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Supply-function equilibrium with taxed benefits

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Supply-function equilibrium models are used to study electricity market auctions with uncertain demand. We study the effects on supply-function equilibrium of a system tax on the observed benefits of suppliers. Such a tax provides an incentive for agents to alter their offers to avoid the tax. We show how this surprisingly can lead to lower prices in equilibrium. The model is extended to a setting in which the agents are taxed on the benefits accruing to them from a transmission line expansion (in order to help fund the line). In these circumstances we study how incentives for agents to alter their bids varies with the relative size of the capacity expansion.

Key words: Supply-function equilibrium, Electricity Markets

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1. Introduction

In electricity market auctions, producers typically submit amounts of generation that they are willing to supply at different prices. These schedules together form a supply curve that is cleared by a system operator to meet demand in a pool, thus yielding a system marginal price. All generation offered at a price equal or below this market price is dispatched. Each generator is then paid the system marginal price for all the energy they are dispatched. This leads to market rents accruing on inframarginal offers (those with offer price below the system marginal price).

The offers of the generators can be modeled by supply functions. In the face of uncertain demand, each agent seeks such a curve to maximize its expected profit, leading to the concept of supply-function equilibrium (SFE) (Klemperer and Meyer 1989). SFE models have been applied to the study of electricity market auctions by a number of authors (Green and Newbery 1992, Holmberg and Newbery 2010). These models deal with demand uncertainty in a natural way, a feature that makes them increasingly useful as intermittent renewable generation grows. To enable solutions to be obtained analytically, SFE models typically assume symmetric players with identical costs and capacities. In electricity markets, demand is inelastic in the short term, so the market is assumed to operate with a price cap. Plant capacities are such that demand exceeds the total supply capacity with some small probability. When this occurs, the market clears at the price cap, and load is shed. Details can be found in the survey by Holmberg and Newbery (2010).

In this paper we study the effects on agent behavior of a system tax levied on the surplus earned by inframarginal rents. Since the true marginal cost functions of the agents are not public knowledge, the rents are computed on the basis of the supply function offered. The imposition of the tax alters the incentives of the agents in choosing what supply functions to offer to the auction. Their offer curves will adjust in such a way to minimize the tax paid, while not sacrificing too much profit. When electricity demand is deterministic, the agents can anticipate the market clearing price and their dispatch quantity. Given this dispatch point, each agent has an incentive to increase the prices of their inframarginal offers so as to reduce the apparent benefit (while maintaining their real benefit). One might then expect all offers to become perfectly elastic at the clearing price up to the anticipated dispatch quantity.

Uncertainty in demand alters this outcome. Agents offering perfectly elastic bids might find if demand is lower than anticipated that they make no money at all. In such circumstances agents do better by offering an increasing supply curve that trades off the amount of tax paid against the need to earn some profit. One might expect this curve to mark up offers to recover the tax through higher prices, however we show that, in equilibrium, this strategy is only applied to the lower end of the supply curve, and at high prices agents may discount their offers.

Our study of such a tax is motivated by a proposal mooted by the New Zealand Electricity Authority to charge electricity market participants for transmission improvements based on the benefits that are deemed to accrue to them from these upgrades (see New Zealand Electricity Authority 2014). Currently most transmission in New Zealand is funded through a charge on load drawn during peak demand periods. Recent expansion to the transmission network has been concentrated in certain regions, and the imbalance of costs and benefits between regions from peak-load charges has prompted a full review of the country's transmission pricing methodology.

Although the details of the new 'beneficiary-pays' scheme are still being negotiated, the proposal is to estimate benefits by running the software used for dispatching the wholesale market and computing locational marginal prices. The key calculations in the proposed scheme are as follows. After the market is dispatched with current transmission assets in place, the benefits of each agent are computed from their bid and offer curves. For a generator this benefit is measured by the rentals earned from inframarginal bids. As we have already remarked, this need not be the true benefit if these bids are marked up above the generator's marginal cost. The dispatch software is then run again using the same bids and offers, but with the transmission assets de-rated to their pre-upgrade levels. The benefits for each agent are then computed under this counterfactual and subtracted from the previous estimates. The agents with positive net benefits contribute to the upgrade cost of the transmission system in proportion to these net benefits. A fuller description is provided in the New Zealand Electricity Authority's consultation paper (2012).

In this paper we use SFE models to study the incentives faced by strategic agents under such a scheme. As typically assumed in electricity market models, we assume that demand is inelastic and subject to an additive shock with a known distribution. This is in contrast to Vives (2011) who consider a SFE model with known elastic demand, and a tax level modeled as a random shock that is added to the intercept of each agent's marginal cost curve. Like many authors Vives (2011) also restricts attention to linear SFE solutions. When marginal costs and demand are linear, linear SFE are uniquely defined (Klemperer and Meyer 1989). Our model assumes inelastic demand which yields nonlinear SFE, which in general are not unique. In this case we select the least competitive SFE, following Newbery (1998).

We find in the simplest case that the presence of a tax on benefits causes strategic agents to mark up prices at low levels of production and to mark these down as production approaches capacity. As a comparison, we compute a symmetric equilibrium for the same model under a price-taking assumption, and show that taxed suppliers mark up prices at all levels of output. When strategic agents are being taxed on the observed benefits of a transmission expansion, their incentives to mark up prices attenuates if the expansion is small and leads to a low probability of lost load. In some examples we show that after-tax prices in equilibrium are less than before-tax prices at all output levels.

In the asymmetric case, SFE models generally require a numerical solution that might be difficult to compute (Anderson 2013), so our analysis is mainly confined to SFE for a symmetric duopoly for which a single ordinary differential equation (ODE) can be solved to yield an equilibrium. Even in the symmetric case existence of pure strategy SFE cannot be taken for granted, as it depends on the level of taxation and the probability distribution of the demand shock. If too much tax is levied, the incentive to avoid it at low output levels outweighs the potential loss in revenue to the extent that a pure-strategy SFE fails to exist. At the extreme a 100% tax corresponds to a pay-as-bid pricing scheme for which pure-strategy SFE exists for only a small class of demand shocks (Anderson et al. 2013). Similarly when the distribution of demand has low variance, the ability to predict the optimal dispatch point provides incentives to mark up prices on all offers below this, and pure-strategy SFE fail to exist.

As mentioned above, asymmetric SFE (where agents have different costs) can only be found numerically. We conduct some numerical experiments that indicate that the effect of different costs on the equilibrium is small, at least in our example model. There are currently no numerical procedures for computing SFE for more realistic settings of different generator capacities, locations and multivariate demand shocks. This remains a tantalizing area for future research.

The paper is laid out as follows. In section 2, we show how a tax on producer surplus gives rise to a best-response problem for which the objective is a convex combination of the best-response objective functions under uniform and pay-as-bid pricing. We then derive a symmetric equilibrium when such a tax is imposed on two agents at the upstream end of a constrained transmission line. Section 3 compares the response to the tax under SFE with a competitive model where all generators act as price takers. Section 4 deals with the setting when the tax is calculated based on the difference between the actual and counterfactual dispatch from the expansion of the transmission line. Section 5 briefly addresses the interaction between cost asymmetries and the benefit-tax. Section 6 describes an example where the demand shock has a small variance and no pure strategy SFE exists. The final section of the paper makes some concluding remarks.

2. Supply-function equilibrium

We assume an electricity pool market with locational marginal pricing. Suppliers submit offer curves of quantity and price to an independent system operator who clears the market by choosing prices at each node to minimize the cost of meeting demand (according to the offer prices), while satisfying transmission capacity constraints. Suppliers are then paid according to some function (the pricing rule) of their offer curve and the local price.

Formally we suppose that each supplier chooses a non-decreasing piecewise differentiable curve $\mathcal{S} = \{(x(t), y(t)), t \in [0, T]\}$ to maximize a functional of the form

$$\Pi(\mathcal{S}) = \int_0^T \left(f(x(t), y(t)) \frac{dx}{dt} + g(x(t), y(t)) \frac{dy}{dt} \right) dt. \tag{1}$$

We assume initially that f and g are continuous and piecewise differentiable functions on a compact region $\Psi \subset \mathbb{R}^2_+$. Proposition 4 will extend the analysis to a case with discontinuous (but piecewise continuously differentiable) f and g. Without loss of generality we set $t = 0$ so that $x(0) = y(0) = 0$, and we set $T = \sup\{t \mid (x(t), y(t)) \in \Psi\}$.

In the electricity market setting, where $\mathcal{S}$ represents a supply curve of quantity and price, $f(q,p)$ and $g(q,p)$ represent the contribution to marginal profit of small increments in quantity and price offered at the point (q,p). Thus $f(q,p)$ and $g(q,p)$ can model different profit expressions arising from either price-taking or price-setting behavior, as well as from different pricing rules. For example, uniform pricing, discriminatory (pay-as-bid) pricing, and taxed producer surplus models all have different objective functionals. Thus the expression (1) generalizes the objective function of Klemperer and Meyer (1989) which assumes uniform pricing and profit equal to revenue minus operating cost.

The first-order optimality condition for a profit-maximizing curve, given competitors' offers, gives a system of ordinary differential equations. In the symmetric case this yields a single ordinary differential equation that we can solve to find the equilibrium supply functions.

Following Anderson and Philpott (2002) we define

$$Z(q,p) = \frac{\partial g}{\partial q} - \frac{\partial f}{\partial p}, \tag{2}$$

$$t_0 = \inf\{t \mid x(t) > 0\}, \tag{3}$$

and let

$$w(\tau) = \int_0^\tau Z(x(t), y(t)) \left(\frac{dx}{dt} + \frac{dy}{dt}\right) dt. \tag{4}$$

The following optimality conditions are the same as those derived in Anderson and Philpott (2002). Suppose

$$I = \left\{t \mid \frac{dx}{dt} > 0, \quad \frac{dy}{dt} > 0\right\}.$$

Theorem 1 **(Necessary optimality conditions for $t \in I$)**. *If $\mathcal{S}^* = \{(x(t), y(t)), t \in [0,T]\}$ is a local maximum of $\Pi(\mathcal{S})$, then $Z(x(t), y(t)) = 0$ for all $t \in I$. Moreover if $Z(q,p)$ is differentiable at $(q,p) = (x(t), y(t))$ for $t \in I$, then we have $\frac{\partial Z}{\partial p}(x(t), y(t)) \geq 0$, and $\frac{\partial Z}{\partial q}(x(t), y(t)) \leq 0$.*

Proof The proof is the same as that in Anderson and Philpott (2002) with an amended definition of Z. □

Theorem 2 **(Necessary optimality conditions for $t \notin I$)**. *Suppose $\mathcal{S}^* = \{(x(t), y(t)), 0 \leq t \leq T\}$ is a non-decreasing piecewise differentiable curve, and let t_0 and w be defined by (3) and (4). The following conditions are necessary for $\mathcal{S}^*$ to be a local maximum of $\Pi(\mathcal{S})$:*

1. *$w(t_0) = w(T)$;*
2. *for every $t \in [0, t_0]$, $w(t) \geq w(t_0)$; and*
3. *for every $t \in [t_0, T]$,*

$$\frac{dx}{dt}(w(t) - w(t_0)) \leq 0 \leq \frac{dy}{dt}(w(t) - w(t_0)).$$

Proof The proof is the same as that in Anderson and Philpott (2002) with an amended definition of Z. □

Theorem 3 **(Sufficient optimality conditions)**. *Let $\mathcal{S}^* = \{(x(t), y(t)), t \in [0,T]\}$ be a continuous piecewise differentiable curve. If both components of $\mathcal{S}^*$ are nondecreasing in t, then a sufficient condition for $\mathcal{S}^*$ to be a global maximum of $\Pi(\mathcal{S})$ is that $Z = 0$ along $\mathcal{S}^*$, and that at every t, $Z(q, y(t)) \geq 0$ for all $q < x(t)$ and $Z(q, y(t)) \leq 0$ for all $q > x(t)$.*

Proof See Appendix. □

Theorems 1, 2, and 3 apply to regions in which $f(q,p)$ and $g(q,p)$ are continuously differentiable functions, so Z is continuous. In what follows we will require optimality conditions for $f(q,p)$ and $g(q,p)$ that are continuously differentiable except possibly at values of (q,p) that satisfy $q+S(p)=J$, where J is an arc capacity and $S(p)$ is a nondecreasing function.

Corollary 1. *Suppose f and g are piecewise continuously differentiable functions on Ψ that are continuously differentiable on $\hat{\Psi}=\{(q,p)\in\Psi : q\neq\hat{q},\ p\in(p_1,p_2)\}$ and $\mathcal{S}^*=\{(x(t),y(t)),\ t\in[0,T]\}$ is a non-decreasing piecewise differentiable curve. Let $Z(q,p)$ be defined on $\hat{\Psi}$ by (2) and let*

$$w^-(\tau)=\int_{t_0}^{\tau}\lim_{\epsilon\downarrow 0}Z(x(t)-\epsilon,y(t))\left(\frac{dx}{dt}+\frac{dy}{dt}\right)dt,$$
$$w^+(\tau)=\int_{\tau}^{T}\lim_{\epsilon\downarrow 0}Z(x(t)+\epsilon,y(t))\left(\frac{dx}{dt}+\frac{dy}{dt}\right)dt.$$

If $\mathcal{S}^*$ is a local maximum of $\Pi(\mathcal{S})$ and $\{(\hat{q},y(t)),\ p_1=y(t_1)\le y(t)\le y(t_2)=p_2\}\subseteq\mathcal{S}^*$, then $w^+(\tau)\le 0$ and $w^-(\tau)\ge 0$ for every $\tau\in(t_1,t_2)$.

Proof The proof follows similarly to the arguments in Anderson and Philpott (2002). On a vertical section of $\mathcal{S}^*$, if $w^-(\hat{\tau})<0$ for some $\hat{\tau}\in(t_1,t_2)$, then a perturbation to the left of the vertical section for $t<\hat{\tau}$ will yield an improvement in $\Pi(\mathcal{S})$. Similarly, if $w^+(\hat{\tau})>0$ for some $\hat{\tau}\in(t_1,t_2)$, then a perturbation to the right of the vertical section for $t>\hat{\tau}$ will yield an improvement in $\Pi(\mathcal{S})$. □

We apply the above theorems to both uniform-price and discriminatory-price auctions, as well as auctions where the market operator taxes a portion of producer surplus. Figure 1 shows the producer surplus and profit under a particular realization of the demand shock. It is the shaded area above the supply curve and below the clearing price. When demand shock ε is realized, there is a specific residual demand curve $RD(\varepsilon)$ faced by a producer. Each producer will be dispatched at the price and quantity where their supply curve $\mathcal{S}$ intersects their residual demand curve.

Uniform-price auction. In a uniform-price auction, the payoff of a firm when dispatched quantity θ at price π is

$$\theta\pi-C(\theta),$$

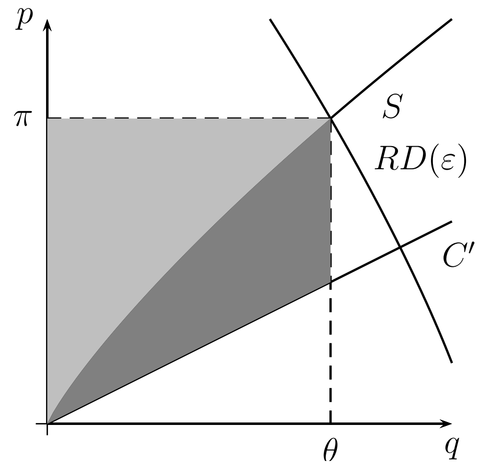

Figure 1 Market clearing under demand realization ε. The lighter shaded area is the observed producer surplus $\sigma(\varepsilon)$, and the darker shaded area is the profit under discriminatory pricing. These sum to $P(\varepsilon)$ the untaxed profit under demand realization ε.

where $C(q)$ is the firm's cost to produce quantity q. We denote the marginal cost by $C'(q)$ and its derivative by $C''(q)$. The expected payoff to a firm offering a curve $\mathcal{S} = \{(x(t), y(t)), t \in [0, T]\}$ is

$$\begin{aligned}\Pi^U(\mathcal{S}) &= \int_{\mathcal{S}} (qp - C(q))\, d\psi(q,p) \\ &= \int_0^T (x(t)\, y(t) - C(x(t))) \left(\frac{dx}{dt}\psi_q + \frac{dy}{dt}\psi_p\right) dt,\end{aligned} \tag{5}$$

where $\psi(q,p)$ is the market distribution function (see Anderson and Philpott 2002), which gives the probability that a supplier is not fully dispatched if they offer the quantity q at price p. The market distribution function can be interpreted as the measure of residual demand curves that pass below and to the left of the point (q,p). The partial derivatives ψ_q and ψ_p give the marginal change in dispatch probability from changing the offer quantity and price. The integrand in Π^U is clearly linear in $\frac{dx}{dt}$ and $\frac{dy}{dt}$, so we can compute the Z function as in (2):

$$\begin{aligned}Z^U(q,p) &= \frac{\partial}{\partial q}\Big((qp - C(q))\,\psi_p\Big) - \frac{\partial}{\partial p}\Big((qp - C(q))\,\psi_q\Big) \\ &= (p - C'(q))\,\psi_p - q\psi_q,\end{aligned}$$

as the terms containing the second-order partial derivative ψ_{qp} cancel. This is the same as the Z function derived in Anderson and Philpott (2002).

Discriminatory-price auction. In a discriminatory-price (pay-as-bid) auction (Anderson et al. 2013), the payoff when a firm is dispatched $\theta(\varepsilon)$ at price $\pi(\varepsilon)$ under demand realization ε is

$$\int_0^{t(\varepsilon)} y(t)\,\frac{dx}{dt}\,dt - C(\theta(\varepsilon)) \tag{6}$$

where $t(\varepsilon)$ satisfies $(x(t(\varepsilon)), y(t(\varepsilon))) = (\theta(\varepsilon), \pi(\varepsilon))$. This is the dark shaded area in Figure 1. The expected payoff is

$$\Pi^D(\mathcal{S}) = \int_0^T (y - C'(x(t)))(1 - \psi(x(t), y(t)))\,\frac{dx}{dt}\,dt.$$

Again this is linear in $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and Theorems 1-3 hold with

$$Z^D(q,p) = (p - C'(q))\,\psi_p - (1 - \psi(q,p)).$$

Apart from a sign change, this is the same as the Z function derived in Anderson et al. (2013).

Tax on producer surplus. Suppose that there is uniform pricing, but some fraction $\alpha \in [0,1]$ of a generator's observed producer surplus is paid as tax. Such a tax is unlikely to be imposed by a real regulator, but we will later see that it enables us to model a proposed beneficiaries-pay transmission pricing scheme that will be analyzed in section 4.

The pretax producer profit under demand realization ε is, as for uniform-pricing,

$$P(\varepsilon) = \pi(\varepsilon)\,\theta(\varepsilon) - C(\theta(\varepsilon)),$$

shown as the entire shaded region in Figure 1. The expected untaxed profit $\Pi^U = \mathbb{E}_\varepsilon[P(\varepsilon)]$.

The system operator is not revealed the producer's true marginal cost and observes only the surplus above the offered supply curve. The observed surplus under demand realization ε is

$$\sigma(\varepsilon) = \pi(\varepsilon)\,\theta(\varepsilon) - \int_0^{t(\varepsilon)} y(t)\,\frac{dx}{dt}\,dt. \tag{7}$$

This is the lightly shaded area in Figure 1.

Under demand realization ε the producer profit after paying a tax at rate α is

$$\begin{aligned} P(\varepsilon) - \alpha\sigma(\varepsilon) &= \pi(\varepsilon)\,\theta(\varepsilon) - C(\theta(\varepsilon)) - \alpha\pi(\varepsilon)\,\theta(\varepsilon) + \alpha\int_0^{t(\varepsilon)} y(t)\,\frac{dx}{dt}\,dt \\ &= (1-\alpha)(\pi(\varepsilon)\,\theta(\varepsilon) - C(\theta(\varepsilon))) \\ &\quad + \alpha\left(\int_0^{t(\varepsilon)} y(t)\,\frac{dx}{dt}\,dt - C(\theta(\varepsilon))\right). \end{aligned}$$

This is a convex combination of $P(\varepsilon)$ and the profit in demand realization ε under pay-as-bid pricing (6).

The producer profit functional Π^A is the expectation of $P(\varepsilon)-\alpha\sigma(\varepsilon)$ over the probability distribution of ε. This will therefore be a convex combination of the expectation Π^U of $P(\varepsilon)$, and the expectation Π^D from a pay-as-bid pricing payoff. Thus

$$\Pi^A(\mathcal{S}) = (1-\alpha)\,\Pi^U(\mathcal{S}) + \alpha\Pi^D(\mathcal{S}).$$

We can apply Theorem 1 to obtain the optimality conditions for the problem faced by a generator maximizing Π^A. These use the scalar field defined by $Z^A(q,p) = (1-\alpha)\,Z^U(q,p) + \alpha Z^D(q,p)$. Thus

$$\begin{aligned} Z^A(q,p) &= (1-\alpha)\left((p-C'(q))\,\psi_p - q\psi_q\right) + \alpha\left((p-C'(q))\,\psi_p - (1-\psi(q,p))\right) \\ &= (p-C'(q))\,\psi_p - (1-\alpha)\,q\psi_q - \alpha\,(1-\psi(q,p)) = 0 \end{aligned} \tag{8}$$

along any optimal supply curve.

2.1. Necessary conditions for SFE

A set of supply curves forms a supply-function equilibrium if each curve optimizes the expected payoff with respect to the market distribution function induced by the other players' offer curves, the transmission constraints, and the demand distribution. The following Theorem provides necessary conditions for a pure-strategy SFE.

Theorem 4 **(Necessary conditions for SFE)**. *Suppose that there is a market where each firm maximizes payoff functional Π^A with tax rate $\alpha \in [0,1]$. Suppose that all firms are located in the same node and have continuous marginal cost curves, that demand is perfectly inelastic in price, and that a price cap is imposed. If a set of curves is a supply-function equilibrium in pure strategies, then – on every interval where every curve has a positive probability of dispatch and in which no firm is at the upper or lower bound of their production quantity – we have:*

1. *no units are offered below marginal cost;*
2. *there are no perfectly elastic segments in the supply functions at any price above marginal cost;*

3. *there are no perfectly inelastic segments in the industry supply curve; and*
4. *no production capacity is withheld in equilibrium.*

Proof See appendix. □

The main consequence of Theorem 4 is that all curves in a pure-strategy symmetric SFE will be continuously differentiable over the range of possible dispatch. There may however be many sets of curves that satisfy the necessary conditions for SFE.

Klemperer and Meyer (1989) showed that under uniform pricing and symmetric firms, the only SFE that exist in the neighborhood of zero output are symmetric. For this reason we restrict our attention to symmetric SFE.

For symmetric SFE, the system of differential equations defined by $Z = 0$ for each producer reduces to a first-order ODE. The general solution to this ODE is parameterized by a single constant of integration, which we can take to be the amount offered at the price cap. We follow Newbery (1998) and focus on the least competitive symmetric SFE. As we have inelastic demand, the least competitive SFE is that in which the total offer reaches the maximum possible demand level at the price cap.

2.2. Example 1

We illustrate the above theorem with a simple example with two symmetric agents, located at node 1 of the two-node network shown in Figure 2. Here there is a price-taking, random and price-inelastic demand ε at node 2. The line connecting the two nodes has capacity K, which is less than the maximum demand at node 2. There is no demand at node 1.

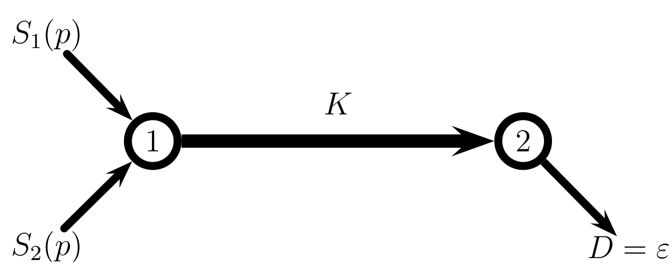

Figure 2 Simple transmission network with two producers and one demand shock. Line has capacity K.

The example we have chosen will be used throughout this paper to illustrate the results we obtain. It has some specific features that enable a symmetric equilibrium to be found. In particular, there are only two nodes, the demand shock is located at node 2 only, and there is no strategic generation at this node. In a more general network setting, the existence of pure-strategy SFE becomes problematic, as discussed in (Holmberg and Philpott 2012).

Suppose there is a proportional tax α imposed on the observed surplus of each agent. We apply the optimality conditions of the previous section to show the existence of a symmetric SFE $S_1(p) = S_2(p) = S(p)$.

PROPOSITION 1 **(Existence of symmetric SFE)**. *Assume the conditions of Theorem 4. Suppose that all firms have the same cost function and are located in node 1, and that at node 2 there is a random demand shock with distribution F and density f. Further suppose that f is bounded away from zero on $[0,K]$, that $F(K)<1$ and that all n firms have production capacity greater than $\frac{K}{n}$. Then the solution $q = S(p)$ to the ODE*

$$(p - C'(S))(n-1) f(nS)S' - (1-\alpha) f(nS)S - \alpha(1 - F(nS)) = 0, \tag{9}$$

subject to the boundary condition

$$S(\overline{p}) = \frac{K}{n}, \tag{10}$$

is a symmetric SFE if and only if the second-order condition

$$-C''(q)(n-1)S'(p) - (1-\alpha) - \frac{\alpha}{n}\frac{\partial}{\partial q}\left[\frac{1-F(nq)}{f(nq)}\right]_{q=S(p)} \leq 0$$

holds at every point along the curve.

Proof Consider player 1's offer curve maximization problem. Suppose the other players' offers aggregate to a total supply function $S_2(p)$. If player 1 offers quantity q at price p, then the market distribution function is the probability that either the total quantity offered $q + S_2(p)$ exceeds the line capacity K or that the combined offers of the two firms $q + S_2(p)$ at price p exceeds the demand shock ε. Thus

$$\psi(q,p) = \Pr\Big(q + S_2(p) > \min(K,\varepsilon)\Big).$$

If the demand shock has cumulative distribution function F and density function f, then we obtain the following piecewise definition for firm 1's market distribution function:

$$\psi(q,p) = \begin{cases} F(q + S_2(p)) & \text{if } q < K - S_2(p) \\ 1 & \text{if } q \geq K - S_2(p). \end{cases}$$

The partial derivatives are $\psi_q = f(q + S_2(p))$ and $\psi_p = f(q + S_2(p))\, S_2'(p) = \psi_q\, S_2'(p)$ when $q \leq K - S_2(p)$ and both are zero otherwise. There is a jump in the value of $\psi(q,p)$ on the curve $q = K - S_2(p)$.

Substituting the market distribution function into (8) yields

$$Z^A(q,p) = (p - C'(q))\, f\big(q + S_2(p)\big)\, S_2'(p) - (1-\alpha)\, q f(q + S_2(p)) - \alpha\Big(1 - F\big(q + S_2(p)\big)\Big) \tag{11}$$

for $q \leq K - S_2(p)$. At points with nonzero probability of dispatch we can divide (11) through by $f(q + S_2(p))$ to obtain

$$\hat{Z}^A(q,p) = (p - C'(q))\, S_2'(p) - (1-\alpha)\, q - \alpha \frac{1 - F\big(q + S_2(p)\big)}{f\big(q + S_2(p)\big)}.$$

By Theorem 1, the curve defined implicitly by the ODE

$$\hat{Z}^A(S(p), p) = 0$$

is a profit-maximizing response if it is non-decreasing and $\frac{\partial}{\partial q}\hat{Z}^A(S(p),p) \leq 0$ for all p, i.e.

$$-C''(q)\, S_2'(p) - (1-\alpha) - \alpha \frac{\partial}{\partial q}\left[\frac{1 - F\big(q + S_2(p)\big)}{f\big(q + S_2(p)\big)}\right]_{q=S(p)} \leq 0. \tag{12}$$

By the assumption of a symmetric equilibrium, the other players' responses are all the same as player 1's. Hence we can take $S_2(p) = (n-1)S(p)$ in (11) and (12). $\square$

The term $G(\varepsilon) = \frac{1-F(\varepsilon)}{f(\varepsilon)}$ is the inverse hazard rate of the distribution. In Holmberg's (2009a) model for pure pay-as-bid pricing, $\alpha = 1$ and marginal costs are constant, so it is necessary that $G' \geq 0$ for (12) to hold. This restricts the analysis to probability distributions that decay faster than the exponential distribution, which has $G' = 0$. If the tax rate α is less than 1, then we are less restricted in our choice of probability distribution for the demand shock. For instance, as shown in the example below, if $\alpha < \frac{1}{2}$, then (12) holds for a uniform distribution.

We now choose some specific problem data to compute an equilibrium for the example. Suppose that the demand shock is uniformly distributed on $[0,\overline{\varepsilon}]$ and $\alpha < \frac{1}{2}$. Assume that the line capacity K is infinitesimally smaller than $\overline{\varepsilon}$. Suppose that each agent has quadratic cost $C(q) = \gamma q^2$, so marginal costs for each agent are the same and are linear. The ODE for the symmetric SFE (9) becomes

$$S'(p) = \frac{(1-3\alpha)S}{p-2\gamma S} + \frac{\alpha\overline{\varepsilon}}{p-2\gamma S} \tag{13}$$

If $\gamma = 0$, then (13) is a first order linear ODE which can be solved using an integrating factor to give, for $\alpha \neq \frac{1}{3}$

$$S^A(p) = kp^{1-3\alpha} - \frac{\alpha\overline{\varepsilon}}{1-3\alpha}, \tag{14}$$

where k is a constant of integration that can be chosen to satisfy the endpoint condition $S^A(\overline{p}) = \frac{K}{2} \approx \frac{\overline{\varepsilon}}{2}$ (for the solution to the special case $\alpha = \frac{1}{3}$, see the appendix). It is straightforward to check that the second-order condition (12) holds.

We can compute the changes in welfare of each agent in equilibrium as the tax is applied. Suppose $K = 1$ and $\overline{\varepsilon} = 1$, that there is a price cap at $\overline{p} = 1$ and constant marginal costs of $\gamma = 0$. Consider first the case where $\alpha = 0$. In perfect competition each generator would offer at price equal to marginal cost and earn no profit. The least competitive SFE, however, is linear with $S^U(p) = \frac{p}{2}$. Since there are two firms, the total supply is $2S^U(p) = p$, and as the market clears when supply

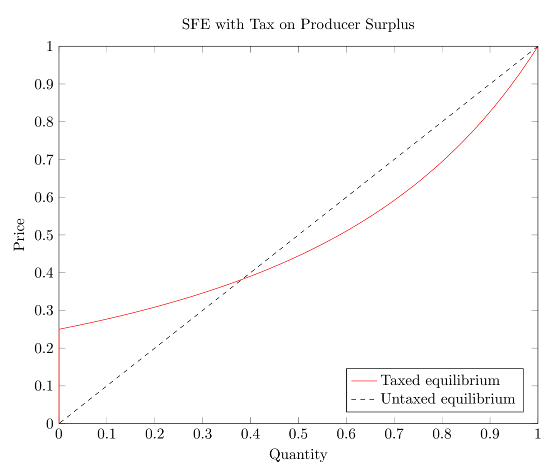

Figure 3 Equilibrium industry supply curves for no tax (dashed) and a 25% tax on observed surplus (solid).

equals demand $2S^U(p) = D(p, \varepsilon) = \varepsilon$, we can write the market price as a function of demand as $p(\varepsilon) = \varepsilon$. The expected consumer surplus (assuming all consumers value electricity at $\overline{p}$) is

$$\begin{aligned} \mathrm{CS} &= \int_0^{\overline{\varepsilon}} \varepsilon\left(\overline{p} - p(\varepsilon)\right) f(\varepsilon)\, d\varepsilon \\ &= 0.1666. \end{aligned}$$

The expected producer revenue is the firm's payoff $\Pi^U = 0.1666$. Their expected observed surplus however, is 0.0833. If a tax is applied to this curve, the firms each pay α of their observed surplus, so if $\alpha = \frac{1}{4}$, then each firm pays 0.0208 in tax, leaving net profit of 0.1458.

Now consider the SFE under the tax with rate $\alpha = \frac{1}{4}$. As shown in Figure 3, the tax gives an incentive for firms to change the shape of their offer curve from the dashed to the solid curve. The new equilibrium curve S^A has equation

$$S^A(p) = \frac{3}{2} p^{\frac{1}{4}} - 1$$

with inverse

$$p(q) = \frac{16}{81}(q+1)^4.$$

It is simple to verify that this is non-decreasing and solves $Z^A = 0$.

At this new equilibrium, the expected consumer surplus is 0.1737; the expected producer profit, before tax, is 0.1632; and the producer surplus observed by the market operator is 0.066. Each producer pays taxes of 0.0165 and so earns 0.1467 net profit.

Table 1 summarizes the changes in producer and consumer surplus between the untaxed and taxed equilibria. The overall effect of the tax in this case is a small transfer of welfare from producers to consumers. Though higher prices are charged at times of low demand, this is offset by lower prices higher up the offer curves. Note that social surplus (the sum of consumer and producer surpluses $\mathrm{CS} + 2\Pi^U$) does not change with the introduction of the tax; this is because demand is inelastic. Also note that expected consumer surplus actually rises once firms adjust to the tax, since the new equilibrium SFE is more competitive for the higher demand realizations. Equivalently, the volume-weighted average price is lower in the taxed equilibrium than in the equilibrium without the tax.

Curve	α	CS	Π^U	Π^A	Tax per firm	Social Surplus	Average Price
S^U	0.25	0.1666	0.1666	0.1458	0.0208	0.5	0.6666
S^A	0.25	0.1737	0.1632	0.1467	0.0165	0.5	0.6137

Table 1 Benefits and taxes under a producer-surplus tax.

In this model where all supply and demand in node 2 is inelastic, there are no congestion rents accruing to the system operator because there is never a price differential between the nodes. All load that cannot be satisfied through the transmission line is lost, and when this happens the market power of the suppliers in node 1 lets them charge the price cap. If there were elastic demand or supply in the downstream node, then there would be positive congestion rent.

3. Modeling all firms as price takers

It is instructive to model the effects of a producer-surplus tax under conditions of perfect competition, i.e. where all generators act as price takers. This can serve as a benchmark to measure the effects of market power on the incentives to adjust supply curves to under the tax.

When players are price takers, the only market information to which they respond is the price. It is as though they face horizontal residual demand curves at fixed prices (that are perfectly elastic). Hence the market distribution function to which the firms react depends only on price.

Let $\Phi(p)$ be the cumulative distribution on prices observed by a given producer (i.e. the market distribution function for a price-taking agent) and $\phi(p) = \Phi'(p)$ its density. Each producer has convex cost function $C(q)$ and seeks to maximize

$$\Pi^P = \int_0^{\overline{p}} (pq - C(q))\,\phi(p) - \alpha q\,(1 - \Phi(p))\;dp$$

Under the parameterization $x(p) = q$, $y(p) = p$ with the substitution $p = t$, we can apply Theorem 1 to obtain the first-order condition

$$Z^P = (p - C'(q))\,\phi(p) - \alpha\,(1 - \Phi(p)) = 0. \tag{15}$$

The parameter α takes us from a uniform-price auction to pay-as-bid as it goes from 0 to 1. The model we construct here generalizes that of Federico and Rahman (2003) who consider the extreme cases of uniform ($\alpha = 0$) and pay-as-bid ($\alpha = 1$) pricing.

Since

$$\frac{\partial}{\partial q} Z^P = -C''(q)\phi(p) \leq 0$$

everywhere, all solutions to $Z^P(S(p), p) = 0$ are maximal curves. The distribution on prices is determined by the system operator. Suppose that the market distribution function arises from a one-dimensional demand shock ε with density f and cumulative distribution function F. With n symmetric producers, the system operator dispatches from the aggregate supply curve by setting $nS(p) = \varepsilon$. This yields a correspondence between ε and the clearing price, so

$$\Phi(p) = F(\varepsilon) = F\left(nS(p)\right). \tag{16}$$

The density is then obtained by the chain rule,

$$\phi(p) = \Phi'(p) = f\left(nS(p)\right) nS'(p). \tag{17}$$

We substitute (16) and (17) into (15) to obtain the ODE

$$\left(p - C'(S)\right) f(nS)\, nS' - \alpha\left(1 - F\left(nS\right)\right) = 0. \tag{18}$$

This represents a fixed point, similar to the rational expectations equilibrium of Lucas and Prescott (1971). The producers choose curves to maximize their profits given the price distribution ϕ. Simultaneously, the system operator dispatches by choosing prices so that supply and demand intersect in each demand realization. In equilibrium, the forecast distribution of prices matches the actual distribution of prices.

Proposition 2. *The industry supply curve in a symmetric price-taking equilibrium depends only on the industry cost function and the distribution of the demand shock.*

Proof Suppose there are n symmetric producers in the market, each with cost function $C(q)$. Denote the industry cost function by

$$C_I(nq) = nC(q).$$

If we make the change of variable $Q = nq$ in the cost function we see that

$$\frac{d}{dQ} C_I(Q) = \frac{dq}{dQ} \frac{d}{dq} nC(q) = \frac{d}{dq} C(q).$$

When the system operator dispatches to get $Q = \varepsilon$, we obtain $\Phi(p) = F(Q(p))$ and $\phi(p) = f(Q(p))\,Q'(p)$. Making all the substitutions into (15) gives the ODE

$$(p - C_I'(Q))\, f(Q)\, Q' - \alpha\,(1 - F(Q)) = 0 \tag{19}$$

for the industry supply curve $Q(p)$. This ODE depends on $C_I(Q)$ and $F(\varepsilon)$, and not on n. □

We obtain a boundary condition for (18) by showing that the price bid for the last unit must equal its marginal cost. This is shown in the following proposition.

PROPOSITION 3. *If the system marginal cost $C_I'(Q)$ is continuous and there is more production capacity than the highest demand realization, then the most expensive unit dispatched in a price-taking equilibrium is offered at marginal cost.*

Proof Let the quantity dispatched in the highest demand realization be $\overline{Q}$ and p^* be the price at which it is offered. Then all units offered at a price above p^* will have zero probability of dispatch. Suppose with a view to contradiction that $p^* > C_I'(\overline{Q})$. Since $C_I'(Q)$ is continuous and there is more production capacity than the highest demand realization, there is some small increment of production whose marginal cost is less than p^* that is offered at a price above p^* (or not offered to the market at all), but whose probability of dispatch is zero. If this increment of production is instead offered at a price between its marginal cost and p^*, then it can obtain a positive markup and a positive probability of dispatch. Because producers are price takers, they behave as though probability of dispatch of their other units is unaffected by this change in offer. This deviation is improving for the owner of that increment, and therefore we cannot have $p^* > C_I'(\overline{Q})$ in equilibrium.

□

3.1. Example 2

To give a non-trivial equilibrium we assume quadratic costs $C(q) = \gamma q^2$ for firms in a duopoly, so that the industry cost function is $C_I(Q) = \frac{\gamma}{2} Q^2$.

Under our uniform demand shock distribution $\varepsilon \sim U[0, \overline{\varepsilon}]$, the equilibrium condition (19) on the industry supply function $Q(p)$ becomes

$$Q' = \alpha \frac{\overline{\varepsilon} - Q}{p - \gamma Q}. \tag{20}$$

The solution to this ODE corresponding to the boundary condition $Q(\bar{\varepsilon}\gamma) = \bar{\varepsilon}$, which we will refer to as the price-taking equilibrium (PTE), happens to be the linear function

$$Q^P(p) = \frac{1+\alpha}{\gamma}p - \alpha\bar{\varepsilon}.$$

If we take the same parameters as in the price-making SFE example above ($\alpha = 0.25$, $K = \bar{\varepsilon} = 1$), and assume linear marginal costs with coefficient $\gamma = \frac{1}{2}$, then we can compare outcomes. Figure 4 shows the solution to (20) alongside the industry supply curve for strategic supply-function equilibrium, which is the solution to (13) in section 2. Note that under the tax, price takers mark up above marginal cost at all quantities except the highest.

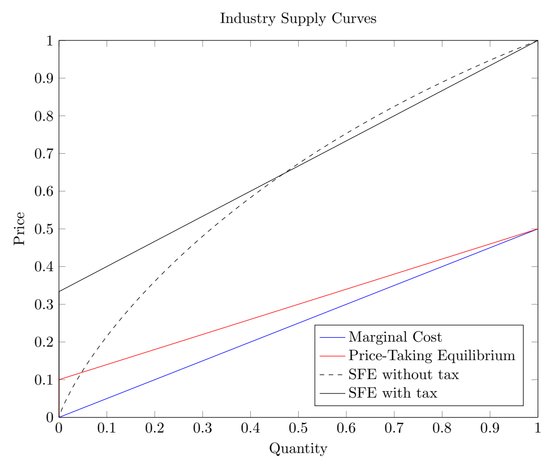

Figure 4 Equilibrium with price-taking producers, and strategic producers competing in supply functions

In Table 2 we summarize the effects of the tax on welfare in the price-taking equilibrium setting. In the absence of the tax, price-taking producers will offer Q^U at marginal cost. We see that the tax causes average prices to rise, reducing consumer welfare. For the producers, pre-tax profit (Π^U) increases so that the net profit under the tax is equal to the profit earned before the tax was imposed (0.0833). Hence the tax on producer surplus has been entirely passed through to consumers.

4. Line capacity expansion

We now consider a model in which the transmission line is expanded from capacity J to capacity K, and a proportional tax on observed benefits is levied to recover the costs of the line expansion. The model is again a simple two-node network as in Figure 2, with symmetric players at one node, and an inelastic demand shock ε at the downstream end of the line.

Curve	α	CS	Π^U	Π^A	Tax	Social Surplus	Average Price
Q^U	0.25	0.3333	0.0833	0.0624	0.0209	0.4166	0.25
Q^P	0.25	0.3166	0.1000	0.0833	0.0167	0.4166	0.30

Table 2 Benefits and taxes under a tax producer surplus, with price-taking producers.

The motivation for the model is a proposal for a new transmission pricing scheme to cover large grid investments in New Zealand. The New Zealand wholesale electricity market is dispatched according to a combined energy and reserve co-optimization in real-time with bids covering half-hour trading periods Alvey et al. (1998). A range of transmission pricing schemes are under consideration by the New Zealand Electricity Authority (2012) that include various combinations of:

- locational 'postage-stamp' charges;
- peak charges based on maximum historical injection or withdrawal; and
- a tax on benefits (surplus), calculated in each trading period.

The benefits-tax scheme differs from other transmission pricing methods promoted as beneficiary-pays in that the benefits are calculated as part of the dispatch, based on actual bids to the market. In fact the only cost information used by the regulator comes from the submitted bid function. Beneficiaries-pay transmission cost recovery schemes in New York see (Hogan 2011) and Argentina (Pollitt 2008) apply charges as locational 'postage-stamp' fees based on an ex-ante analysis of benefits arising from network expansion. Locational charges, once the rate has been announced, function as an additional cost of production and so their effect on the spot market is just a uniform markup across all levels of output. We shall see that a tax on differences in observed surpluses leads to a less simple adjustment of equilibrium bids.

Presently, in the New Zealand wholesale market, suppliers submit bid stacks (that we model as curves). In the delivery period, demand and intermittent supply are realized. The system operator solves the dispatch problem to satisfy demand at least cost based on the bids submitted and subject to transmission capacity constraints. The solution gives price and production levels for all generators. The beneficiaries-pays scheme proposed as the 'SPD method' by the Authority (New Zealand Electricity Authority 2014) makes an adjustment to payments at dispatch by solving a counterfactual dispatch problem. The dispatch is solved a second time, with transmission lines derated to their pre-expansion capacity. The difference in producer (or consumer) surplus between the two dispatches is calculated and market participants are charged a portion of this difference. In our model, the portion α of observed benefits to be charged is declared in advance. Observe that consumers are also charged, but we assume their bids in the spot market are inelastic so they have no way of strategically responding to the charge in the short term.

4.1. Payoffs under beneficiaries-pay tax

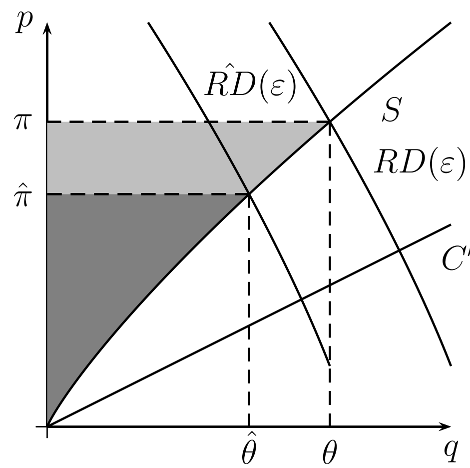

Figure 5 Difference in producer surplus between actual (θ) and counterfactual ($\hat{\theta}$) dispatch is shaded

The benefit that a producer is deemed to derive from a network expansion is the difference between producer surplus in dispatch with the actual configuration and producer surplus in dispatch with the counterfactual network configuration. In Figure 5, the dispatch point depends on the demand realization ε and the network configuration, so the two network configurations give two dispatch points, an actual (θ, π) and a counterfactual $(\hat{\theta}, \hat{\pi})$. These in turn give rise to two distinct realizations of producer surplus $\sigma(\varepsilon)$ and $\hat{\sigma}(\varepsilon)$, defined by (7).

The firm pays a portion α of the difference between the observed surplus in the actual network and observed surplus in the counterfactual network; i.e. the tax is

$$\alpha\left(\sigma(\varepsilon) - \hat{\sigma}(\varepsilon)\right).$$

This gives profit net of tax of

$$R(\varepsilon) = P(\varepsilon) - \alpha(\sigma(\varepsilon) - \hat{\sigma}(\varepsilon)).$$

The generator constructs an offer curve to maximize this tax-adjusted profit. Since $R(\varepsilon)$ is the linear combination of three terms, we can express the expectation as a linear combination of the individual expectations. The expectations of $P(\varepsilon)$ and $\sigma(\varepsilon)$ are evaluated against the market distribution function ψ assuming a full line capacity, whereas $\hat{\sigma}(\varepsilon)$ is evaluated using the counterfactual market distribution function $\hat{\psi}$ assuming the unexpanded capacity. The expected profit over the entire supply curve is

$$\begin{aligned}
\Pi^L(\mathcal{S}) &= \mathbb{E}[P] - \alpha(\mathbb{E}[\sigma] - \mathbb{E}[\hat{\sigma}]) \\
&= \int_0^T (xy - C(x)) \left(\frac{dy}{dt}\psi_p + \frac{dx}{dt}\psi_q\right) dt - \alpha\left(\int_0^T x\Big(1 - \psi(x,y)\Big)\frac{dy}{dt}\,dt - \int_0^T x\Big(1 - \hat{\psi}(x,y)\Big)\frac{dy}{dt}\,dt\right) \\
&= \int_0^T \left((xy - C(x)) \left(\frac{dy}{dt}\psi_p + \frac{dx}{dt}\psi_q\right) - \alpha q\left(\hat{\psi}(x,y) - \psi(x,y)\right)\frac{dy}{dt}\right) dt.
\end{aligned}$$

Observe that this can be written in the form of (1) with:

$$f(q,p) = (pq - C(q))\,\psi_q, \quad \text{and} \tag{21}$$

$$g(q,p) = (pq - C(q))\,\psi_p - \alpha q\left(\hat{\psi}(q,p) - \psi(q,p)\right). \tag{22}$$

Applying Theorem 1, the resulting Z function is

$$Z^L(q,p) = (p - C'(q))\,\psi_p - q\psi_q - \alpha\left(q\left(\hat{\psi}_q - \psi_q\right) + \hat{\psi} - \psi\right). \tag{23}$$

4.2. Example 3

We now return to the network of section 2. As the demand shock occurs only in the downstream node, $\left(\hat{\psi} - \psi\right)$ is nonzero only when

$$J < q + S_2(p) \le K,$$

in which case $\hat{\psi} = 1$. Hence $Z^L(q,p)$ can be thought of as piecewise defined: equal to $Z^U(q,p)$ when $\hat{\psi} - \psi = 0$ and $Z^A(q,p)$ otherwise.

We assume that marginal costs are zero for both firms, that the price cap is 1 and that the lower bound on the uniformly distributed demand shock is zero.

As $Z^L(q,p)$ is discontinuous along the line $J-q-S_2(p)=0$, we can no longer rely on Theorems 3 and 4 directly. The following propositions give sufficient conditions for a supply function to maximize the functional $\Pi^L(\mathcal{S})$, and rule out equilibria with vertical or horizontal segments at the counterfactual line capacity.

PROPOSITION 4 **(Optimal bidding with discontinuous payoff)**. *Suppose f and g are as in equations (21) and (22) with a jump discontinuity on a curve defined by $q+S_2(p)=J$. Let $\mathcal{S}^*=\{(x(t),y(t)),\ t\in[0,T]\}$ be a continuous, piecewise differentiable, and strictly increasing curve that crosses $q+S_2(p)=J$ exactly once, at $t=T_1$. The following conditions are sufficient for $\mathcal{S}^*$ to maximize $\Pi^L(\mathcal{S})$:*

1. *$Z^L(x(t),y(t))=0$ at all points along $\mathcal{S}^*$ except possibly at $t=T_1$; and*
2. *at every t, $Z^L(q,y(t))\geq 0$ for all $q<x(t)$ and $Z^L(q,y(t))\leq 0$ for all $q>x(t)$.*

Proof The proof is by a simple decomposition argument. Consider the problem of choosing separate maximal curves for (1) over $[0,T_1]$ and $[T_1,T]$ with the boundary conditions (upper and lower respectively) $x(T_1)+S_2(y(T_1))=J$. Conditions 1 and 2 are, by Theorem 3, sufficient for optimality in these subproblems.

To pass from separately maximizing over the two subintervals to globally maximizing over all of $[0,T]$, we add the constraint that the curve be continuous at T_1. As our candidate solution satisfies the additional constraint, it is still optimal. $\square$

PROPOSITION 5. *A supply function $\mathcal{S}$ with a vertical segment at $q=\frac{J}{2}$ cannot be a symmetric equilibrium for firms maximizing $\Pi^L(\mathcal{S})$.*

Proof Suppose each supplier offers a curve containing the segment $\{(\frac{J}{2},p): p_1\leq p\leq p_2\}$. Recalling Corollary 1, it is easy to show that $Z^L=-q<0$ in the region $\{(q,p): 0<q<\frac{J}{2},\ p_1<p<p_2\}$, so $w^-(t)<0$ for any t between the endpoints of the vertical section. This violates the local optimality conditions in Corollary 1. $\square$

PROPOSITION 6. *A supply function $\mathcal{S}$ with a horizontal segment at a price above marginal cost cannot be a symmetric equilibrium for firms maximizing* $\Pi^L(\mathcal{S})$.

Proof If the horizontal segment is entirely within the Z^U or the Z^A region, then Theorem 4 can be applied directly, since it is a local condition that is true for any $\alpha \in [0,1]$. On the other hand, suppose the horizontal segment crosses the line $q = \frac{J}{2}$; in this case, the horizontal segment must lie on the boundary between the Z^U and Z^A regions. In the counterfactual dispatch, the line of capacity J can either be congested or not. When the line is congested in the counterfactual, a producer offering an infinitesimally lower price over the horizontal segment has an additional effect on its after-tax profit. Not only does the producer's dispatch probability increase by a finite amount, its dispatch in the counterfactual also increases. This decreases the observed benefits, thereby increasing the after-tax profit; thus this remains a profitable deviation. However, when there is no congestion in the counterfactual, there are no observed benefits, and therefore no additional effects from the taxation. Nevertheless, the first-order effect of undercutting remains, and is profitable. $\square$

Propositions 5 and 6 imply that a symmetric SFE must have quantity as a continuous and increasing function of price. Thus we can find an equilibrium by solving the ODE $Z^L(S(p), p) = 0$ with the boundary condition $S(\overline{p}) = \frac{K}{2}$, as in Proposition 1.

The first-order condition for a symmetric SFE, analogous to (9), becomes

$$pS'\frac{1}{\overline{\varepsilon}} - S\frac{1}{\overline{\varepsilon}} = 0 \text{ for } S < \frac{J}{2} \tag{24}$$

$$pS'\frac{1}{\overline{\varepsilon}} - (1-\alpha)\,S\frac{1}{\overline{\varepsilon}} - \alpha\left(1 - \frac{2S}{\overline{\varepsilon}}\right) = 0 \text{ for } S \geq \frac{J}{2}. \tag{25}$$

We assume zero marginal costs, so (24) and (25) can both be solved by the same method as (13) above. Equation (25) has general solution

$$S(p) = k_1 p^{1-3\alpha} - \frac{\alpha\overline{\varepsilon}}{1-3\alpha}, \tag{26}$$

with k_1 a constant of integration. To pass through the price cap we require

$$k_1 = \left(\frac{K}{2} + \frac{\alpha\overline{\varepsilon}}{1-3\alpha}\right)\overline{p}^{3\alpha-1}.$$

Equation (24) is an ODE for $S(p) < \frac{J}{2}$ with general solution

$$S(p) = k_2 p,$$

where k_2 is another constant of integration. For continuity of the curve, we choose $k_2 = \frac{J}{2p^*}$, where p^* solves $2S(p) = J$ in (26). Our equilibrium candidate is thus

$$S^L(p) = \begin{cases} \left(\frac{K}{2} + \frac{\alpha\bar{\varepsilon}}{1-3\alpha}\right)\left(\frac{p}{\bar{p}}\right)^{1-3\alpha} - \frac{\alpha\bar{\varepsilon}}{1-3\alpha} & \text{if } p \geq p^* \\ \frac{J}{2}\frac{p}{p^*} & \text{if } p < p^*. \end{cases}$$

4.2.1. Example Consider an equilibrium with the following choice of parameters:

$$\begin{aligned} \bar{\varepsilon} &= 1 && \text{maximum shock;} \\ K &= 0.8 && \text{expanded line capacity;} \\ J &= 0.2 && \text{original line capacity;} \\ \alpha &= \tfrac{1}{4} && \text{tax rate.} \end{aligned} \tag{27}$$

We can plot the values of Z^L, as defined in (23), to see that condition 2 of Proposition 4 is indeed satisfied for the equilibrium. In Figure 6, we see that $Z_q < 0$ everywhere and that $Z = 0$ along the supply curve. Note that the discontinuity in the integrand of the objective occurs when $J - q - S_2(p) = 0$, and that the supply curve intersects this decreasing curve once, at the kink where $q = \frac{J}{2}$.

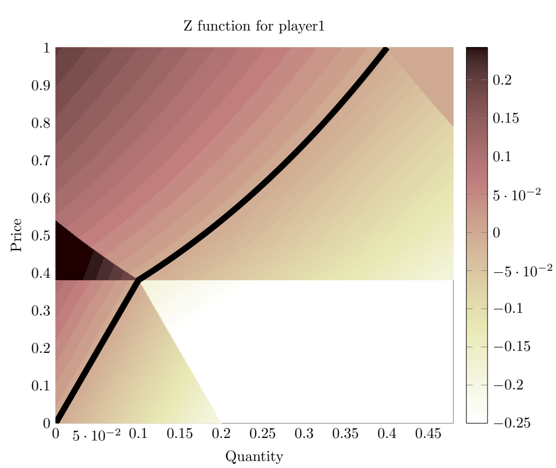

Figure 6 Plot of contours of $Z^L(q,p)$ when competitor is playing the $\alpha = \frac{1}{4}$ curve from Figure 7

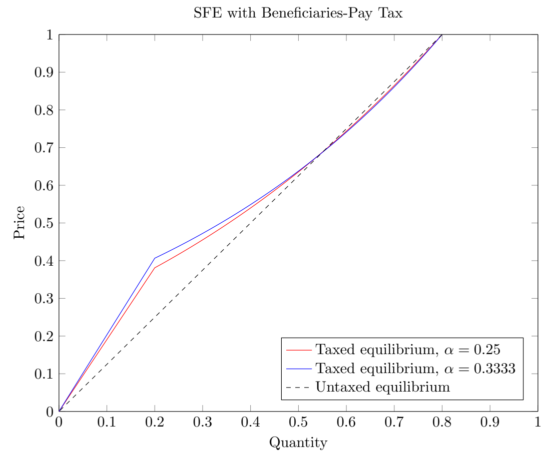

Figure 7 Plot of untaxed equilibrium industry supply curve (dashed) and taxed equilibrium industry supply curve (solid) when maximum demand is 1 and $\alpha = \frac{1}{4}$ (red) and $\alpha = \frac{1}{3}$ (blue).

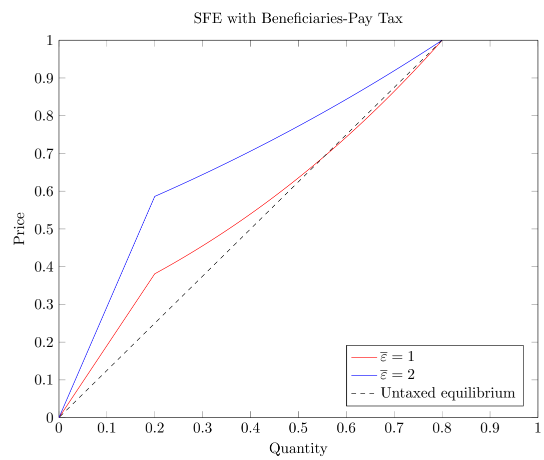

Figure 8 Plot of untaxed equilibrium offer (dashed) and taxed equilibrium offer (solid) when the maximum demand is 1 (red) and 2 (blue)

The industry supply curves in supply-function equilibria for two different choices of α are plotted in Figure 7. The degree to which the taxed equilibrium is marked up above the untaxed equilibrium depends on the range of the demand shock. If the range of the demand shock is large, then there is a high probability that the expanded line will be congested, which increases the probability of an observed benefit. This means that the equilibrium offers try to avoid taxation by flattening the offer curves in regimes where the unexpanded line would be congested. This can be observed in Figure 8. As the probability of lost load increases, small quantities are marked up more, since their contribution to the tax paid rises relative to the profits earned when they are at the margin.

When we look at what choice of market parameters leads to prices being marked down at the highest output levels, we find it depends only on the number of players and the hazard rate of the demand shock distribution at the point where the transmission constraint binds. Proposition 7 demonstrates this.

PROPOSITION 7. *If $\frac{K}{n} > \frac{1-F(K)}{f(K)}$, then prices at the highest demand outcomes will be lower under the taxed SFE than under the untaxed SFE. Moreover, if the demand shock is distributed so that $\frac{1-F(\varepsilon)}{f(\varepsilon)}$ is everywhere non-increasing and $\frac{K}{n} < \frac{1-F(K)}{f(K)}$, then prices will be higher at all demand outcomes under the taxed SFE.*

Proof In (9), the slope of the symmetric SFE at the price cap is

$$S'_\alpha\left(p, \frac{K}{n}\right) = \frac{1}{\left(\overline{p} - C'(\frac{K}{n})\right)(n-1)}\left((1-\alpha)\frac{K}{n} + \alpha\frac{1-F(K)}{f(K)}\right).$$

Without the tax, $\alpha = 0$, the slope at the price cap is

$$S'_0\left(p, \frac{K}{n}\right) = \frac{1}{\left(\overline{p} - C'(\frac{K}{n})\right)(n-1)}\frac{K}{n}.$$

If $\frac{K}{n} > \frac{1-F(K)}{f(K)}$, then $(1-\alpha)\frac{K}{n} + \alpha\frac{1-F(K)}{f(K)} < \frac{K}{n}$, so $S'_\alpha\left(\frac{K}{n}\right) < S'_0\left(\frac{K}{n}\right)$, yielding the first part of the result.

If the demand shock is distributed so that $\frac{1-F(\varepsilon)}{f(\varepsilon)}$ is everywhere non-increasing and $\frac{K}{n} < \frac{1-F(K)}{f(K)}$, then for $q \leq \frac{K}{n}$, we have $\frac{1-F(nq)}{f(nq)} \geq \frac{1-F(K)}{f(K)} > \frac{K}{n} \geq q$. Hence $S'_\alpha(p,q) - S'_0(p,q) = \alpha\left(\frac{1-F(nq)}{f(nq)} - q\right) > 0$, yielding the second part of the result. $\square$

Recall from section 2 that $\frac{1-F(\varepsilon)}{f(\varepsilon)}$ is the inverse hazard rate of the distribution. Proposition 7 states that prices will be marked up at all outputs by strategic players under a tax if the probability of demand exceeding the expanded line capacity is large enough and the inverse hazard rate is non-increasing. Any probability distribution with a density that decays less quickly than the exponential distribution will have a non-increasing inverse hazard rate; in particular, the uniform and normal distributions possess this property. This condition is the opposite of the condition in Holmberg (2009a) that is necessary for pure strategy SFE in pay-as-bid auctions.

When $J \approx K$, it can happen that the marking-down at the highest output levels is carried across the entire symmetric SFE. This is due to the continuity condition in Proposition 4. In Figure 9 are shown equilibria for different values of J. Observe that for small increases in line capacity (from $J = 0.6$ to $K = 0.8$) the blue curve is under the dashed curve at every point, so prices are (slightly) lower in every demand outcome.

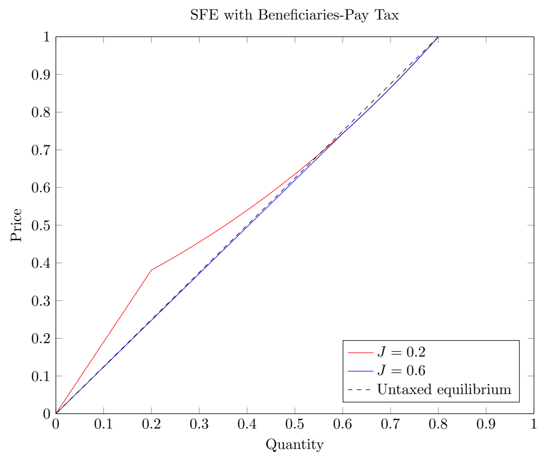

Figure 9 Plot of untaxed equilibrium offer (dashed) and taxed equilibrium offer (solid) when $J = 0.2$ (red) and $J = 0.4$ (blue)

4.2.2. Welfare calculations We may calculate consumer and producer welfare for different levels of tax. Taking the base level parameters (27) and repeating the analysis of section 2, we obtain the values of Table 3. Again, the total surplus does not change.

Curve	α	CS	Π^U	Π^A	Tax per firm	Social Surplus	Average Price
S^U	0.25	0.1067	0.1067	0.0861	0.0206	0.32	0.6866
S^L	0.25	0.1003	0.1098	0.0921	0.0170	0.32	0.6666

Table 3 Benefits and taxes under a tax on line-expansion benefits.

We see a slight decrease in consumer surplus as the very slight discounting at the top of the offer curve is not sufficient to offset the heavy markups around $q = \frac{J}{2}$. For small expansions in line capacity this effect diminishes.

We can measure the change in consumer surplus, profits and tax collected as the magnitude of the line expansion varies. We will analyze the consequences of varying the pre-expansion line capacity J to illustrate the effects of the size of the expansion on strategic behavior.

We keep K constant at 0.8 and vary J from 0 to K, to cover a range of scenarios, from a completely new line to a zero increase in line capacity. In this variation the system operator chooses J, the baseline network capacity. The change in welfare after the tax is imposed depends on the size of the counterfactual line J, as well as the probability of line congestion in the actual system $1-\frac{K}{\bar{\varepsilon}}$. The plots for a low probability of line congestion ($\bar{\varepsilon}=1$, giving 20% probability) are shown in figure 10. Solid curves represent welfare under equilibria where producers take the tax into account and dashed curves measure the same thing for equilibria where agents ignore the tax. Note that when $J \approx K = 0.8$, the mark-down effect dominates so that there is actually a reduction in price levels in the post-tax SFE, leading to a slight gain in consumer surplus and slight reductions in producer profits and transmission charges collected, compared to the equilibrium when no tax is charged.

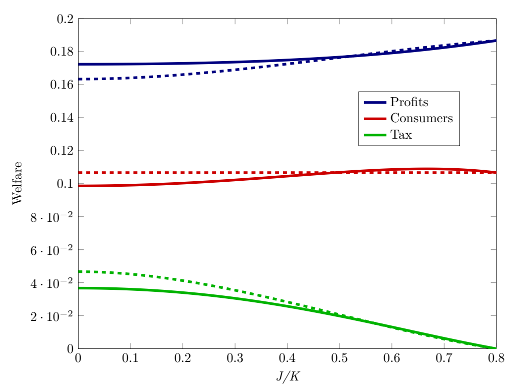

Figure 10 Welfare of distribution as J varies, with (solid) and without (dashed) strategic reaction to tax. Probability of line congestion is 0.2.

4.3. Price-taking players

We can apply the piecewise solution method to a duopoly of price-taking producers. Because the market distribution function for price takers has no quantity component, the dispatch price is not necessarily continuous with respect to the demand shock. For outputs in node 1 less than J, there is no tax applied and so all units are offered at marginal cost. At J, the price jumps up and follows a solution to (20) up to the point where the last unit is offered at marginal cost.

Figure 11 shows the price-taking equilibrium for the base parameters (27), with the corresponding strategic SFE. As in section 3, we assume marginal cost functions $C_i'(q_i) = q_i$, for an industry marginal cost function $C_I'(Q) = \frac{1}{2}Q$. We see that price-taking producers will mark up their offers at all output levels where benefits are observed.

Note that since the highest clearing price achieved in node 1 is 0.4, there will be congestion rents accruing to the system operator whenever demand exceeds the expanded line capacity $K = 0.8$. Under the untaxed price-taking equilibrium, when all supply is offered at marginal cost, the volume-weighted average price paid to producers is 0.24 and the average price paid by load is 0.36. Under the taxed price-taking equilibrium, these prices rise to 0.27 for producers and 0.39 for load. The expected congestion rent is 0.12 in both cases.

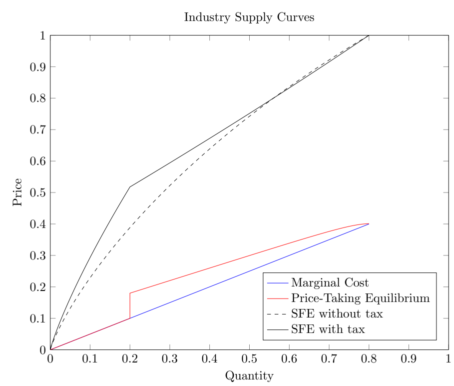

Figure 11 Industry supply curves under benefits-tax on line expansion, comparing strategic SFE and price-taking equilibrium.

5. Asymmetric Players

All of the examples we have used have had symmetric players competing in supply functions. It is reasonable to ask whether the same outcomes occur when the players are not symmetric.

We compare SFE in duopoly with two pairs of linear marginal cost functions, chosen so that industry cost function is invariant. One is symmetric

$$C_1'(q_1) = q_1 \quad C_2'(q_2) = q_2$$

and the other is asymmetric

$$C_1'(q_1) = \frac{3q_1}{4} \quad C_2'(q_2) = \frac{3q_2}{2}.$$

The demand shock and line expansion parameters are at the base levels (27) of example 3 above.

As in the symmetric case there is a one-parameter family of SFE that are monotone and that have zero mark-up at zero output (Holmberg 2009b). We use a spline collocation method similar to Anderson and Hu (2008) to solve the system of ODEs given by the first-order optimality conditions for the least competitive SFE, in which the line capacity constraint binds at the price cap.

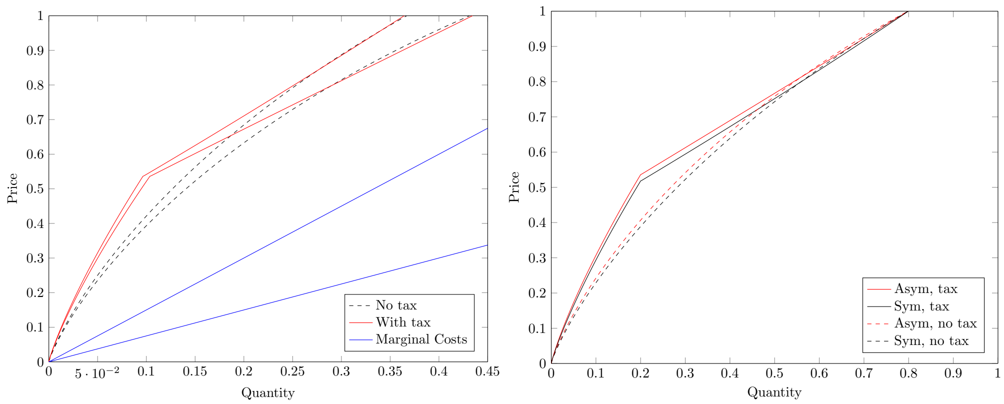

Figure 12 Equilibrium with producers competing in supply functions, having symmetric or asymmetric marginal cost functions. The asymmetric SFE is on the left and the industry supply curves for symmetric and asymmetric SFE are on the right.

Figure 12 shows the industry supply curves for SFE under these two divisions of marginal cost. We see that cost asymmetry makes the industry slightly less competitive. This is in line with other models of oligopoly. Furthermore, the magnitude of mark-up resulting from strategic reaction to the line-benefit tax is comparable. In the symmetric case the average price rises from 0.755 to 0.775 after the tax is imposed and in the asymmetric case the average price rises from 0.767 to 0.786. Therefore the strategic effects of a line-benefits tax are not greatly exacerbated by the asymmetry of producers in the market.

6. Normally distributed demand shock

Until now we have assumed the distribution of the demand shock to be uniform. In real-world markets, firms can have quite accurate information about likely demand levels; this can be represented in an SFE model by a concentrated distribution of the demand shock.

Figures 13-15 show the symmetric solutions to (8) for the network in Figure 2 when the demand shock ε is normally distributed. In Figure 13 the demand shock in node 2 has a mean of 0.3 and a

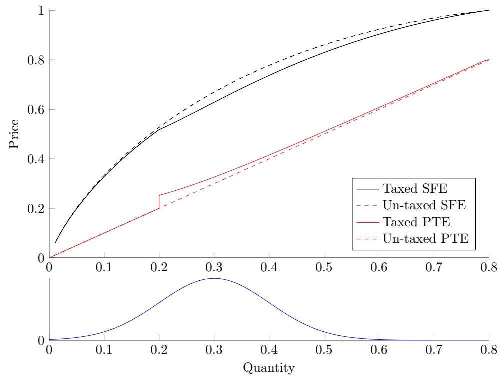

Figure 13 Equilibrium solutions for Example 1 with normally distributed $\varepsilon \sim N(0.3, 0.1)$.

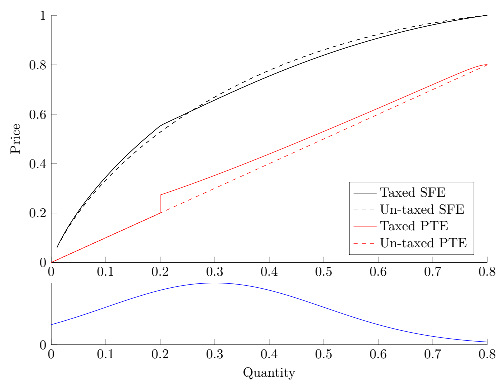

Figure 14 Equilibrium solutions for Example 1 with normally distributed $\varepsilon \sim N(0.3, 0.2)$.

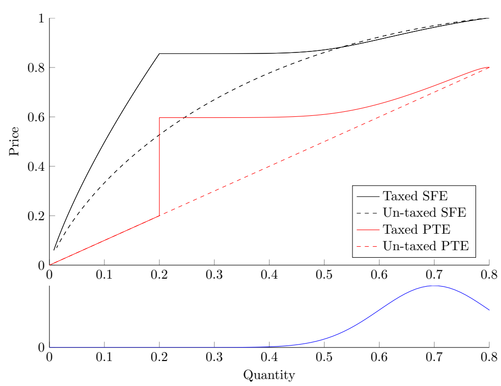

Figure 15 First-order solutions for Example 1 with normally distributed $\varepsilon \sim N(0.7, 0.1)$. The solid black curve is not an SFE.

standard deviation of 0.1. In Figure 14 the mean is the same but the distribution is more spread out with a standard deviation of 0.2. The marginal cost functions are as in section 3, with the transmission line expanding from $J = 0.2$ to $K = 0.8$ as in the previous section.

As the uniform-price SFE is independent of the demand-shock distribution, it is the same in all three cases. Comparing Figures 13 and 14, we see that markups in response to the tax are greater when there is greater variance in the demand shock. The strategic SFE candidates in both these cases satisfy Theorems 1 and 4 and Proposition 4.

Figure 16 shows the value of Z^L for the symmetric candidate solutions in Figures 13 and 15. In the left-hand plot we see that Z^L is positive everywhere to the left of the candidate curve and negative everywhere to the right. Therefore the curve is a symmetric SFE. In the right-hand plot we see that for offer quantities between 0.1 and 0.3, Z^L is negative to the left of the candidate solution and positive to the right. This violates the second-order optimality condition of Theorem 1; in this region the candidate is a local *minimum* of the profit functional. The same phenomenon has been observed in discriminatory-price SFE by Holmberg (2009a). Moreover, Anderson et al. (2013) find mixed strategy equilibria where firms mix over bids that are perfectly inelastic for small quantities and have finite slope for larger quantities.

Table 4 shows the change in welfare when the line-benefit tax is applied to the SFE in Figure 13. We see that the overall marking down of the offer curves is reflected in a lower average price after the tax than before.

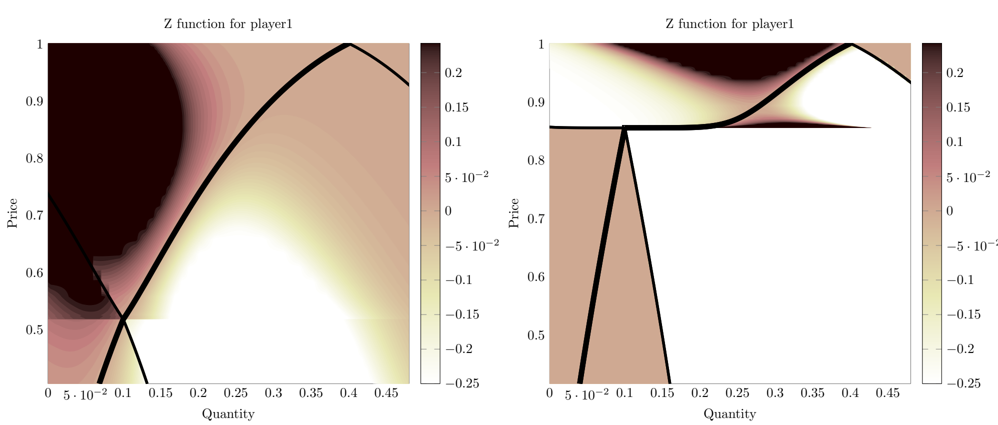

Figure 16 Plot of Z field for firm 1 in strategic SFE with normally distributed shock. On the left $\varepsilon \sim N(0.3, 0.1)$, and on the right $\varepsilon \sim N(0.7, 0.1)$.

Curve	α	CS	Π^U	Π^A	Tax per firm	Social Surplus	Average Price
S^U	0.25	0.0915	0.0791	0.0743	0.0049	0.250	0.6950
S^L	0.25	0.1019	0.0739	0.0697	0.0042	0.250	0.6600

Table 4 Benefits and taxes from SFE under a tax on line-expansion benefits. Normal shock with mean 0.3 and standard deviation 0.1

7. Conclusion

This work has examined the incentives of firms to adjust their offering strategies (in equilibrium) when a charge is applied as a percentage of either perceived profits (where the regulator believes that the firm offers at marginal cost), or observed benefits of an investment in transmission assets (e.g. a line capacity upgrade). In a deterministic setting one may think that there would be an incentive to conceal one's observed benefits by increasing the offers up to the dispatch point. However in a setting where the dispatch point is not known in advance (uncertain residual demand), we have shown that a balance must be struck between concealing the benefits and maximizing the (untaxed) profit. This new balance does not always exhibit higher mark-ups than the untaxed regime.

In regions of quantity-price space where the tax applies, producers optimize functionals that are a convex combination of uniform and pay-as-bid profit functionals. For a tax rate below a certain threshold a symmetric SFE exists that, compared to the equilibrium without the tax, has generally higher markups at low offer quantities but possibly smaller markups near the capacity constraint.

We discovered a counter-intuitive effect of the 'beneficiary-pays' charge in a duopoly setting. When the size of the line upgrade is small – and the probability of line-congestion is low – the consumer surplus can increase when the charge is applied, since firms submit offer curves that are strictly lower than the untaxed curves. Moreover, due to their competition, firms in fact receive a lower profit and actually pay more tax than they would if they had chosen the same supply curves that comprise the untaxed equilibrium.

There remain several obstacles to the use of our SFE model as a quantitative tool. To calculate the symmetric SFE we must solve a first-order non-linear ODE; however to calculate asymmetric SFE the ODE becomes a system with order equal to the number of asymmetric firms, which often can only be solved numerically. In some special cases (e.g. symmetric firms in radial networks) closed-form SFE can be found (Holmberg and Philpott 2012). However in real settings (like the New Zealand electricity market) agents typically offer at several locations, networks are meshed, and the nodal demand distributions are correlated. Any of these circumstances on its own are enough to make the existence of SFE problematic.

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Appendix

Proof of Theorem 3 By assumption, the field Z is continuous. We can apply Green's Theorem, as in Anderson and Philpott (2002), to obtain a sufficient condition for optimality. By Green's theorem, the integral of $\left(f\left(q,p\right)\frac{dq}{dt}+g\left(q,p\right)\frac{dp}{dt}\right)$ around any simple closed curve $(q(t),p(t))$ in the anticlockwise direction is equal to the integral of Z over the area enclosed by the curve. If $Z=0$ along $\mathcal{S}^*$, then there is an improving deviation to the left only if $Z<0$ somewhere in that region. Similarly, there is an improving deviation to the right only if $Z>0$ somewhere in that region. Thus the conditions guarantee that there are no improving deviations, therefore $\mathcal{S}^*$ is maximal. $\square$

Proof of Theorem 4 The case $\alpha=0$ was proved as a series of propositions in Holmberg (2008). Assume henceforth that $\alpha>0$. Using market distribution functions and Theorems 1 and 2, we can find improving deviations in wherever the conditions are not met. Theorem 1 allows us to show the existence of improving deviations without having to construct them explicitly as Holmberg does. Anderson (2013) addresses the case of a pool market with uniform pricing, where demand has positive elasticity and the firms have maximum output constraints that bind at the highest demand levels.

1. Suppose that at some point t_0, with nonzero probability of dispatch, some firm offers below marginal cost. This means that $y(t) < C'(x(t))$ on some interval about t_0. However, ψ_p and ψ_q are both non-negative, which implies that $Z^A \leq -\alpha(1-\psi(x,y))$ along some part of the curve. Thus the curve cannot be a local maximum unless $\psi(x(t),y(t)) = 1$, which contradicts the assumption of positive probability of dispatch.

2. Suppose that some firm offers a curve with a perfectly elastic segment above marginal cost. Then the residual demand curve of every other firm will have a perfectly elastic segment. Because we assume inelastic demand, the slopes of each firm's residual demand curve is determined only by the slopes of its competitors' supply curves, i.e. $\psi_p = \psi_q \sum_{j\neq i} S'_j$, for all players i. Consider two cases:

   (a) Suppose a firm offers a perfectly elastic segment at p^*, while its competitors are offering continuous curves. The competitors find themselves optimizing against a market distribution function with a jump at $p = p^*$. The price derivative ψ_p has a point mass while the other market distribution function terms in Z^A remain finite-valued. This gives Z^A a line of positive mass along $p = p^*$. Hence if at least one competitor is offering above marginal cost at this price, they can improve their payoff by making a small undercutting deviation.

   (b) At least two players offer perfectly elastic segments at the same price. Then, whatever the tie-break rule for dispatching units offered at the same price, at least one player will do better by offering the same tranche at an slightly lower price. This lower price offer is an improving deviation for that firm so the set of supply functions could not be an equilibrium.

In both cases, some supplier can obtain a finite gain in dispatch probability for an infinitesimal loss in revenue by undercutting by an infinitesimal amount.

3. Suppose that the industry supply curve is perfectly inelastic over some range of prices. As the supply functions making up this industry supply curve are all non-decreasing, they must all be perfectly inelastic over this interval. This implies that ψ_p is zero for every player, so at least one player will have $Z^A < 0$ over some interval and can marginally improve their profit by withholding a small amount.

4. Suppose some firm withholds some amount of production from the market. This is not profit-maximizing as, given the assumption that demand will exceed the capacity of supply with some probability, it could always gain some revenue from the unit by offering it at the price cap. Units offered at the price cap do not affect the revenue earned by other units lower down the supply schedule as they are never inframarginal and so never affect the producer surplus.

□

Solution to (13) **when** $\alpha = \frac{1}{3}$**:** Observe that the term $(3\alpha - 1)S(p)$ vanishes when $\alpha = \frac{1}{3}$. In that case the differential equation

$$pS'(p) + (3\alpha - 1)S(p) = \alpha\bar{\varepsilon}$$

becomes

$$S'(p) = \frac{\alpha\bar{\varepsilon}}{p},$$

so the symmetric equilibrium supply functions are

$$S(p) = \alpha\bar{\varepsilon}\log\frac{p}{\bar{p}} + \frac{K}{2}.$$