_{j, ji \in \mathcal{A}} f_{ji} = q_i - d_i \;\; [\pi_i] && \forall i \in \mathcal{N} \\
& -K_{ij} \leq f_{ij} \leq K_{ij} \;\; [\eta_{ij}^1, \eta_{ij}^2] && \forall ij \in \mathcal{A} \\
& x_i \geq 0 && \forall i \in \mathcal{N}.
\end{aligned}$$

The objective of $\mathrm{P}(q)$ is to minimize the area under the competitive fringe supply functions. This is equivalent to maximizing the total welfare of consumers and generators where there is demand elasticity (as opposed to a fringe). The first constraint is a node balance constraint for each node and the corresponding duals give the nodal prices. The second constraint ensures that the flow on any arc does not exceed that arc's capacity. The final constraint ensures that the competitive fringes do not produce negative amounts of electricity. Note that $\mathrm{P}(q)$ is not feasible for all $q$, e.g. if $\sum_{i \in \mathcal{N}} (q_i - d_i) > 0$. At this point we have not placed any restriction on the sign of any element of $q$, however, in subsequent sections $q$ will be non-negative.

The optimality conditions of $\mathrm{P}(q)$ are given by the following Karush-Kuhn-Tucker (KKT) conditions:

$$\begin{aligned}
-x_i + \sum_{j, ij \in \mathcal{A}} f_{ij} - \sum_{j, ji \in \mathcal{A}} f_{ji} = q_i - d_i \quad & \forall i \in \mathcal{N} \\
\pi_i - \pi_j + \eta_{ij}^1 - \eta_{ij}^2 = 0 \quad & \forall ij \in \mathcal{A} \\
0 \leq \tfrac{1}{a_i} x_i - \pi_i \perp x_i \geq 0 \quad & \forall i \in \mathcal{N} \\
0 \leq K_{ij} - f_{ij} \perp \eta_{ij}^1 \geq 0 \quad & \forall ij \in \mathcal{A} \\
0 \leq K_{ij} + f_{ij} \perp \eta_{ij}^2 \geq 0 \quad & \forall ij \in \mathcal{A}.
\end{aligned} \tag{1}$$

Note that in absence of capacity constraints, the nodal prices are the same across all nodes in the network, as power can be bought from anywhere without restriction.

We will now derive two technical lemmas arising from the optimality conditions for the economic dispatch model. The first lemma is a simple consequence of the optimality conditions above. It states that if any line is congested from node $i$ to $j$ at the optimal solution to the dispatch problem, then $\pi_i < \pi_j$.

LEMMA 1. *Suppose that $(x, f)$ satisfies the constraints of* P$(q)$ *and* $0 \leq \frac{1}{a_i}x_i - \pi_i \perp x_i \geq 0, \forall i \in \mathcal{N}$. *Then $(x, f)$ solves* P$(q)$ *if and only if, for every $ij \in \mathcal{A}$, either*

$$(K_{ij} - f_{ij}) \perp (\pi_j - \pi_i)_+, \tag{2}$$

*or*

$$(K_{ij} + f_{ij}) \perp (\pi_j - \pi_i)_-, \tag{3}$$

*where* $(\pi_j - \pi_i)_+ = \max\{\pi_j - \pi_i, 0\}$, *and* $(\pi_j - \pi_i)_- = -\min\{\pi_j - \pi_i, 0\}$.

*Proof.* The result follows from the fact that P$(q)$ is a convex optimization problem, rendering the conditions (1) necessary and sufficient for optimality of P$(q)$. If there exists an instance of $x$, $f$ and $\pi$ satisfying the conditions of the lemma, and either (2) or (3) then defining $\eta^1_{ij} = (\pi_j - \pi_i)_+$, $\eta^2_{ij} = (\pi_j - \pi_i)_-$, gives (1) demonstrating the optimality of $x$ and $f$. Conversely if $x$ and $f$ solve P$(q)$ then there exists $\pi$, $\eta^1$ and $\eta^2$ satisfying (1). If $\eta^1_{ij}$ and $\eta^2_{ij}$ are both strictly positive then redefining

$$\eta^1_{ij} = \eta^1_{ij} - \min\{\eta^1_{ij}, \eta^2_{ij}\}, \quad \eta^2_{ij} = \eta^2_{ij} - \min\{\eta^1_{ij}, \eta^2_{ij}\},$$

gives $\eta^1_{ij}$ and $\eta^2_{ij}$ satisfying (1) with $\eta^1_{ij} = (\pi_j - \pi_i)_+$, $\eta^2_{ij} = (\pi_j - \pi_i)_-$. This gives

$$(K_{ij} - f_{ij}) \perp (\pi_j - \pi_i)_+,$$

and

$$(K_{ij} + f_{ij}) \perp (\pi_j - \pi_i)_-$$

by virtue of the last two conditions in (1). □

Observe, in the above proof, that the optimal dual variables $\eta^1$, $\eta^2$ are defined in terms of $\pi$. Furthermore, from (1), the optimal flows $f$ and dispatches $x$ can be determined uniquely from the optimal prices $\pi$.

In the next lemma, we show that when there is known to be congestion in the network, uncongested subtrees can be solved independently of one another.

LEMMA 2. *Consider* P$(q)$ *for some arbitrary but fixed $q$; now let $\mathcal{F}$ and $\mathcal{G}$ be any disjoint subsets of $\mathcal{A}$ and set $f_{ij} = K_{ij}$, $\forall ij \in \mathcal{F}$, and $f_{ij} = -K_{ij}$, $\forall ij \in \mathcal{G}$. Now for each $i \in \mathcal{N}$ define*

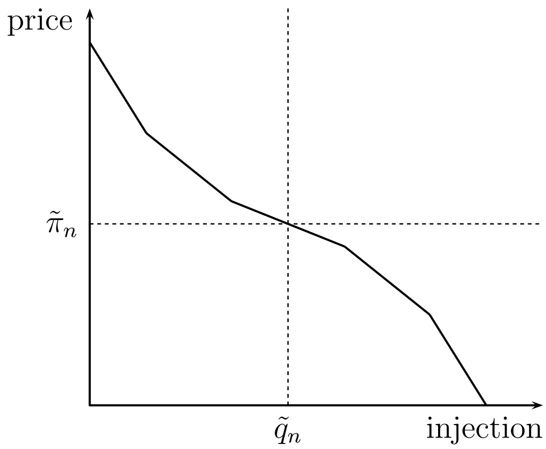
$$\hat{q}_i = q_i - \sum_{\{j, ij \in \mathcal{F} \cup \mathcal{G}\}} f_{ij} + \sum_{\{j, ji \in \mathcal{F} \cup \mathcal{G}\}} f_{ji}$$

*to be a new set of injections, augmented by fixed flows $f_{ij}$, $ij \in \mathcal{F} \cup \mathcal{G}$. If the optimal solution to* P$(\hat{q})$ *for each connected component of $(\mathcal{N}, \mathcal{A} \setminus (\mathcal{F} \cup \mathcal{G}))$ gives $\pi$ with $\pi_i \leq \pi_j$, $ij \in \mathcal{F}$, $\pi_i \geq \pi_j$, $ij \in \mathcal{G}$, then these solutions together with $f_{ij}$, $ij \in \mathcal{F} \cup \mathcal{G}$ solve* P$(q)$.

*Proof.* By construction the solution is easily shown to satisfy the optimality conditions for each arc in $(\mathcal{N}, \mathcal{A} \setminus (\mathcal{F} \cup \mathcal{G}))$. The remaining optimality conditions pertain to arcs in $\mathcal{F} \cup \mathcal{G}$, which hold by lemma 1. □

Since P$(q)$ is a strictly convex quadratic program, it has a unique solution. The preceding lemma has shown that by correctly predicting the congested arcs in the network, the optimal solution to the dispatch problem can be found by separating the dispatch problem into subtrees and solving each subtree independently. However, as there are a finite number of ways that the sets $\mathcal{F}$ and $\mathcal{G}$ defined in lemma 2 can be constructed, by enumerating the possible choices of these sets we are guaranteed to find the unique solution to P$(q)$.

**Figure 1** Residual demand curve for generator $n$.




## 2.2. Prices nodal pool markets over radial networks

In this section, we establish some properties of the residual demand curve at node $n$, i.e. the price at node $n$ as a function of the injection $q_n$ at that node. To establish these properties, it is convenient to adopt the convention throughout this section that all arcs in the radial network are directed towards node $n$; this adjusted set of arcs will be denoted by $\mathcal{A}_n$. We first establish that the residual demand curve faced by each generator is piecewise linear. Furthermore, if $\tilde{q}$ is a vector of injections such that no line is constrained at the optimal solution to the economic dispatch problem $\mathrm{P}(\tilde{q})$ (i.e. prices are the same at all nodes), then the residual demand curve faced by the generator at node $n$ is convex for all $q_n < \tilde{q}_n$ (and concave for all $q_n > \tilde{q}_n$); in fact this curve is piecewise linear, as we assume linear fringes. (See figure 1.)

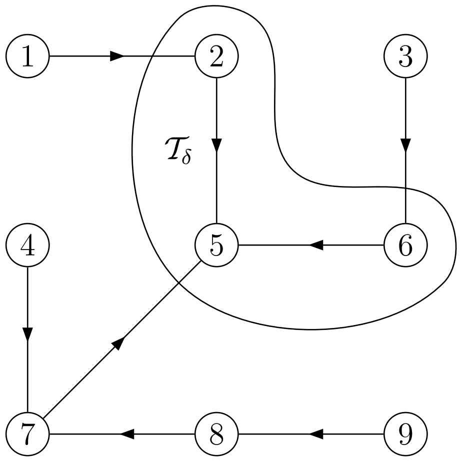
To establish this result, we consider a "decomposition scheme" for the network. A *decomposition* $\delta$ for node $n$ is determined by choosing a subtree $\mathcal{T}_\delta$ of the network rooted at node $n$. We denote the nodes and arcs within $\mathcal{T}_\delta$ by $\mathcal{N}_\delta$ and $\mathcal{A}_\delta$ respectively. The network is therefore decomposed into $\mathcal{T}_\delta$ and several other subtrees, each rooted at the tail node of an arc with its head node in $\mathcal{N}_\delta$. We denote by $\mathcal{B}_\delta$ the set of arcs that link these subtrees. (See figure 2.)

For each node $n$, we denote by $\mathcal{D}_n$ the set of all decompositions pertaining to $n$. Given a decomposition $\delta \in \mathcal{D}_n$ and a vector of injections $q$, we compute nodal prices $\pi_i^\delta$ by setting $f_{ij} = K_{ij}$ for arcs $ij \in \mathcal{B}_\delta$, and solving a modified dispatch problem, $\mathrm{P}^\delta(q)$, which enforces these flows, but ignores arc capacity constraints for the arcs in $\mathcal{A}_\delta$. We are interested in these decompositions because each defines a possible solution to the KKT conditions in (1). Later in this section, we will show that a generator only ever has incentive to congest lines toward their own node. So by considering all the decompositions in $\mathcal{D}_n$, we are effectively enumerating all possible solutions to the problem $\mathrm{P}(q)$ where the $n^{\text{th}}$ component of $q$ decreases below $\tilde{q}_n$. In corollary 1, we consider the optimal solution to $\mathrm{P}^\delta(q)$ and define the necessary and sufficient conditions such that it also solves $\mathrm{P}(q)$.

The dispatch problem for decomposition $\delta$ is

$$
\begin{aligned}
\mathrm{P}^\delta(q) := \min \; & \sum_{i\in\mathcal{N}} \tfrac{1}{2a_i} {x_i}^2 \\
\text{s.t.} \; & -x_i + \sum_{j,ij\in\mathcal{A}_n} f_{ij} - \sum_{j,ji\in\mathcal{A}_n} f_{ji} = q_i - d_i \;\; [\bar{\pi}_i^\delta] && \forall i \in \mathcal{N} \\
& f_{ij} = K_{ij} && \forall ij \in \mathcal{B}_\delta \\
& -K_{ij} \le f_{ij} \le K_{ij} && \forall ij \in \mathcal{A}_n \setminus (\mathcal{A}_\delta \cup \mathcal{B}_\delta) \\
& x_i \ge 0 && \forall i \in \mathcal{N}.
\end{aligned}
$$

**Figure 2** A decomposition for node 5; here $\mathcal{N}_\delta = \{2,5,6\}$ and $\mathcal{B}_\delta = \{12,36,75\}$.




When $\mathrm{P}^\delta(q)$ is feasible the dual variables of the node balance constraints, $\bar{\pi}_i^\delta$, give the nodal prices associated with decomposition $\delta$.

Corollary 1. *Suppose that $(x,f,\pi)$ satisfies the optimality conditions for $\mathrm{P}^\delta(q)$, for some $\delta \in \mathcal{D}_n$ and some arbitrary but fixed, $q$ and $n$. Then $(x,f,\pi)$ solves $\mathrm{P}(q)$ with $\mathcal{A} = \mathcal{A}_n$, if and only if $\pi_i \le \pi_j, \forall ij \in \mathcal{B}_\delta$, and $-K_{ij} \le f_{ij} \le K_{ij}, \forall ij \in \mathcal{A}_n$.*

*Proof.* Consider the optimal solution to $\mathrm{P}^\delta(q)$. For this to solve $\mathrm{P}(q)$ it is necessary that it satisfies the KKT conditions (1), which give:

$$\pi_i \le \pi_j, \quad \forall ij \in \mathcal{B}_\delta \quad \text{and}$$

$$-K_{ij} \le f_{ij} \le K_{ij}, \quad \forall ij \in \mathcal{A}_\delta.$$

These conditions are also sufficient by lemma 2, with $\mathcal{F} = \mathcal{B}_\delta$ and $\mathcal{G} = \emptyset$. □

Due to the radial structure of the network, the problem $\mathrm{P}(q)$ can be decoupled in such a way that the dispatch for the sub-tree $\mathcal{T}_\delta$ can be computed independently of the rest of the network. The dispatch problem for $\mathcal{T}_\delta$ is given by

$$\begin{aligned}
\mathrm{P}^{\mathcal{T}_\delta}(q) := \min \ & \sum_{i\in\mathcal{N}_\delta} \tfrac{1}{2a_i} {x_i}^2 \\
\text{s.t.} \ & -\sum_{i\in\mathcal{N}_\delta} x_i = \sum_{i\in\mathcal{N}_\delta} q_i - \sum_{i\in\mathcal{N}_\delta} d_i + \sum_{ij\in\mathcal{B}_\delta} K_{ij} \ [\bar{\pi}_n^\delta] \\
& x_i \ge 0 \qquad\qquad\qquad\qquad\qquad \forall i \in \mathcal{N}_\delta.
\end{aligned}$$

The KKT conditions of $\mathrm{P}^{\mathcal{T}_\delta}(q)$ can be written as

$$-\sum_{i\in\mathcal{N}_\delta} x_i = \sum_{i\in\mathcal{N}_\delta} q_i - \sum_{i\in\mathcal{N}_\delta} d_i + \sum_{ij\in\mathcal{B}_\delta} K_{ij} \tag{4}$$

$$0 \le x_i \perp \frac{x_i}{a_i} - \bar{\pi}_n^\delta \ge 0 \quad \forall i \in \mathcal{N}_\delta. \tag{5}$$

Lemma 3. *Suppose that $\mathrm{P}(q)$ is feasible for some arbitrary but fixed $q$; for any $\delta$, $\mathrm{P}^\delta(q)$ is feasible if and only if $\mathrm{P}^{\mathcal{T}_\delta}(q)$ is also feasible.*

*Proof.* Clearly if $\mathrm{P}^{\mathcal{T}_\delta}(q)$ is not feasible, $\mathrm{P}^{\delta}(q)$ cannot be feasible either. Now suppose $\mathrm{P}^{\mathcal{T}_\delta}(q)$ is feasible, and let $(\hat{x},\hat{f})$ be a feasible solution to $\mathrm{P}(q)$. We know, from (1), that $\hat{f}_{ij} \le K_{ij}, \forall ij \in \mathcal{B}_\delta$, and as there is no upper bound on $x$, we can construct a feasible solution,$(x,f)$, to $\mathrm{P}^{\delta}(q)$ with

$$x_i = \begin{cases} \hat{x}_i + \sum\limits_{j,ij\in\mathcal{B}_\delta} \left(K_{ij} - \hat{f}_{ij}\right), & \forall i, ij \in \mathcal{B}_\delta, \\ \hat{x}_i, & \forall i \in \mathcal{N} \setminus \left(\mathcal{N}_\delta \cup \{i \,|\, ij \in \mathcal{B}_\delta\}\right), \end{cases}$$
$$f_{ij} = \begin{cases} K_{ij}, & \forall ij \in \mathcal{B}_{\delta,} \\ \hat{f}_{ij}, & \forall ij \in \mathcal{A}_n \setminus \left(\mathcal{A}_\delta \cup \mathcal{B}_\delta\right), \end{cases}$$

and the remainder of the solution defined by a feasible solution to $\mathrm{P}^{\mathcal{T}_\delta}(q)$. $\square$

We define, for decomposition $\delta$:

$$\pi_n^\delta = \begin{cases} \dfrac{\sum\limits_{i\in\mathcal{N}_\delta} d_i - \sum\limits_{i\in\mathcal{N}_\delta} q_i - \sum\limits_{ij\in\mathcal{B}_\delta} K_{ij}}{\sum\limits_{i\in\mathcal{N}_\delta} a_i}, & \sum\limits_{i\in\mathcal{N}_\delta} d_i - \sum\limits_{i\in\mathcal{N}_\delta} q_i - \sum\limits_{ij\in\mathcal{B}_\delta} K_{ij} \ge 0, \\ -\infty, & \sum\limits_{i\in\mathcal{N}_\delta} d_i - \sum\limits_{i\in\mathcal{N}_\delta} q_i - \sum\limits_{ij\in\mathcal{B}_\delta} K_{ij} < 0, \end{cases} \tag{6}$$

to be the nodal price at all nodes in $\mathcal{N}_\delta$.

LEMMA 4. *Given decomposition $\delta \in \mathcal{D}_n$, $\mathrm{P}^{\mathcal{T}_\delta}(q)$ is feasible if and only if*

$$\sum_{i\in\mathcal{N}_\delta} d_i - \sum_{i\in\mathcal{N}_\delta} q_i - \sum_{ij\in\mathcal{B}_\delta} K_{ij} \ge 0.$$

*In this case the unique primal solution is $x_i = a_i \pi_n^\delta$ and $\bar{\pi}_n^\delta = \pi_n^\delta$ is a corresponding dual solution. Moreover if $\pi_n^\delta > 0$, it is the unique dual solution.*

*Proof.* From the constraints of $\mathrm{P}^{\mathcal{T}_\delta}(q)$, it is easy to see that $x$ is feasible if and only if

$$\sum_{i\in\mathcal{N}_\delta} d_i - \sum_{i\in\mathcal{N}_\delta} q_i - \sum_{ij\in\mathcal{B}_\delta} K_{ij} \ge 0.$$

Now we will consider two cases. First if

$$\sum_{i\in\mathcal{N}_\delta} d_i - \sum_{i\in\mathcal{N}_\delta} q_i - \sum_{ij\in\mathcal{B}_\delta} K_{ij} = 0,$$

then $x_i = 0, \forall i \in \mathcal{N}_\delta$. Here condition (5) becomes

$$\bar{\pi}_n^\delta \le 0, \quad \forall i \in \mathcal{N}_\delta,$$

which is satisfied by $\bar{\pi}_n^\delta = \pi_n^\delta = 0$.

On the other hand, if

$$\sum_{i\in\mathcal{N}_\delta} d_i - \sum_{i\in\mathcal{N}_\delta} q_i - \sum_{ij\in\mathcal{B}_\delta} K_{ij} > 0,$$

then $x_i = a_i \pi_n^\delta > 0, \forall i \in \mathcal{N}_\delta$. Here condition (5) becomes

$$\frac{x_i}{a_i} - \bar{\pi}_n^\delta = 0, \quad \forall i \in \mathcal{N}_\delta, \tag{7}$$

and this system of equations has a unique solution, $\bar{\pi}_n^\delta = \pi_n^\delta$. $\square$

Lemma 4 shows that $\pi_n^\delta$ is an optimal dual solution to $\mathrm{P}^{\mathcal{T}_\delta}(q)$ so long as this problem is feasible. However, if $\mathrm{P}^{\mathcal{T}_\delta}(q)$ were infeasible the dual solution would be undefined. Infeasibility results when the total injection into the the set of nodes $\mathcal{N}_\delta$ exceeds the total demand at these nodes. This would mean that the marginal value of electricity should be some large negative value. To reflect this, we have set $\pi_n^\delta = -\infty$ when $\mathrm{P}^{\mathcal{T}_\delta}(q)$ is infeasible.

Recall that figure 1 shows a piecewise linear convex curve to the left of $\tilde{q}_n$. Theorem 1 shows that this curve can be obtained from $\max_{\delta\in\mathcal{D}_n}\pi_n^\delta$. The following lemma, which we use in the proof of theorem 1, shows that the price within some uncongested node set will increase if it is connected to a more expensive node set.

Lemma 5. *Suppose two disjoint uncapacitated radial networks with node sets $\mathcal{N}_1$ and $\mathcal{N}_2$ have optimal dispatches, and node $i\in\mathcal{N}_1$ has price $\pi_i$ while node $j\in\mathcal{N}_2$ has price $\pi_j<\pi_i$. Then connecting node $i$ and node $j$ with a line of infinite capacity gives a new price ${\pi_j}'>\pi_j$.*

*Proof.* We define the total demand within node sets $\mathcal{N}_1$ and $\mathcal{N}_2$ to be $D_1$ and $D_2$ respectively. Since $\pi_i>\pi_j$ we have

$$\frac{D_1}{\sum\limits_{k\in\mathcal{N}_1} a_k} > \frac{D_2}{\sum\limits_{k\in\mathcal{N}_2} a_k}.$$

Thus

$$\begin{aligned}(D_1+D_2)\sum_{k\in\mathcal{N}_2} a_k - D_2\sum_{k\in\mathcal{N}_2} a_k &= D_1\sum_{k\in\mathcal{N}_2} a_k\\ &> D_2\sum_{k\in\mathcal{N}_1} a_k,\end{aligned}$$

yielding

$$(D_1+D_2)\sum_{k\in\mathcal{N}_2} a_k > D_2\sum_{k\in\mathcal{N}_1} a_k + D_2\sum_{k\in\mathcal{N}_2} a_k.$$

Dividing by $\sum\limits_{k\in\mathcal{N}_2} a_k \times \sum\limits_{k\in\mathcal{N}_1\cup\mathcal{N}_2} a_k$ gives

$${\pi_j}' = \frac{D_1+D_2}{\sum\limits_{k\in\mathcal{N}_1\cup\mathcal{N}_2} a_k} > \frac{D_2}{\sum\limits_{k\in\mathcal{N}_2} a_k} = \pi_j$$

as required. $\square$

We will now examine the shape of the residual demand curve for generator $n$ in two parts. We first consider the range whereby $q_n<\tilde{q}_n$ and $q_i=\tilde{q}_i,\ \forall i\in\mathcal{N}\setminus\{n\}$. As the injection $q_n$ decreases from $\tilde{q}_n$, we show in lemma 6 that arcs only ever congest towards node $n$. Therefore one of the decompositions defined above will provide the optimal solution to the economic dispatch problem. We prove in theorem 1 that this decomposition has price $\pi_n^*$ equal to $\max_{\delta\in\mathcal{D}_n}\pi_n^\delta$ where $\pi_n^\delta$ is the price at node $n$ with injection $q_n$ under the decomposition scheme $\delta\in\mathcal{D}_n$. Therefore the convex part of the residual demand curve faced by generator $n$ is the upper envelope of a finite number of linear curves. For the range $q_n\geq\tilde{q}_n$ we will argue that the residual demand curve is piecewise linear and concave.

Lemma 6. *Consider the vector of injections $q=\tilde{q}$ for which there is no line congested at the optimal solution to $\mathrm{P}(q)$. Suppose now that for some node, $n$, $q_n$ is decreased (leaving all other injections fixed at $\tilde{q}$). As $q_n$ decreases the dispatch problem remains feasible, $\pi_n$ is non-decreasing, and the flow in every line is non-decreasing in the direction of node $n$.*

*Proof.* Note that initially the network contains no congested lines. The price at node $n$ is a decreasing function of the injection at node $n$, (as $\pi_n$ is the subgradient of a convex quadratic program). Therefore, at least until any flows reach their bounds, as $q_n$ decreases the nodal prices at all nodes increase uniformly. Since at optimality

$$\pi_i = \frac{1}{a_i} x_i$$

each fringe dispatch $x_i$ increases as $q_n$ decreases, by a total amount that equals the change in $q_n$. Because the network has a radial structure, the changes in line flows in the direction of $n$ are unique and non-negative.

Now, if any flow $f_{ij}$ hits a bound $K_{ij}$ at some value $q_n = \bar{q}_n$ then this defines a decomposition $\delta \in \mathcal{D}_n$ with $ij \in \mathcal{B}_\delta$. By lemma 2 we may fix the flows in $\mathcal{B}_\delta$ and solve each subtree in $\delta$ separately to give fringe dispatch $\bar{x}$, say. It is easy to verify that as $q_n$ decreases below $\bar{q}_n$, we may write down a solution to (1) that has $x_j = \bar{x}_j$, $j \notin \mathcal{N}_\delta$, and flows $f_{ij}$, $ij \in \mathcal{B}_\delta$ remaining at their bounds. This is because as $q_n$ decreases, $\pi_j$ increases to $\pi_n^\delta$, for $j \in \mathcal{N}_\delta$, and for $j \notin \mathcal{N}_\delta$, $\pi_j$ remains fixed at its current value, so the complementary slackness conditions for $ij \in \mathcal{B}_\delta$ continue to be satisfied. The reduction in $q_n$ means that the total demand for $j \in \mathcal{N}_\delta$ exceeds supply, and so for each $j \in \mathcal{N}_\delta$, $x_j$ increases to $a_j \pi_n^\delta$, resulting in a unique increase in flow towards node $n$ (where the deficit is occurring).

As $q_n$ decreases we obtain a sequence of decompositions $\delta_1, \delta_2, \dots$ with $\mathcal{N}_{\delta_{k+1}} \subset \mathcal{N}_{\delta_k}$, so we may apply this argument recursively for each $k$ to yield the result. □

The results above show that the price at node $n$ is a non-increasing piecewise linear function of $q_n$. We now show that this is convex in the range $q_n < \tilde{q}_n$ and concave in the range $q_n > \tilde{q}_n$. To do this we first prove a simple lemma that characterizes the solution to $\mathrm{P}(q)$ when no arcs are congested in the direction away from node $n$.

LEMMA 7. *Suppose at the solution to* $\mathrm{P}(q)$ *that* $f_{ij} > -K_{ij}$, $\forall ij \in \mathcal{A}_n$. *This solution is the same as the solution to* $\mathrm{P}'(q)$ *given below*

$$\begin{aligned} \mathrm{P}'(q) := \min \ & \sum_{i\in\mathcal{N}} \tfrac{1}{2a_i} {x_i}^2 \\ \text{s.t.} \ & -x_i + \sum_{j,ij\in\mathcal{A}_n} f_{ij} - \sum_{j,ji\in\mathcal{A}_n} f_{ji} = q_i - d_i \ [\pi_i] \ \forall i \in \mathcal{N} \\ & f_{ij} \le K_{ij} \qquad \forall ij \in \mathcal{A}_n \\ & x_i \ge 0 \qquad \forall i \in \mathcal{N}. \end{aligned}$$

*Proof.* By assumption the constraint $f_{ij} \ge -K_{ij}$ in $\mathrm{P}(q)$ is not binding. It is clear that if a constraint to a convex problem is not active at the optimal solution, then the removal of the constraint has no bearing on the optimal solution. Therefore we can create a problem $\mathrm{P}'(q)$ without this constraint, which has the same optimal solution. □

The KKT conditions for the modified dispatch problem, $\mathrm{P}'(q)$ are

$$\begin{aligned} -x_i + \sum_{j,ij\in\mathcal{A}_n} f_{ij} - \sum_{j,ji\in\mathcal{A}_n} f_{ji} = q_i - d_i \quad & \forall i \in \mathcal{N} \\ 0 \le \tfrac{1}{a_i} x_i - \pi_i \perp x_i \ge 0 \quad & \forall i \in \mathcal{N} \\ 0 \le \pi_j - \pi_i \perp K_{ij} - f_{ij} \ge 0 \quad & \forall ij \in \mathcal{A}_n. \end{aligned} \tag{8}$$

THEOREM 1. *Recall the vector of injections* $\tilde{q}$ *for which there is no line congested at the optimal solution to* $\mathrm{P}(\tilde{q})$. *Now consider any vector of injections,* $q$, *such that* $q_n < \tilde{q}_n$ *and* $q_i = \tilde{q}_i$, $\forall i \in \mathcal{N} \setminus \{n\}$. *The nodal price at node* $n$, *from the optimal solution to* $\mathrm{P}(q)$ *is given by* $\pi_n^* = \max_{\delta\in\mathcal{D}_n} \pi_n^\delta$, *where* $\pi_n^\delta$ *is defined by equation* (6).

*Proof.* As there are only a finite number of decompositions in $\mathcal{D}_n$, we know that the maximum nodal price $\pi_n^*$ is attained for some decomposition. Furthermore since the economic dispatch problem is a strictly convex quadratic program it has a unique optimal primal solution, and so there is a unique solution $\pi$ satisfying (1). As $q_n < \tilde{q}_n$ and $q_i = \tilde{q}_i$, $\forall i \in \mathcal{N} \setminus \{n\}$, from lemmas 6 and 7, we know the solution to (1) exists and is the same as to the solution to (8). We will proceed by demonstrating that any decomposition $\delta \in \mathcal{D}_n$ with $\pi_n^\delta = \pi_n^*$ satisfies (8). We will therefore have that the (unique) solution to (8) will satisfy $\pi_n^\delta = \pi_n^*$.

Suppose that we have a decomposition $\delta \in \mathcal{D}_n$ with $\pi_n^\delta = \pi_n^*$. By construction, we know that within each subtree of $\mathcal{D}_n$, the conditions (8) are satisfied. Therefore if the solution to $\mathrm{P}^\delta(q)$ does not satisfy (8) then it must be the case that either

(A) a flow on an arc in $\mathcal{A}_\delta$ exceeds its capacity, or
(B) there is a congested arc $ij \in \mathcal{B}_\delta$ where $\pi_i^\delta > \pi_j^\delta$, or
(C) the dispatch problem $\mathrm{P}^{T_\delta}(q)$ is infeasible.

For (A), suppose that $f_{lm} > K_{lm}$ where $lm \in \mathcal{A}_\delta$. Removing $lm$ from $\mathcal{A}_\delta$ creates a new decomposition $\delta'$ with $\mathcal{N}_{\delta'}$ consisting of the node set containing node $n$, and setting $f_{lm} = K_{lm}$. The net flow into $\mathcal{N}_{\delta'}$ is thus smaller than it was for decomposition $\delta$, and so $\pi_n^{\delta'}$ is greater than $\pi_n^\delta$ which violates the assumption that $\pi_n^\delta$ is maximal over all decompositions.

Now consider case (B). Here again we will create a new decomposition where the price at node $n$ will increase from $\pi_n^\delta$. We define $\mathcal{N}^i$ to be the set of nodes in $\mathcal{N} \setminus \mathcal{N}_\delta$ that are connected to node $i$ via an uncongested path including node $i$, and construct the decomposition $\gamma$ with $\mathcal{N}_\gamma = \mathcal{N}^i \cup \mathcal{N}_\delta$ (see figure 3). By taking into account the flow on congested arcs into and out of these node sets, we find that the effective demands for $\mathcal{N}^i$ and $\mathcal{N}_\delta$ are

$$\begin{aligned} D_1 &= \sum_{k\in\mathcal{N}^i} d_k - \sum_{k\in\mathcal{N}^i} q_k^U + K_{ij} - \sum_{lm\in\mathcal{B}_\gamma\setminus\mathcal{B}_\delta} K_{lm}, \quad \text{and} \\ D_2 &= \sum_{k\in\mathcal{N}_\delta} d_k - \sum_{k\in\mathcal{N}_\delta\setminus\{n\}} q_k^U - q_n - K_{ij} - \sum_{lm\in\mathcal{B}_\delta\setminus\{ij\}} K_{lm}, \end{aligned}$$

respectively. As

$$D_1 + D_2 = \sum_{k\in\mathcal{N}_\gamma} d_k - \sum_{k\in\mathcal{N}_\gamma\setminus\{n\}} q_k^U - q_n - \sum_{lm\in\mathcal{B}_\gamma} K_{lm},$$

and $\pi_i^\delta > \pi_j^\delta$ it follows from lemma 5 that $\pi_n^\gamma = \pi_j^\gamma > \pi_j^\delta = \pi_n^\delta$, which violates the assumption that $\pi_n^\delta$ is maximal over all decompositions.

Finally for case (C), we represent the nodal price for an infeasible decomposition by $-\infty$. Any feasible dispatch, by lemma 4 has positive prices, and as we know there exists a decomposition for which the dispatch is feasible, this violates the assumption that $\pi_n^\delta$ is maximal over all decompositions. □

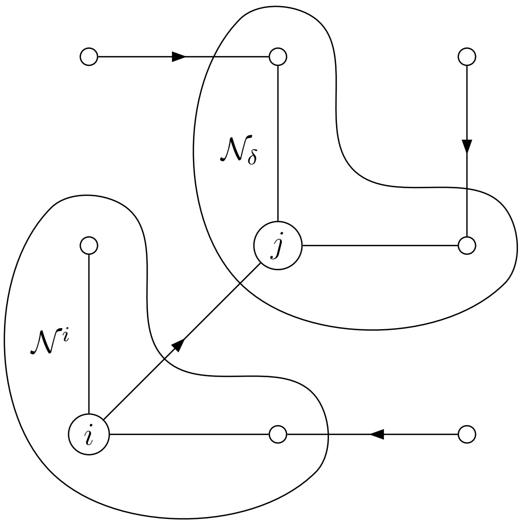
THEOREM 2. *Suppose that for some vector of injections $\tilde{q}$, the arcs are uncongested at the optimal dispatch for the economic dispatch problem. The residual demand curve faced by the generator at node $n$ is a convex piecewise linear function, for $q_n \le \tilde{q}_n$.*

*Proof.* The proof is a direct consequence of theorem 1, and the fact that the pointwise maximum of a set of linear functions is a convex function. □

COROLLARY 2. *Suppose that for some vector of injections $\tilde{q}$, the arcs are uncongested at the optimal dispatch for the economic dispatch problem. The residual demand curve faced by the generator at node $n$ is a concave piecewise linear function, for $q_n \ge \tilde{q}_n$.*

*Proof.* It is easy to show, so long as $\mathrm{P}(q)$ is feasible, that analogous results to lemma 6 and theorem 1 hold for $q_n \ge \tilde{q}_n$. However, in this case the price is a pointwise minimum of a set of linear functions giving a concave function. □

**Figure 3** Decomposition $\delta$, with $\pi_j^\delta > \pi_i^\delta$.




Note that so far we have established the shape of the residual demand curve, faced by the generator located at node $n$. For any player, $n$, in our radial network the residual demand curve is a convex, piecewise linear curve in all quantities $q_n \le q_n^U$, and is a concave, piecewise linear curve in all quantities $q_n \ge q_n^U$.

## 2.3. Cournot equilibrium

We now turn our attention to the existence of Cournot equilibria in tree networks. We will define the strategy space for the $n^{\text{th}}$ generator to be $q_n \in [0, \infty)$, and its cost function to be $c_n q_n$. First, we will show that there always exists an equilibrium for this game if the line capacities are infinitely large. We then derive necessary and sufficient conditions on the line capacities which ensure that their presence does not create an incentive for any generator to deviate from this equilibrium. Borenstein *et al.* in Borenstein et al. (2000) demonstrated a similar analysis in the context of a two node network.

In the absence of arc capacities in the network, we will show in theorem 3 that a unique Nash-Cournot equilibrium always exists. This equilibrium is found when all generators' simultaneously maximize their respective profits as a function of their injection.

We first calculate the price at all nodes, $\pi_n^U(q)$, from equation (6), where $U$ is the decomposition for which $\mathcal{N}_U = \mathcal{N}$ and $\mathcal{B}_U = \emptyset$, (as the lines have infinite capacity this is the only feasible decomposition).

Now, instead of using $\pi_n^U$, it is helpful to define a modified (finite) price function:

$$\hat{\pi}_n^U(q) = \frac{\sum_{i\in\mathcal{N}} d_i - \sum_{i\in\mathcal{N}} q_i}{\sum_{i\in\mathcal{N}} a_i}, \tag{9}$$

and show that with this price function a unique equilibrium exists. Observe that $\hat{\pi}_n^U$ is a linear function of $q_n$, in contrast to $\pi_n^U$ defined in (6). However, we will show (in theorem 3) that the equilibrium with prices $\hat{\pi}_n^U$ is also a unique equilibrium in the game where prices are $\pi_n^U$.

Generator $n$'s profit function in this modified game is

$$\hat{\rho}_n^U = q_n \left( \frac{\sum_{i\in\mathcal{N}} d_i - \sum_{i\in\mathcal{N}} q_i}{\sum_{i\in\mathcal{N}} a_i} - c_n \right).$$

This modified Cournot game where all generators are maximizing their profit function with the restriction $q \geq 0$ can be written as the following linear complementarity problem (LCP),

$$0 \leq \frac{2q_n + \sum_{i \in \mathcal{N} \setminus \{n\}} q_i - \sum_{i \in \mathcal{N}} d_i}{\sum_{i \in \mathcal{N}} a_i} + c_n \perp q_n \geq 0, \quad \forall n \in \mathcal{N}.$$

This can be written in the form,

$$0 \leq Mq + b \perp q \geq 0, \tag{10}$$

where

$$M = \frac{I_{|\mathcal{N}|} + J_{|\mathcal{N}|}}{\sum_{i \in \mathcal{N}} a_i} \text{ and } b = -\frac{\sum_{i \in \mathcal{N}} d_i}{\sum_{i \in \mathcal{N}} a_i} e + c,$$

and $I_{|\mathcal{N}|}$ is the identity matrix and $J_{|\mathcal{N}|}$ is a square matrix of ones, each having $|\mathcal{N}|$ rows.

By theorem 3.1.6 from Cottle et al. (1992), we are guaranteed that the LCP given in (10) has a unique solution, since $M$ is a symmetric positive definite matrix. The solution to this LCP is the unique unconstrained equilibrium, $q^U$. For example, in the case where all generators are generating a positive amount, the equilibrium injection for the generator at node $n$ will be of the form:

$$q_n^U = \frac{\sum_{i \in \mathcal{N}} d_i + \sum_{i \in \mathcal{N}} c_i \sum_{i \in \mathcal{N}} a_i}{|\mathcal{N}| + 1} - c_n \sum_{i \in \mathcal{N}} a_i.$$

Lemma 8. *Any equilibrium to the Cournot game with prices $\pi^U$ defined by* (6) *for $\delta = U$, has $\pi^U \geq 0$.*

*Proof.* If any generator injects 0 units of electricity then their profit will be 0, therefore any equilibrium must satisfy the condition that every generator's profit must be greater than or equal to 0. Hence at equilibrium we must have either $\pi^U \geq 0$ or $q_i = 0, \forall i \in \mathcal{N}$. This can be shown easily by observing that $q_i > 0$ and $\pi^U < 0$ would lead to a negative profit. For the potential equilibrium where $q_i = 0, \forall i \in \mathcal{N}$, we can calculate

$$\pi^U = \begin{cases} \frac{\sum_{i \in \mathcal{N}} d_i}{\sum_{i \in \mathcal{N}} a_i}, & \sum_{i \in \mathcal{N}} d_i \geq 0, \\ -\infty, & \sum_{i \in \mathcal{N}} d_i < 0, \end{cases}$$

which will be positive so long as $\sum_{i \in \mathcal{N}} d_i > 0$, which is true by assumption. □

Theorem 3. *Any equilibrium in the Cournot game with prices $\pi^U$ defined by* (6) *is also an equilibrium in the game with prices $\hat{\pi}^U$ defined by* (9)*.*

*Proof.* Let $q^U$ be an equilibrium in the game with prices $\pi^U$. From lemma 8 we have that $\pi^U$ is non-negative at equilibrium. Now consider the same equilibrium point but now with prices $\hat{\pi}^U$. Observe that any deviation $q_n$ for generator $n$ that results in $\hat{\pi}_n^U \geq 0$ is not profitable for $n$, since in this case $\hat{\pi}_n^U = \pi_n^U$ and $q^U$ is an equilibrium. Alternatively, if it results in $\hat{\pi}_n^U < 0$ then $\hat{\rho}_n^U < 0$ so it is also not a profitable deviation. Hence there is no incentive to deviate from the equilibrium $q^U$ when prices are given by $\hat{\pi}_n^U$. □

From theorem 3 we know that the unique solution to the LCP given by equation (10) is the unique Cournot equilibrium in the game with the price function $\pi^U$. We will refer to this equilibrium as the unconstrained equilibrium, $q^U$.

We now derive conditions on the line capacities that ensure that the strategies defined by the solution to (10) remain an equilibrium when line capacities are present. Our approach is to show that given the unconstrained equilibrium, no generator has any incentive to change its strategy.

Suppose then that each arc $ij$ has a capacity $K_{ij}$. We first need to ensure that the line flows in the unconstrained equilibrium $(f_{ij}^U)$ are supported. For this, the necessary and sufficient conditions are

$$K_{ij} \geq \left| f_{ij}^U \right|. \tag{11}$$

In what follows we will derive further conditions which are necessary and sufficient to ensure that no generator has incentive to deviate from the unconstrained equilibrium. These conditions together with (11) ensure the existence of the unconstrained equilibrium.

First, note that because of the concavity of the residual demand curve faced by generator $n$ when $q_n > q_n^U$ (see figure 1) there is no incentive for generator $n$ to deviate to a quantity higher than $q_n^U$. Therefore we only need to consider deviations in the range $0 < q_n < q_n^U$.

Since any withholding by generator $n$ will not improve its profit unless some line becomes congested, we investigate all possible congested states of the network by considering all decompositions $\delta \in \mathcal{D}_n$. Each decomposition $\delta$, has an associated residual demand curve. We consider the residual demand curve associated with $\delta$ and find the point $(q_n^{\delta *}, \pi_n^{\delta *})$ that maximizes the profit $\rho_n^\delta$, for the generator at node $n$. Although not all decompositions will occur as a consequence of withholding at node $n$, at least one of them will correspond to the congested state of the network when $q_n = q_n^{\delta *}$.

Deviating so as to congest lines corresponding to decomposition $\delta$ gives the following profit function:

$$\begin{aligned}\rho_n^\delta &= q_n^\delta \left(\pi^\delta - c_n\right)\\ &= \begin{cases} q_n^\delta \left( \dfrac{\sum_{i\in\mathcal{N}_\delta} d_i - q_n^\delta - \sum_{i\in\mathcal{N}_\delta\setminus\{n\}} q_i^U - \sum_{ij\in\mathcal{B}_\delta} K_{ij}}{\sum_{i\in\mathcal{N}_\delta} a_i} - c_n \right), & \sum\limits_{i\in\mathcal{N}_\delta} d_i - q_n^\delta - \sum\limits_{i\in\mathcal{N}_\delta\setminus\{n\}} q_i^U - \sum\limits_{ij\in\mathcal{B}_\delta} K_{ij} \geq 0,\\ -\infty, & \sum\limits_{i\in\mathcal{N}_\delta} d_i - q_n^\delta - \sum\limits_{i\in\mathcal{N}_\delta\setminus\{n\}} q_i^U - \sum\limits_{ij\in\mathcal{B}_\delta} K_{ij} < 0. \end{cases}\end{aligned}$$

There is no incentive to deviate to a point yielding a negative price, we will therefore only consider deviations such that

$$\sum_{i\in\mathcal{N}_\delta} d_i - q_n^\delta - \sum_{i\in\mathcal{N}_\delta\setminus\{n\}} q_i^U - \sum_{ij\in\mathcal{B}_\delta} K_{ij} \geq 0.$$

This yields the following optimization problem for generator $n$:

$$\begin{aligned} &\max\ q_n^\delta \left( \frac{\sum_{i\in\mathcal{N}_\delta} d_i - q_n^\delta - \sum_{i\in\mathcal{N}_\delta\setminus\{n\}} q_i^U - \sum_{ij\in\mathcal{B}_\delta} K_{ij}}{\sum_{i\in\mathcal{N}_\delta} a_i} - c_n \right)\\ &\text{s.t.}\quad 0 \leq q_n^\delta \leq \sum_{i\in\mathcal{N}_\delta} d_i - \sum_{i\in\mathcal{N}_\delta\setminus\{n\}} q_i^U - \sum_{ij\in\mathcal{B}_\delta} K_{ij} \end{aligned}$$

This is a maximization problem with a strictly concave objective function, so the following KKT conditions are necessary and sufficient to determine the unique solution.

$$\begin{aligned} 0 \leq \frac{2q_n^\delta + \sum\limits_{i\in\mathcal{N}_\delta\setminus\{n\}} q_i^U - \sum\limits_{i\in\mathcal{N}_\delta} d_i + \sum\limits_{i\in\mathcal{B}_\delta} K_{ij}}{\sum\limits_{i\in\mathcal{N}_\delta} a_i} + c_n + \mu \perp q_n^\delta \geq 0,\\ 0 \leq \sum_{i\in\mathcal{N}_\delta} d_i - q_n^\delta - \sum_{i\in\mathcal{N}_\delta\setminus\{n\}} q_i^U - \sum_{i\in\mathcal{B}_\delta} K_{ij} \perp \mu \geq 0. \end{aligned}$$

As we are looking for a solution which yields a positive profit, we require that $q_n^\delta > 0$ and $\pi_n^\delta > 0$. These conditions imply that any solution leading to a profitable deviation must yield $\mu = 0$ and

$$\frac{2q_n^{\delta *} + \sum_{i\in\mathcal{N}_\delta\setminus\{n\}} q_i^U - \sum_{i\in\mathcal{N}_\delta} d_i + \sum_{i\in\mathcal{B}_\delta} K_{ij}}{\sum_{i\in\mathcal{N}_\delta} a_i} + c_n = 0.$$

This gives

$$q_n^{\delta *} = \frac{1}{2}\left(\sum_{i\in\mathcal{N}_\delta} d_i - \sum_{i\in\mathcal{N}_\delta\setminus\{n\}} q_i^U - \sum_{ij\in\mathcal{B}_\delta} K_{ij} - c_n \sum_{i\in\mathcal{N}_\delta} a_i\right), \tag{12}$$

and

$$\begin{aligned}\pi^{\delta *} &= \frac{\sum_{i\in\mathcal{N}_\delta} d_i - q_n^{\delta *} - \sum_{i\in\mathcal{N}_\delta\setminus\{n\}} q_i^U - \sum_{ij\in\mathcal{B}_\delta} K_{ij}}{\sum_{i\in\mathcal{N}_\delta} a_i} \\ &= \frac{q_n^{\delta *}}{\sum_{i\in\mathcal{N}_\delta} a_i} + c_n. \end{aligned} \tag{13}$$

For feasibility, we require that $q_n^{\delta *} \geq 0$. (The upper bound constraint is automatically satisfied by $q_n^{\delta *}$, defined by (12).) Solving for the deviation profit gives

$$\rho_n^{\delta *} = \frac{\left(q_n^{\delta *}\right)^2}{\sum_{i\in\mathcal{N}_\delta} a_i}. \tag{14}$$

There is incentive to deviate to this decomposition only if

$$\rho_n^{\delta *} > \rho_n^U \text{ and } q_n^{\delta *} \geq 0.$$

The complement of this gives sufficient conditions under which there is no incentive to deviate to this decomposition, namely

$$\rho_n^U \geq \rho_n^{\delta *} \text{ or } q_n^{\delta *} < 0.$$

The first inequality is

$$\rho_n^U \geq \rho_n^{\delta *}$$

which, from (14), gives

$$\rho_n^U \sum_{i\in\mathcal{N}_\delta} a_i \geq \left(q_n^{\delta *}\right)^2,$$

yielding

$$-\sqrt{\rho_n^U \sum_{i\in\mathcal{N}_\delta} a_i} \leq q_n^{\delta *} \leq \sqrt{\rho_n^U \sum_{i\in\mathcal{N}_\delta} a_i}.$$

The union of this condition with $q_n^{\delta *} < 0$ gives

$$q_n^{\delta *} \leq \sqrt{\rho_n^U \sum_{i\in\mathcal{N}_\delta} a_i},$$

which with (12) gives

$$\sqrt{\rho_n^U \sum_{i\in\mathcal{N}_\delta} a_i} \geq \frac{1}{2}\left(\sum_{i\in\mathcal{N}_\delta} d_i - \sum_{i\in\mathcal{N}_\delta\setminus\{n\}} q_i^U - \sum_{ij\in\mathcal{B}_\delta} K_{ij} - c_n \sum_{i\in\mathcal{N}_\delta} a_i\right).$$

This yields the following inequality on the line capacities $K_{ij}$:

$$\sum_{ij\in\mathcal{B}_\delta} K_{ij} \geq \sum_{i\in\mathcal{N}_\delta} d_i - \sum_{i\in\mathcal{N}_\delta\setminus\{n\}} q_i^U - c_n \sum_{i\in\mathcal{N}_\delta} a_i - 2\sqrt{\rho_n^U \sum_{i\in\mathcal{N}_\delta} a_i}. \tag{15}$$

If we impose inequalities analogous to (15) for all nodes $n$ and for all possible decompositions $\delta \in \mathcal{D}_n$, we have a set of sufficient conditions that guarantees that there is no incentive for any generator to deviate from the unconstrained Nash-Cournot equilibrium. We call the set of arc capacities that satisfy these inequalities along with the conditions given by (11) the *competitive capacity set*.

The set of constraints defining the competitive capacity set is also necessary. That is, for any combination of arc capacities $\{K_{ij}\}_{ij\in\mathcal{A}}$ that lie outside the competitive capacity set, either the arc capacities are not large enough to support the equilibrium flows, or there exists a generator who has an incentive to deviate from the unconstrained Cournot equilibrium. To see this, observe that if any inequality in (15) is violated, then there is a generator $n$ and a decomposition $\delta \in \mathcal{D}_n$ giving

$$\rho_n^{\delta *} = q_n^{\delta *}\left(\pi_n^\delta - c_n\right) > \rho_n^U.$$

Now by theorem 1 an injection of $q_n^\delta < q_n^U$ yields a price

$$\pi_n^* = \max_{\delta\in\mathcal{D}_n} \pi_n^\delta,$$

and so the profit actually made by generator $n$ injecting $q_n^\delta$ (while others inject $q_i^U$) is

$$\rho_n = q_n^{\delta *}\left(\pi_n^* - c_n\right) \geq \rho_n^{\delta *} > \rho_n^U,$$

showing that generator $n$ can deviate profitably from $q_n^U$.

Note that there is a trivial set of constraints on the capacities that constitute sufficient conditions for the existence of the unconstrained Nash-Cournot equilibrium. This set is given by the inequalities

$$\sum_{ij\in\mathcal{B}_\delta} K_{ij} \geq \sum_{i\in\mathcal{N}_\delta} d_i - \sum_{i\in\mathcal{N}_\delta\setminus\{n\}} q_i^U, \quad \forall\delta \in \mathcal{D}_n, \forall n \in \mathcal{N}.$$

These ensure that no decomposition has a positive price, hence there can be no incentive for any generator to deviate. Furthermore note that this characterization is weaker than (15). This is clear, because if $c_n = 0$ then $\rho_n^U > 0$, therefore we have that:

$$c_n \sum_{i\in\mathcal{N}_\delta} a_i + 2\sqrt{\rho_n^U \sum_{i\in\mathcal{N}_\delta} a_i} > 0.$$

Hence the inequalities given by (15) give smaller right-hand sides, and thus a larger competitive capacity set, than the trivial set of constraints above.

**Figure 4** Three-node radial network.

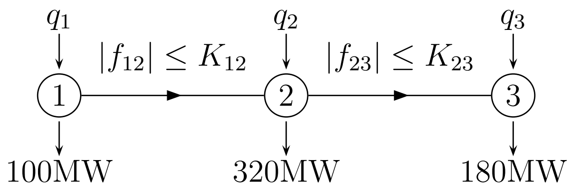
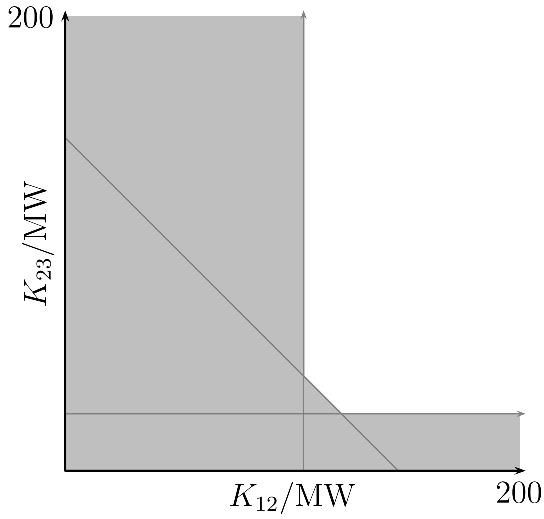



**Figure 5** Competitive capacity set (given by unshaded region).




*Example* To illustrate the competitive capacity set, consider the three node example shown in figure 4. This has identical fringes with slope 1 at all nodes, and zero marginal cost for each generator. The unconstrained equilibrium is

$$q_1^U = q_2^U = q_3^U = 150,\ \pi^U = 50,\ f_{12}^U = 100,\ f_{23}^U = -20.$$

The competitive capacity set (shown by the unshaded region in figure 5) is defined by the intersection of the following constraints.

$$\begin{aligned} K_{12} &\geq 105.05, \\ K_{23} &\geq 25.05, \\ K_{12} + K_{23} &\geq 146.79. \end{aligned}$$

Generators have an incentive to deviate when the line capacities do not lie in this set. For example, consider the capacities $K_{12} = 106$ and $K_{23} = 26$. At this point the condition that the sum of the capacities must exceed 146.79 is violated, and we find that the profit for generator 2 to deviate by congesting both lines towards node 2 is 8836, whereas its equilibrium profit is only 7500. This means that generator 2 has incentive to deviate, hence $q^U$ is not an equilibrium.

# 3. Extension to DC load flow models

## 3.1. Networks with loops

The above results apply to tree networks which have the crucial property that the set of congested lines increases as a single generator withholds at some node. Most real electricity networks have loops which are required for reliability. When loops are present the flows of electricity in the network arrange themselves according to Kirchhoff's laws for DC load flow. This is usually modelled using a voltage phase angle $\theta_i$ at each node $i$ and a line reactance $l_{ij}$. The flow in $ij$ must then satisfy

$$l_{ij} f_{ij} = \theta_i - \theta_j.$$

This gives the following version of the dispatch model.

$$\begin{aligned}
\mathrm{P}(q) := \min \ & \sum_{i \in \mathcal{N}} \tfrac{1}{2a_i} {x_i}^2 & \\
\text{s.t.} \ & -x_i + \sum_{j, ij \in \mathcal{A}} f_{ij} - \sum_{j, ji \in \mathcal{A}} f_{ji} = q_i - d_i & \forall i \in \mathcal{N} \\
& -K_{ij} \le f_{ij} \le K_{ij} & \forall ij \in \mathcal{A} \\
& l_{ij} f_{ij} = \theta_i - \theta_j & \forall ij \in \mathcal{A}.
\end{aligned}$$

Observe that for radial networks (with no loops), we may remove the voltage phase angles and their constraints from the DC load flow formulation, as we did in section 2.

With the addition of loops to the dispatch problem, lemma 6 is no longer valid, as there is no longer a single path between every pair of nodes. We show later that theorem 2, which states that the residual demand curve for generator $n$ is convex for $q_n < q_n^U$, is no longer true. Without this result, the set of arc capacities that are necessary and sufficient for the existence of the unconstrained equilibrium are more difficult to compute. We must now enumerate all possible ways that the complementarity conditions for $\mathrm{P}(q)$ with loops can be satisfied. These will be referred to as *KKT regimes*. For each generator $n$ and KKT regime $r$ we compute the set of arc capacities precluding a profitable deviation from the unconstrained Cournot equilibrium $q^U$, namely

$$\mathcal{K}_n^r = \{K \mid \text{there is no feasible } q_n \text{ in KKT regime } r \text{ with profit } \rho_n(q_n) > \rho_n(q_n^U)\}.$$

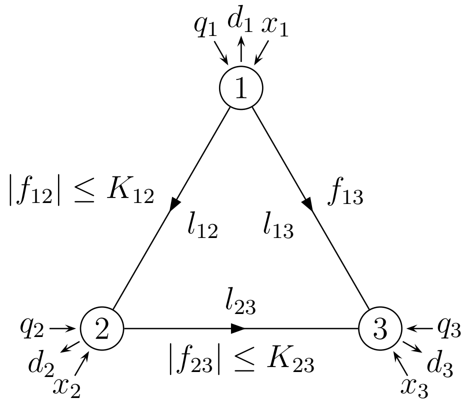
The set of arc capacities that are necessary and sufficient to ensure generator $n$ cannot unilaterally increase its profit above what would have been achieved in the unconstrained equilibrium can therefore be written as

$$\bigcap_r \mathcal{K}_n^r.$$

We will refer to this as the *non-deviation set* for generator $n$.

Instead of the set of arc capacities that are necessary and sufficient for the existence of the unconstrained equilibrium being determined by a number of linear inequalities (as is the case for radial networks), the set must be constructed from the intersection of the non-deviation sets for all generators, which may yield a non-convex result. Observe that the calculation of this set is much more cumbersome than its counterpart for radial networks, since there are $\left(2^{|\mathcal{N}|} \times 3^{|\mathcal{A}|}\right) - 1$ possible KKT regimes, which would represent a significant computational task. As an illustration

**Figure 6** Three-node network; two lines have capacity constraints.




of this procedure, consider the three-node network shown in figure 6. When voltage phase angles are eliminated, we have the following dispatch problem:

$$\min \ \tfrac{1}{2a_1}{x_1}^2 + \tfrac{1}{2a_2}{x_2}^2 + \tfrac{1}{2a_3}{x_3}^2$$

$$\begin{array}{ll}
-x_1 + f_{12} + f_{13} = q_1 - d_1 & [\pi_1] \\
-x_2 - f_{12} + f_{23} = q_2 - d_2 & [\pi_2] \\
-x_3 - f_{13} - f_{23} = q_3 - d_3 & [\pi_3] \\
l_{12}f_{12} - l_{13}f_{13} + l_{23}f_{23} = 0 & [\lambda] \\
|f_{12}| \leq K_{12} & [\eta^1_{12}, \eta^2_{12}] \\
|f_{23}| \leq K_{23} & [\eta^1_{23}, \eta^2_{23}] \\
x_1, x_2, x_3 \geq 0. &
\end{array}$$

As the above problem is convex, we can replace it with its equivalent KKT system, shown below:

$$\begin{aligned}
-x_1 + f_{12} + f_{13} &= q_1 - d_1 \\
-x_2 - f_{12} + f_{23} &= q_2 - d_2 \\
-x_3 - f_{13} - f_{23} &= q_3 - d_3 \\
l_{12}f_{12} - l_{13}f_{13} + l_{23}f_{23} &= 0 \\
\pi_1 - \pi_2 + l_{12}\lambda + \eta^1_{12} - \eta^2_{12} &= 0 \\
\pi_1 - \pi_3 - l_{13}\lambda &= 0 \\
\pi_2 - \pi_3 + l_{23}\lambda + \eta^1_{23} - \eta^2_{23} &= 0 \\
0 \leq x_1 - a_1\pi_1 &\perp x_1 \geq 0 \\
0 \leq x_2 - a_2\pi_2 &\perp x_2 \geq 0 \\
0 \leq x_3 - a_3\pi_3 &\perp x_3 \geq 0 \\
0 \leq K_{12} - f_{12} &\perp \eta^1_{12} \geq 0 \\
0 \leq K_{12} + f_{12} &\perp \eta^2_{12} \geq 0 \\
0 \leq K_{23} - f_{23} &\perp \eta^1_{23} \geq 0 \\
0 \leq K_{23} + f_{23} &\perp \eta^2_{23} \geq 0.
\end{aligned}$$

The unconstrained equilibrium (assuming zero marginal cost) for this problem is given by

$$q^U_1 = q^U_2 = q^U_3 = \frac{d_1 + d_2 + d_3}{4}.$$

**Figure 7** Residual demand curve for three-node network.

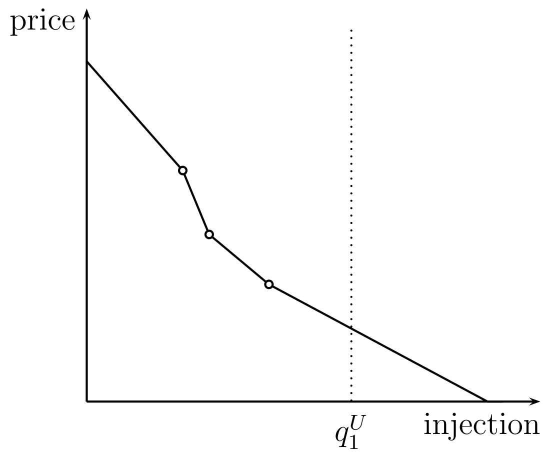
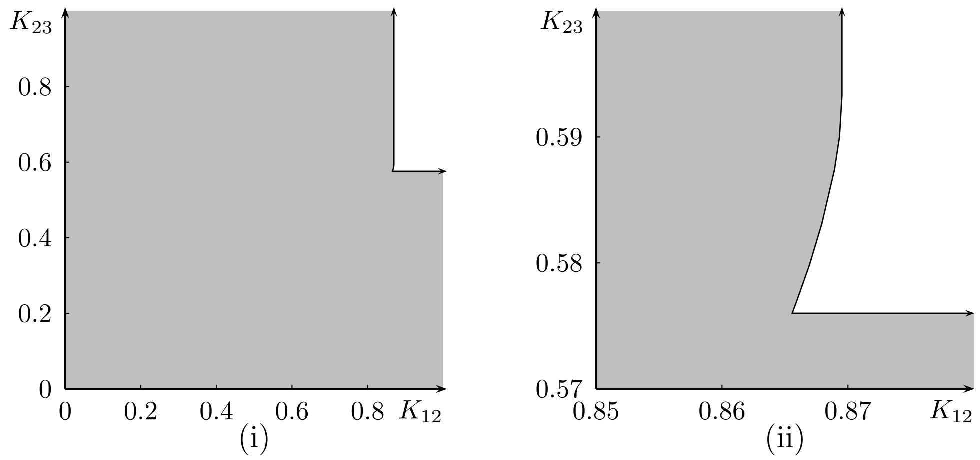



**Figure 8** (i) Combined deviation set for generator 1. (ii) Zoomed in.




Now by setting $q_2 = q_2^U$ and $q_3 = q_3^U$, we can construct the residual demand curve for generator 1, for a given $K_{12}$ and $K_{23}$, by considering different KKT regimes; each regime corresponds to a different piece of the residual demand curve. Note that this curve, shown in figure 7, is not convex for $q_1 \le q_1^U$.

We can find the non-deviation set for generator 1, for the situation where $a_1 = 1.44$, $a_2 = 3.4$, $a_3 = 1.44$, $d_1 = 3.04$, $d_2 = 0.76$, $d_3 = 1.5$, $l_{12} = 0.178$, $l_{13} = 0.104$, $l_{23} = 0.104$, by enumerating $\mathcal{K}_1^r$ for all $r$. The intersection of these sets is the non-deviation set for generator 1, which is shown as the unshaded region in figure 8 (i). If we zoom in on the corner of this set, shown in 8 (ii), we can see that the non-deviation set for this generator is not convex.

### 3.2. Networks with losses

Up to this point this paper has been concerned with networks in which the transmission lines have no losses. Here we digress briefly to consider this issue. Thermal losses in electricity transmission networks are usually modelled as quadratic functions of the flow on the line, which can be represented in the dispatch model as follows.

$$-x_i + \sum_{j,ij\in\mathcal{A}} f_{ij} - \sum_{j,ji\in\mathcal{A}} (f_{ji} - r{f_{ji}}^2) \le q_i - d_i \quad \forall i \in \mathcal{N}$$

Observe that the flow conservation constraint is now an inequality implying the option of free disposal of energy at any node (see e.g. Philpott and Pritchard (2004)). This is required to ensure a convex dispatch problem. In most circumstances (e.g. when electricity prices are positive) this free disposal will be zero at optimality. We will assume that this is the case in our discussion here.

The presence of quadratic losses creates a limit on how much power can be sent along the line; this limit is due to the marginal loss becoming equal to 1, so that any increment of flow sent disappears. This capacity limit decreases as the loss coefficient $r$ increases. It is asserted by Borenstein et al. (2000) that for large enough values of $r$, the unconstrained equilibrium will not exist due to the effects of the implied capacity limit. We will show, in a two-node symmetric example, that an equilibrium always exists regardless of the magnitude of the loss coefficient $r$. We model the losses as they are modelled by Borenstein et al. (2000) (the losses are incurred at the node that is receiving power, i.e. the losses are proportional to the square of sent power)

In this setting, equations (16) below provide the candidate Cournot equilibrium quantities.

$$\begin{aligned} q_1 = q_2 = q^C &= \frac{3+6r-\sqrt{9+4r+4r^2}}{8r}, \\ \pi_1 = \pi_2 &= \frac{2r-3+\sqrt{9+4r+4r^2}}{8r}, \\ f &= 0. \end{aligned} \tag{16}$$

The optimal revenue for each generator is

$$\begin{aligned} \bar{R} &= \pi_1 q^C \\ &= \frac{1}{8} - \frac{9}{32r^2} - \frac{1}{4r} + \frac{3\sqrt{9+4r+4r^2}}{32r^2} + \frac{\sqrt{9+4r+4r^2}}{16r}. \end{aligned}$$

We will show in what follows that for any value of $r$, no generator has incentive to change their offer from the candidate equilibrium quantity $q^C$. As this is a symmetric example it suffices to show that generator 1 has no incentive to deviate from injecting the equilibrium quantity $q^C$ when generator 2 offers in $q_2 = q^C$. There are two types of deviation for generator 1, injecting more than $q^C$ and injecting less than $q^C$. The direction of flow along the line will be different for these two cases therefore they must be considered separately. We assume that the line has infinite capacity.

*Injecting more electricity* If generator 1 were to deviate from the candidate equilibrium by injecting more electricity at node 1, then a flow $f \geq 0$ on the line from node 1 to 2 will result. This gives the dispatch problem:

$$\begin{array}{lrl} \min & \frac{1}{2}{x_1}^2 + \frac{1}{2}{x_2}^2 & \\ \text{s.t.} & -x_1 + f \leq q_1 - 1 & [\pi_1] \\ & -x_2 - f + rf^2 \leq q_2 - 1 & [\pi_2] \\ & x_1, x_2 \geq 0, & \end{array}$$

which has the following optimality conditions when prices are positive.

$$\begin{aligned} -\pi_1 + f &= q_1 - 1 \\ -\pi_2 - f + rf^2 &= q_2 - 1 \\ \pi_1 - \pi_2(1-2rf) &= 0 \\ \pi_1, \pi_2 &\geq 0. \end{aligned}$$

For generator 1 to have a profitable deviation, we require $\pi_1 > 0$, which from the optimality conditions above, implies that $\pi_2 > 0$ and $(1-2rf) > 0$. The maximum revenue that generator 1 can obtain by increasing his offer from $q^C$ is given by the optimal value of

$$\begin{array}{rrl} \mathrm{S}\,(q_2): & \max & q_1\pi_1 \\ & \text{s.t.} & -\pi_1 + f = q_1 - 1 \\ & & -\pi_2 - f + rf^2 = q_2 - 1 \\ & & \pi_1 - (1-2rf)\,\pi_2 = 0 \\ & & \pi_1, \pi_2 \geq 0. \end{array}$$

We can remove the inequality constraints on the prices and replace them by $0 \leq f \leq \frac{1}{2r}$ to give the following relaxation of $\mathrm{S}\,(q_2)$

$$\begin{array}{rrl} \hat{\mathrm{S}}\,(q_2): & \max & q_1\pi_1 \\ & \text{s.t.} & -\pi_1 + f = q_1 - 1 \\ & & -\pi_2 - f + rf^2 = q_2 - 1 \\ & & \pi_1 - (1-2rf)\,\pi_2 = 0 \\ & & 0 \leq f \leq \frac{1}{2r}. \end{array} \tag{17}$$

Let $q_2 = q^C$, the equilibrium quantity defined by (16). Then the first 3 constraints in (17) may be expressed as

$$q_1 = 1 + f + \frac{(1-2rf)\left(3 - 2r + 8rf - 8r^2f^2 - \sqrt{9+4r+4r^2}\right)}{8r},$$

$$\pi_1 = -\frac{(1-2rf)\left(3 - 2r + 8rf - 8r^2f^2 - \sqrt{9+4r+4r^2}\right)}{8r},$$

and

$$\pi_2 = -\frac{3 - 2r + 8rf - 8r^2f^2 - \sqrt{9+4r+4r^2}}{8r}.$$

The objective function of $\hat{\mathrm{S}}\,(q_2)$ becomes

$$\begin{aligned} \hat{R}(f) = {} & \frac{1}{8} - \frac{9}{32r^2} - \frac{1}{4r} + \frac{3\sqrt{9+4r+4r^2}}{32r^2} + \frac{\sqrt{9+4r+4r^2}}{16r} \\ & + \left(\frac{11}{8} + \frac{r}{2} - \frac{r^2}{2} - \left(\frac{9}{8} + \frac{1}{4}r\right)\sqrt{9+4r+4r^2}\right) f^2 \\ & + \left(3r + 2r^2 + 2r\sqrt{9+4r+4r^2}\right) f^3 \\ & + \left(-12 - 2r^3 - r^2\sqrt{9+4r+4r^2}\right) f^4 \\ & + 12r^3f^5 - 4r^4f^6, \end{aligned}$$

which we seek to maximize over $f \in [0, \frac{1}{2r}]$.

It is easy to verify that $\hat{R}(f)$ over this range has a unique global maximum at $f = 0$ with value $\bar{R} > 0$. This is because $\hat{R}(\frac{1}{2r}) = 0$, and $\left.\frac{d\hat{R}}{df}\right|_{f=0} = 0$, $\left.\frac{d^2\hat{R}}{df^2}\right|_{f=0} < 0$, and $\frac{d\hat{R}}{df}$ has at most one zero in $(0, \frac{1}{2r})$. It follows that the optimal revenue $R(f)$ from any deviation $q_1 > q_1^C$ giving flow $f > 0$ satisfies

$$R(f) \leq \hat{R}(f) \leq \hat{R}(0) = \bar{R}.$$

*Injecting less electricity* If generator 1 were to deviate from the candidate equilibrium by injecting less electricity at its node, a flow on the line from node 2 to 1 will be induced. This results in the dispatch problem:

$$\begin{array}{lll} \min & \frac{1}{2}{x_1}^2 + \frac{1}{2}{x_2}^2 & \\ \text{s.t.} & -x_1 + f + rf^2 \le q_1 - 1 & [\pi_1] \\ & -x_2 - f \le q_2 - 1 & [\pi_2] \\ & x_1, x_2 \ge 0, & \end{array}$$

which has again the following optimality conditions when prices are positive.

$$\begin{aligned} -\pi_1 + f + rf^2 &= q_1 - 1 \\ -\pi_2 - f &= q_2 - 1 \\ \pi_1 (1 + 2rf) - \pi_2 &= 0 \\ \pi_1, \pi_2 &\ge 0. \end{aligned}$$

For generator 1 to have a profitable deviation when $q_2 = q^C$ we require $\pi_1 > 0$. We also require $f > -\frac{1}{2r}$ for feasible solutions. As above consider the relaxation

$$\begin{array}{ll} \hat{\mathrm{S}}(q_2): \max & q_1\pi_1 \\ \text{s.t.} & -\pi_1 + f + rf^2 = q_1 - 1 \\ & -\pi_2 - f = q_2 - 1 \\ & \pi_1 (1 + 2rf) - \pi_2 = 0 \\ & 0 \le -f \le \frac{1}{2r} \end{array}$$

Once we substitute $q_2 = q^C$ and eliminate variables $q_1, \pi_1$, and $\pi_2$, we have that

$$\begin{aligned} \hat{R}(f) = & -\frac{1}{64r^2 (1+2rf)^2}[18 + 16r - 8r^2 - (6+4r)\sqrt{9+4r+4r^2} \\ & + \left(72 + 64r - 32r^2 - (24+16r)\sqrt{9+4r+4r^2}\right) rf \\ & + \left(200 + 80r - 24\sqrt{9+4r+4r^2}\right) r^2 f^2 \\ & + \left(240 - 32r - 16\sqrt{9+4r+4r^2}\right) r^3 f^3 \\ & + 128 r^4 f^4]. \end{aligned}$$

As in the previous case we have that $\left.\frac{d\hat{R}}{df}\right|_{f=0} = 0$, and $\left.\frac{d^2\hat{R}}{df^2}\right|_{f=0} < 0$, and $\frac{d\hat{R}}{df} > 0$ in $(-\frac{1}{2r}, 0)$. It follows that $\hat{R}(f)$ has a unique global maximum over $(-\frac{1}{2r}, 0]$ at $f = 0$ with value $\bar{R} > 0$. It follows that the optimal revenue $R(f)$ from any deviation $q_1 < q_1^C$ giving flow $f < 0$ satisfies

$$R(f) \le \hat{R}(f) \le \hat{R}(0) = \bar{R}.$$

As the generators are symmetric, a Nash-Cournot equilibrium in this example always exists regardless of the magnitude of the loss coefficient on the line.

## 4. Application to the New Zealand market

In this section, we will calculate the minimum transfer capacities to ensure the existence of unconstrained Cournot equilibria in a simplified representation of the New Zealand electricity market. This work was part of a project undertaken by New Zealand's grid owner Transpower seeking to quantify the future competition benefits (in 2010, 2015 and 2020) of expanding the capacity of the transmission line in the upper part of New Zealand's North Island (between locations Whakamaru

**Figure 9** Two-node network.

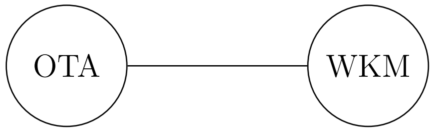



**Table 1** Strategic generators and their capacities.

| | | Capacity (MW) | | |
|---|---|---|---|---|
| Generator | Location | 2010 | 2015 | 2020 |
| Otahuhu B | OTA | 390 | 390 | 390 |
| Huntly + E3P | OTA | 1413 | 1413 | 1413 |
| Waikato Hydro | WKM | 776 | 776 | 776 |
| Taranaki Combined Cycle + New Plymouth | WKM | 665 | 665 | 665 |
| Clutha Dam | WKM | 700 | 790 | 790 |
| Waitaki System | WKM | 2718 | 3008 | 3183 |

**Table 2** Aggregated expected wind and geothermal power.

| | Wind/Geothermal (MW) | | |
|---|---|---|---|
| Node | 2010 | 2015 | 2020 |
| OTA | 10 | 70 | 70 |
| WKM | 701 | 1081 | 1081 |

and Otahuhu). The New Zealand Electricity Commission (that regulates the electricity industry) uses a form of cost-benefit analysis called the Grid Investment Test (2006) to prioritize new investments in transmission capacity, and competition benefits can be included in this analysis if they can be estimated.

In order to perform a cost-benefit analysis of grid upgrades, it is necessary to consider multiple demand scenarios (for example peak and off-peak) each with some frequency or likelihood. It is straightforward to construct a competitive capacity set which ensures that the unconstrained Nash-Cournot equilibrium exists for all scenarios; in fact this set can be shown to be polyhedral. However, a policy which ensures that the line capacities lie in this set may be too conservative. A more practical approach could involve assigning some penalty to an expansion plan falling outside the competitive capacity set for a particular demand scenario. However, for purposes of illustration, we focus here on only one peak demand scenario and its corresponding competitive capacity set.

The transmission system in our model is a two-node network, with one node (OTA) at Otahuhu and the other node (WKM) at Whakamaru.

In order to apply this model, we need to categorize generators into one of three types: strategic; part of a competitive fringe; or assumed to offer in however much they can produce at \$0 (e.g. wind / geothermal). We assume there are six major strategic generators in the New Zealand market, and have located them at either OTA or WKM based on their position in the New Zealand grid, as shown in Table 1.

As plant outputs are limited to their generation capacities, we need to use a modified approach to analyze the situation, and remove the limitation of one generator per node. Wind farms and geothermal plants are assumed to have \$0 short-run marginal costs and have no ability to act strategically. We therefore subtract their production directly from the demand. In the case of wind farms we assume that on average they are running at 40% of capacity and that geothermals are running at 100% of capacity. The expected amount of wind power and geothermal power available is shown in Table 2.

**Table 3** Aggregated demand forecasts.

| | Demand Forecast (MW) | | |
|---|---|---|---|
| Node | 2010 | 2015 | 2020 |
| OTA | 2288 | 2631 | 2987 |
| WKM | 5005 | 5447 | 5830 |

**Table 4** Short-run marginal costs.

| | Marginal Cost ($/MWhr) | | |
|---|---|---|---|
| Fuel Type | 2010 | 2015 | 2020 |
| Gas | 56.56 | 63.61 | 78.42 |
| Coal | 46.35 | 48.98 | 51.60 |

The demand and generation assumptions have been derived from scenarios developed by the New Zealand Electricity Commission and Transpower; updated fuel costs (gas and coal) have been derived from the New Zealand Ministry of Economic Development's latest Energy Outlook New Zealand Ministry of Economic Development (2007). The demand forecast we used is shown in Table 3.

The short-run marginal costs (excluding the proposed carbon charge) for gas and coal plants, that were used in the model, are shown in Table 4.

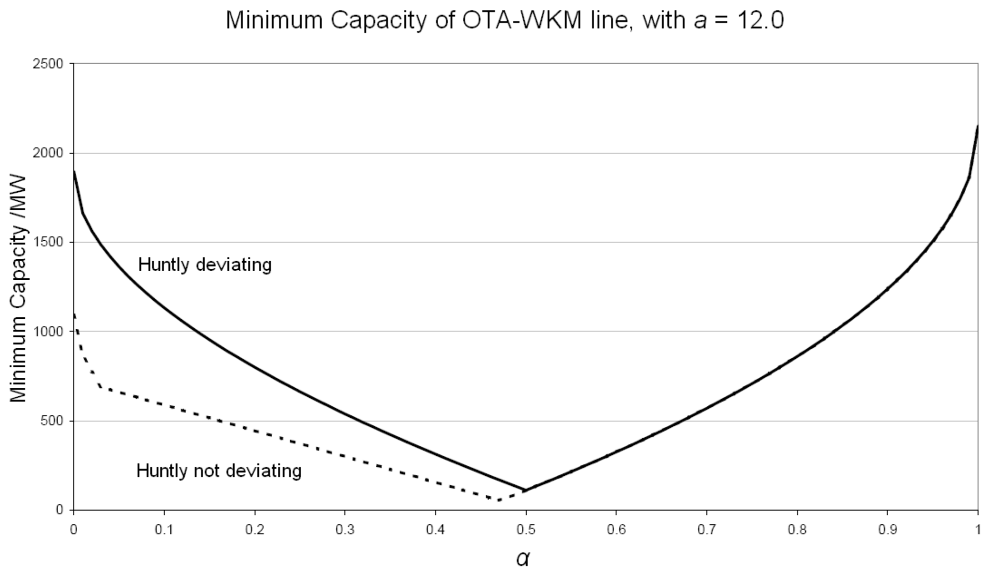
All remaining generators form part of a competitive fringe which offers a linear supply function. There is one competitive fringe at each node, and the prices paid to the generators are determined by their slopes. The slope at the OTA node is $a_1$ and the slope at the WKM node is $a_2$. We choose $a_1 = \alpha \times a$ and $a_2 = (1 - \alpha) \times a$, for a range of $\alpha$ and $a$.

The value of $a$ determines the price paid to all generators in the unconstrained equilibrium, and the value of $\alpha$ affects the incentives generators have to deviate from the unconstrained equilibrium. For example, if the fringe slope at one node is small and the fringe slope at the other node is large, then the generators located at the node with the smaller fringe slope will be more willing to attempt to congest the line toward themselves to get a higher price.

When $\alpha$ is small, we require a large line to prevent the large generator Huntly at OTA from withholding too much. However, when $\alpha$ is large the Waitaki System at the WKM node sets the size of the line. A plot of the required line size as a function of $\alpha$ for $a = 12.0$ is shown as the solid line in figure 10.

Huntly is New Zealand's largest thermal generator, and contributes significantly to the base load. Given that it will typically be contracted for a high proportion of its generation, it is instructive to examine the situation when Huntly chooses not to deviate from its unconstrained equilibrium generation. A plot of the required line size as a function of $\alpha$ for $a = 12.0$ is shown as a dashed line in figure 10. With Huntly constrained, it is the Otahuhu B plant that sets the size of the line when $\alpha$ is very small. However, since it is a smaller plant than Huntly it cannot always take advantage of the situation. It turns out that in this situation, for $0.04 \leq \alpha \leq 0.47$ the size of line required is equal to the flow on the line in equilibrium.

**Figure 10** Required capacity on OTA–WKM line in 2010.



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