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Value of transmission capacity in electricity markets with risk averse agents

Corey Kok
Electric Power Optimization Centre, University of Auckland, Auckland, New Zealand, ckok006@aucklanduni.ac.nz

Andy Philpott
Electric Power Optimization Centre, University of Auckland, Auckland, New Zealand, a.philpott@auckland.ac.nz

Golbon Zakeri
Electric Power Optimization Centre, University of Auckland, Auckland, New Zealand, g.zakeri@auckland.ac.nz

We consider a model of risk-averse electricity capacity investment that includes electricity generators, retailers, industrial consumers, and an independent system operator as individual agents in a perfectly competitive game. These agents each simultaneously make their operational and contract decisions to maximise their individual risk-adjusted profit across the potential scenarios, given the other agents' decisions. We prove the existence of equilibrium for our model under some relatively nonrestrictive assumptions. The model is applied to investigate the value of increased transmission capacity. Even when the new transmission capacity is free, we show that increasing its capacity can lead to lower risk-adjusted system welfare when there are insufficient contract instruments for trading risk. In contrast, when a liquid market for contracts is included in the model, increases in transmission capacity is welfare enhancing and lead to better capacity investment decisions made by the generation agents.

Key words: electricity markets, risk, transmission expansion

History:

1. Introduction

Investment and divestment in generation capacity is a topical issue in electricity systems worldwide. There are many types of model that attempt to predict generation capacity investment, each using specific assumptions about how these decisions are made. The simplest models assume a social planning paradigm and treat capacity investment as an optimisation problem. The earliest examples of this are Booth (1972) and Mass and Gibrat (1957). More recent studies incorporate real-world constraints. For example, Sen et al. (1994) develop a two-stage model that integrates demand, capacity investment, and budget constraints. Economies of scale and fixed costs are modelled by adding integer variables to the first stage of a two-stage model (see Barahona et al. (2005), Eppen et al. (1989), Riis and Lodahl (2002), Riis and Andersen (2002)).

Another approach to capacity investment is to use agent-based models Bunn and Oliveira (2001), in which agents react independently to market signals. Thus a generator will expand generation plants if the revenue it is predicted to earn exceeds the long run cost of the plant. Agents within the model typically assume that no other generation plant will be constructed and that their generation plant will not impact spot market prices. These models can display “boom and bust” cycles of too much generation and insufficient generation, when agents simultaneously decide to build generation plants, significantly dropping spot market price, leading to an extended period without much investment at all.

More sophisticated approaches treat capacity investment as a non-cooperative game. The simplest of these assume agents are perfectly competitive (as in Murphy and Smeers (2005), Zttl (2010)), making both capacity and generation decisions simultaneously within a spot market. A second type of model assumes agents compete with their capacity decisions in an oligopolistic manner Murphy and Smeers (2005), simultaneously deciding to build capacity and selling long-term contracts for this generation. The most complicated models treat capacity investment as a bi-level problem (for example Chuang et al. (2001), Murphy and Smeers (2005), Zttl (2010)). Agents compete in a Cournot manner when choosing capacity investment decisions in the first stage, and in the second stage, agents act perfectly competitively given the capacity decisions.

In Wang et al. (2009) agents with incomplete information make decisions in a bi-level game. In the top level, the Independent System Operator (ISO) chooses prices that clear the market competitively. In the bottom level, agents select capacity investment decisions and submit capacity, generation, and reserve bids.

One of the first capacity investment equilibrium models using coherent risk measures is described in Ehrenmann and Smeers (2011). They consider each potential generation plant individually and compute the risk-adjusted profit that the plant individually earns using conditional value at risk (as defined by Rockafellar and Uryasev (2000)). Using a model with an energy-only market, and a capacity market, it is shown that modelling plant owners as risk neutral misses some critical structural differences between energy-only and capacity markets.

Uncertainty in spot market prices makes financial contracts a vital tool for market participants. They help balance extremes in profit for all involved agents and can help incentivise capacity decisions that are closer to socially optimal. Limited contract liquidity (by constraining contract trades within the model) is compared to unconstrained contract trading in de Maere dAertrycke and Smeers (2013). Different types of contracts are compared in de Maere dAertrycke et al. (2017) where it is shown that models without contracts give less efficient capacity decisions than those having them. This outcome is mirrored in our analysis.

We model a game where agents represent generation companies that may have a fleet of generation and make decisions of capacity investment and generation. There are also retailers that must meet all demand at a fixed price, or miss out on revenue and pay a penalty. We assume all agents have access to complete information about potential future outcomes and their probability distributions. These agents are also risk-averse and choose capacity investment to maximise their risk-adjusted profit.

The contributions of the paper are as follows. As in de Maere dAertrycke et al. (2017), we demonstrate the beneficial effect of contracts on the efficiency of competitive equilibrium for investment. Our complementarity models provide a computational tool for quantifying this effect. Using these we show by example how extra transmission capacity (even if is free) might decrease risk-adjusted welfare in the absence of contracts. These models illustrate the importance of modeling risk and competition in planning investments in electricity systems.

The paper is laid out as follows. In section 2 we describe the models in more detail and list the sets, parameters, and variables that we use in our models. We also formulate a risk-averse social plan. In section 3 we present the formulation of our equilibrium model and give a theorem that guarantees existence of an equilibrium. In section 4 we present results from an example where due to the generation agents being risk-averse, a free and lossless transmission line can lead to lower system welfare through inefficient capacity investment decisions caused by risk aversion. In section 5, we present results that show that allowing agents to trade financial contracts can be a useful tool to improve agent and system welfare, and can be sufficient to help ensure a transmission line is useful for the system as a whole. In section 6 we make conclusions.

2. Minimising risk-adjusted social cost

We consider an electricity transmission network with varying levels of demand at each node. We make decisions around capacity investment, generation, transmission, and demand curtailment to meet this demand at minimum social cost. To model increases in capacity, we use a combination of existing and potential new generation plants. Each generator (existing and potential) has a given location in the network at which they inject electricity. We assume that generation plants are built without delay. We aggregate electricity demand into types. Each demand type has a representative node, giving the location in the network at which this demand must be satisfied. Each demand type is either an industrial plant that purchases generation directly from the spot market or has their demand met by a retailer that promises to meet demand at an agreed-upon fixed rate. In both cases, demand is satisfied by purchasing directly from the spot market.

Transmission lines allow us to send power from where there is surplus cheap electricity to nodes of insufficient supply. Each transmission line has a nominal maximum capacity, beyond which the line begins to overheat. We approximate losses as increasing quadratically with the flow through the line. Loops may exist within the transmission network, so we must ensure that electricity dispatch satisfies Kirchhoff's laws for a DC-Load Flow approximation. We model the central planner (and later the generators, retailers, and transmission) as risk-averse. We compare a risk-averse social plan in this setting, which determines all of the agent's actions for them, to an equilibrium model. We assume that each agent (and the central planner) uses a risk measure ρ, that transforms random variables to real numbers. We assume that ρ is *coherent* as defined in Artzner et al. (1999). This means it has a dual representation ($\mathcal{M}$) (see Theorem 4.16 of Föllmer and Schied (2011)), whereby

$$\rho(\boldsymbol{Z}) = \max_{\mathbb{Q}\in\mathcal{M}}(\mathbb{E}_{\mathbb{Q}}[\boldsymbol{Z}]). \tag{1}$$

$\mathcal{M}$ is called the *risk set* of the coherent risk measure. We use this dual representation when proving the existence of equilibrium in Appendix A.

To conclude, we summarise the relevant sets, parameters, and variables that we use in formulation of the social plan as well as the equilibrium problem. We use the convention that calligraphic letters are sets, Roman type text denotes parameters, math-type text denotes variables and indices, and bold is used denote parameters and variables that are not fully indexed.

2.1. Set definitions

$\mathcal{A}$:= Set of agents that participate in generation, retail, or industrial demand or a combination of these three $(a \in \mathcal{A})$.

$\mathcal{I}$:= Set of nodes at which electricity can be injected or consumed $(i \in \mathcal{I})$.

$\mathcal{K}$:= Set of existing generation plants (that may be expanded) and possible new generation plants $(k \in \mathcal{K})$.

$\mathcal{H}$:= Set of demand types (e.g. industrial, retail) $(h \in \mathcal{H})$

$\mathcal{L}$:= Set of directed lines connecting the nodes $((i,j) \in \mathcal{L})$.

$\mathcal{E}$:= Set that indexes loops that exist in transmission network $(e \in \mathcal{E})$.

$\mathcal{L}_e$:= Set of arcs that exist in loop e $((i,j) \in \mathcal{L}_e)$

Ω := Set of (discrete) scenarios that may occur $(\omega \in \Omega)$

$\mathcal{B}$:= Set of load blocks to occur within each scenario $(b \in \mathcal{B})$

2.2. Agent variable definition

$Z^a(\omega)$:= The disbenefit each agent observes within each potential scenario.

$z^a_{i,k}$:= The capacity of each plant k (MW).

$x^a_{i,k}$:= The level of construction (or expansion if it already exists) that each plant k undergoes (MW).

$y^a_{i,k}(\omega, b)$:= The level of generation output by plant k at node i in scenario ω and load block b (MW).

$q^a_{i,h}(\omega, b)$:= The level of load curtailment that consumer h experiences at node i in scenario ω and load block b (MW).

2.3. Transmission agent variable definition

$L_i(\omega, b)$:= The transmission losses accumulated at node i.

$f_{i,j}(\omega, b)$:= The level of generation in MW sent from node i to node j.

2.4. Parameter definition

xC_k := Per Megawatt cost of constructing or expanding each generation plant.

oC_k := Per Megawatt cost of keeping each generation plant operational.

$\mathrm{gC}_k(\omega, b)$:= Marginal cost of electricity generation.

$\mathrm{k}^a_{i,k}$:= The existing capacity for each plant.

$\mathrm{u}^a_{i,k}$:= The upper bound of capacity investment for each plant.

$\mathrm{d}^a_{i,h}(\omega, b)$:= The quantity of demand of each type and owner observed in each scenario/load block.

$\mathrm{m}_k(\omega, b)$:= Multiplicative factor which modifies the available capacity from a generation plant depending on the scenario and load block.

$\mathrm{n}_k(\omega)$:= Multiplicative factor which determines the average generation that could be maintained over the load blocks without putting too much pressure on the power source or the plant itself.

r_h := The per unit revenue earned from retail consumers of each type.

v_h := The per unit cost of curtailing demand of each type.

$\mathrm{f}^+_{i,j}$:= The maximum transmission (in MW) from node i to node j.

$\mathrm{f}^-_{i,j}$:= The negative of the maximum transmission (in MW) from node j to node i.

$\mathrm{c}_{i,j}$:= Multiplier that is used to calculate the loss across a transmission line connecting nodes i and j.

$\mathrm{s}_{i,j,e}$:= The reactance of line $(i,j) \in \mathcal{L}$ with respect to loop e. This value is negative if $(i,j) \in \mathcal{L}$ but $(j,i) \in \mathcal{L}_e$.

$\mathrm{T}(\omega, b)$:= Time (in hours) spent in each load block given the scenario (known by all agents)

2.5. Problem formulation

We formulate a social plan that describes the problem of capacity investment, generation, and electricity curtailment to minimise the risk-adjusted social disbenefit, calculated using the social risk measure ρ.

The objective for the system is to minimise the risk-adjusted social cost $\rho(\sum_{a\in\mathcal{A}} \boldsymbol{Z}^a)$. The net disbenefit $Z^a(\omega)$ observed by each agent, a, in each scenario, ω, is defined by equation (2). This equation excludes any payments between agents on the spot market. As we calculate social welfare by first calculating the sum of each agent's welfare, these payments between agents cancel when calculating the risk-adjusted social

PROBLEM 1. Social disbenefit minimisation objectives and constraints

$$\mathcal{S}: \min_{\substack{\boldsymbol{z}\geq\mathbf{0},\\ \boldsymbol{x}\geq\mathbf{0},\\ \boldsymbol{y}\geq\mathbf{0},\\ \boldsymbol{q}\geq\mathbf{0},\\ \boldsymbol{f}.}} \quad \rho\left(\sum_{a\in\mathcal{A}} \boldsymbol{Z}^a\right)$$

$$\begin{aligned}
Z^a(\omega) &= \sum_{i\in\mathcal{I},k\in\mathcal{K}} \mathrm{xC}_k\cdot x^a_{i,k} + \sum_{i\in\mathcal{I},k\in\mathcal{K}} \mathrm{oC}_k\cdot z^a_{i,k}\\
&+ \sum_{b\in\mathcal{B}} \mathrm{T}(\omega,b)\cdot \sum_{i\in\mathcal{I},k\in\mathcal{K}} \mathrm{gC}_k(\omega,b)\cdot y^a_{i,k}(\omega,b)\\
&- \sum_{b\in\mathcal{B}} \mathrm{T}(\omega,b)\cdot \sum_{i\in\mathcal{I},h\in\mathcal{H}} \mathrm{r}_h\cdot(\mathrm{d}^a_{i,h}(\omega,b)-q^a_{i,h}(\omega,b))\\
&+ \sum_{b\in\mathcal{B}} \mathrm{T}(\omega,b)\cdot \sum_{i\in\mathcal{I},h\in\mathcal{H}} \mathrm{v}_h\cdot q^a_{i,h}(\omega,b) \qquad \forall\omega\in\Omega, \qquad (2)
\end{aligned}$$

s.t.

$$x^a_{i,k} \leq \mathrm{u}^a_{i,k} \qquad \forall a\in\mathcal{A}, i\in\mathcal{I}, k\in\mathcal{K}, \qquad (3)$$

$$z^a_{i,k} \leq x^a_{i,k} + \mathrm{k}^a_{i,k} \qquad \forall a\in\mathcal{A}, i\in\mathcal{I}, k\in\mathcal{K}, \qquad (4)$$

$$y^a_{i,k}(\omega,b) \leq \mathrm{m}_k(\omega,b)\cdot z^a_{i,k} \qquad \forall i\in\mathcal{I}, k\in\mathcal{K}, \omega\in\Omega, b\in\mathcal{B}, \qquad (5)$$

$$\textstyle\sum_{b\in\mathcal{B}} \mathrm{T}(\omega,b)\cdot y^a_{i,k}(\omega,b) \leq \mathrm{n}_k(\omega)\cdot z^a_{i,k} \qquad \forall a\in\mathcal{A}, i\in\mathcal{I}, k\in\mathcal{K}, \omega\in\Omega, \qquad (6)$$

$$\sum_{(i,j)\in\mathcal{L}_e} \mathrm{s}_{i,j,e}\cdot f_{i,j}(\omega,b) = 0 \qquad \forall e\in\mathcal{E}, \omega\in\Omega, b\in\mathcal{B}, \qquad (7)$$

$$q^a_{i,h}(\omega,b) \leq \mathrm{d}^a_{i,h}(\omega,b) \qquad \forall a\in\mathcal{A}, i\in\mathcal{I}, h\in\mathcal{H}, \omega\in\Omega, b\in\mathcal{B}, \qquad (8)$$

$$L_i(\omega,b) = \sum_{\substack{j:\\ ((i,j)\cup(j,i))\in\mathcal{L}}} \frac{\mathrm{c}_{i,j}}{2}\cdot(f_{i,j}(\omega,b))^2 \qquad \forall i\in\mathcal{I}, \omega\in\Omega, b\in\mathcal{B}, \qquad (9)$$

$$f_{i,j}(\omega,b) \leq \mathrm{f}^+_{i,j} \qquad \forall (i,j)\in\mathcal{L}, \qquad (10)$$

$$f_{i,j}(\omega,b) \geq \mathrm{f}^-_{i,j} \qquad \forall (i,j)\in\mathcal{L}, \qquad (11)$$

$$\begin{aligned}
\textstyle\sum_{a\in\mathcal{A},h\in\mathcal{H}} \mathrm{d}^a_{i,h}(\omega,b) &\leq \textstyle\sum_{a\in\mathcal{A},k\in\mathcal{K}} y^a_{i,k}(\omega,b)\\
&+ \textstyle\sum_{a\in\mathcal{A},h\in\mathcal{H}} q^a_{i,h}(\omega,b)\\
&+ \textstyle\sum_{\substack{j\in\mathcal{I}:\\ (j,i)\in\mathcal{L}}} f_{j,i}(\omega,b)\\
&- \textstyle\sum_{\substack{j\in\mathcal{I}:\\ (i,j)\in\mathcal{L}}} f_{i,j}(\omega,b)\\
&- L_i(\omega,b) \qquad \forall i\in\mathcal{I}, \omega\in\Omega, b\in\mathcal{B}. \qquad (12)
\end{aligned}$$

disbenefit. Excluding the spot market payments, this equation includes all sources of costs and revenue associated with participating on the electricity market as either a generator, retailer, or an industrial consumer (or some combination of the three).

In the first line of equation (2) we have physical capacity investment cost $\mathbf{xC} \cdot \boldsymbol{x}$, and the operation and maintenance cost $\mathbf{oC} \cdot \boldsymbol{z}$. Recall that $\boldsymbol{x}$ measures capacity expansion and $\boldsymbol{z}$ measures capacity after possible expansion of construction.

In the second term, we define the cost of generation $\mathbf{gC} \cdot \boldsymbol{y}$, with $\mathbf{gC}$ giving the marginal cost of each generation plant, and $\boldsymbol{y}$ the output of each generation plant. We then multiply this by the time $T(\omega, b)$ spent in each load block b, and sum across all load blocks to get the total cost of generation in each scenario.

In the third line, we subtract the revenue earned from meeting demand. The revenue earned per unit of met consumer demand is $\mathbf{r}$. Here $\mathbf{r}$ can also represent the per unit short-run profit from running an industrial plant like an aluminium smelter. The demand met is given by $\mathbf{d} - \boldsymbol{q}$. The exogenous demand of each consumer is given by $\mathbf{d}$, and $\boldsymbol{q}$ defines how much demand of each type is curtailed. The overall revenue earned is given by $\mathbf{r}(\mathbf{d} - \boldsymbol{q})$. We then multiply the retail revenue (or industrial short-run profit) earned in each load block by the time $T(\omega, b)$ spent in each load block b to calculate the overall revenue earned from demand in scenario ω.

In the fourth term of (2), we define the total cost of curtailing demand $\boldsymbol{q}$. This penalty is added to the lost revenue $\mathbf{r} \cdot \boldsymbol{q}$ earned from not meeting demand. The total amount of curtailed demand is given by $\boldsymbol{q}$. Thus, the overall penalty is $\mathbf{v} \cdot \boldsymbol{q}$. Again, we multiply this by the time $T(\omega, b)$ spent within load block, b and sum.

The different values of v_h and r_h yield a non increasing demand curve at each node i, which defines the total consumption at i as the price of energy increases. The minimum consumption zero will occur first at some price that does not exceed $\bar{\mathrm{P}} = \max_h(\mathrm{r}_h + \mathrm{v}_h)$.

In constraints (3) through (12) we define the physical generation constraints. Equation (3) limits the capacity investment, $\boldsymbol{x}$ of each generation plant to be at most $\mathbf{u}$. Equation (4) limits the capacity, $\boldsymbol{z}$ of each generation plant to the sum of expanded and existing capacity $\boldsymbol{x} + \mathbf{k}$, allowing for divestment (at no cost). Equation (5) limits the power output, $\boldsymbol{y}$, depending on the capacity, $\boldsymbol{z}$, and a multiplicative adjustment,

$\mathbf{m}$, that depends on both the scenario and load block. This models long-term uncertainty (e.g. the long-term average wind speed) and fluctuations (e.g. day-to-day changes in wind speed) that cause a generation plant to not be run at 100% capacity. In equation (6), we limit the total energy output across load blocks, depending on the capacity, $\boldsymbol{z}$, and a multiplicative adjustment, $\mathbf{n}$, that depends only on the scenario. For example, a hydroelectric plant depends on the inflows, which does not limit the power output, but does limit how long they will be able to maintain this level of generation.

In equations (9) through (11) we define the constraints on the transmission lines. We use a DC-Load Flow approximation of the active power flows in the transmission lines Downward (2011). This assumes the voltage magnitudes at all nodes are approximately equal, and their phase angle differences are small. Assuming that the resistance is much smaller than the reactance, the flow in a transmission line is proportional to the difference between the voltage angles at its endpoints.

$$f_{i,j} = \frac{\theta_i - \theta_j}{s_{i,j}}. \tag{13}$$

We can then rearrange (13) to give

$$\sum_{(i,j)\in\mathcal{L}_e} \mathrm{s}_{i,j,e} \cdot f_{i,j} = 0. \quad \forall e \in \mathcal{E}, \tag{14}$$

where the parameter $\mathrm{s}_{i,j,e}$ is negative if the orientation of an arc $(i,j) \in \mathcal{L}$ is in the opposite direction to the modelled loop direction, giving constraint (7).

In (9) we define the losses from transmission and assign them to nodes. Losses are assumed to be proportional to the square of the flow giving $\mathrm{c}_{i,j}(f_{i,j})^2$. For simplicity, we measure flow at the midway point of the line and assume that half of the losses are accumulated at the origin node, and half at the destination node. The total losses accumulated at a node i is then half of the losses that occur on connecting transmission lines, giving

$$\sum_{\substack{j:\\ ((i,j)\cup(j,i))\in\mathcal{L}}} \frac{\mathrm{c}_{i,j}}{2} \cdot (f_{i,j})^2.$$

Recalling that $f_{i,j}$ measures the flow at the midpoint of the transmission line, the constraint (10) imposes thermal limits on transmission. As the set $\mathcal{L}$ is directed, the constraint (11) gives the negative of the maximum transmission from node j to node i (with $(i,j) \in \mathcal{L}$) due to the thermal limits.

Finally, in equation (12), we define the constraint that ensures that for each scenario and load-block, that demand is met at all nodes. First, on the left-hand side of equation (12), we define the demand, $\mathbf{d}$. Allowing for free disposal of power, net supply of power to a node (generation plus curtailment of demand plus net transmission to a node minus losses) must be equal to or exceed demand.

3. Perfectly competitive risk-averse equilibrium

In this section, we formulate the equilibrium problem describing the interactions between the agents. Each gentailer chooses their capacity investment, generation, curtailment, and contract decisions to maximise their risk-adjusted profit. Simultaneously, the system operator chooses how to transmit power between nodes, and the spot market and contract markets are cleared.

3.1. Competitive equilibrium

We model agents competitively making decisions to minimise their individual risk-adjusted disbenefit. These agents participate as price takers within a perfectly competitive market, where prices are set by an auctioneer to ensure supply is equal to demand. We represent this auctioneer through complementarity conditions Facchinei and Pang (2007).

3.2. Contracts

An electricity derivative contract compensates for extremes in payoffs (both positive and negative) from an underlying position that generates value. For example, an electricity retailer may be concerned with potentially high wholesale market prices and wishes to be insured against this outcome.

We model the trading of derivative contracts by first introducing a set of financial instruments, $c \in \mathcal{C}$. These instruments are provided by a contract auctioneer who chooses the prices p_c of the instruments to clear the contract market. Each instrument c has a random return, $\mathbf{W}_c$. If an agent chooses to buy w_c units of each contract c then the risk-adjusted disbenefit for this agent is defined by

$$\rho(\boldsymbol{Z} + \sum_{c \in \mathcal{C}} w_c \cdot (p_c - \mathbf{W}_c)). \tag{15}$$

Thus, depending on $\boldsymbol{Z}$, $\boldsymbol{p}$, and $\boldsymbol{W}$, an agent will typically be able to reduce $\rho(\boldsymbol{Z})$ by acquiring a portfolio of contracts $\boldsymbol{w}$.

Specific examples of contracts are *Arrow-Debreu* securities and *contracts for differences*. Arrow-Debreu securities Arrow and Debreu (1954) have $\mathcal{C} \subseteq \Omega$, and have $W_c(\omega, \boldsymbol{\pi}(\omega)) = 1$ if $c = \omega$ and $W_c(\omega, \boldsymbol{\pi}(\omega)) = 0$ otherwise. If $\mathcal{C} = \Omega$, then we have a *complete* market. A contract for differences (CFD) is a financial derivative that uses the observed prices to determine its value. In this case, a contract at node i has $\mathrm{W}_c(\omega, \boldsymbol{\pi}(\omega)) = \sum_{b \in \mathcal{B}} \mathrm{T}(\omega, b)\pi_i(\omega, b)/(\sum_{b \in \mathcal{B}} \mathrm{T}(\omega, b))$. Continuing with the conventions used for our social cost minimisation model, we now define additional sets, variables and parameters to allow these agents to trade financial contracts with one another.

3.3. Additional notation

$\mathcal{C}$:= Set of financial contracts that are available for agents to buy or sell market $(c \in \mathcal{C})$.

$\Psi^a(\omega)$:= The disbenefit each agent observes within each potential scenario including net payoff from each contract.

w_c^a := The number (possibly negative) of contracts of type $c \in \mathcal{C}$ purchased by agent a.

$\pi_i(\omega, b)$:= The spot price for electricity per MWh at node i in each scenario ω and load block b.

p_c := Spot market price for contract c.

$\mathrm{W}_c(\omega, \boldsymbol{\pi}(\omega))$:= The revenue earned per unit of contract c purchased, in scenario ω. This may depend partially or completely on the observed spot market prices during a scenario.

In defining equilibrium, we adopt the mixed-complementarity notation of Ferris and Munson (2000). Given lower bounds $\boldsymbol{\ell} \in \{\mathbb{R} \cup \{-\infty\}\}^n$, upper bounds $\boldsymbol{u} \in \{\mathbb{R} \cup \{\infty\}\}^n$, and a function $F : \mathbb{R}^n \to \mathbb{R}^n$, a mixed complementarity problem (MCP) seeks $\boldsymbol{x} \in \mathbb{R}^n$ such that *precisely* one of the following holds for each $i \in \{1, \ldots, n\}$:

$$x_i = \ell_i, F_i(x) \geq 0, \tag{16}$$

$$\ell_i < x_i < u_i, F_i(x) = 0, \tag{17}$$

$$x_i = u_i, F_i(x) \leq 0. \tag{18}$$

We write these conditions compactly using the $\perp$ (said: "perpendicular") notation:

$$F(\boldsymbol{x}) \perp \boldsymbol{\ell} \leq \boldsymbol{x} \leq \boldsymbol{u}.$$

Each of the four different combinations of finite/infinite lower/upper bounds on x,

$$0 = F(\boldsymbol{x}) \perp \boldsymbol{x},$$

$$0 \leq F(\boldsymbol{x}) \perp \boldsymbol{x} \geq \boldsymbol{\ell},$$

$$0 \geq F(\boldsymbol{x}) \perp \boldsymbol{x} \leq \boldsymbol{u},$$

$$F(\boldsymbol{x}) \perp \boldsymbol{\ell} \leq \boldsymbol{x} \leq \boldsymbol{u}.$$

gives rise to a different MCP.

3.4. Problem formulation

We can now formulate each agent's minimisation problem, which together form the risk-averse competitive equilibrium. We also formulate the market clearing conditions. This follows a similar structure to the definition of the social plan above.

In Appendix A, we reformulate this equilibrium problem as a game, representing the market clearing conditions by agents that choose prices. We use the KKT conditions in Appendix B to show that both sets of problems are equivalent, and prove that the game has a Nash equilibrium which must also be a competitive equilibrium.

PROBLEM 2. Agent objectives and constraints ($\forall a \in \mathcal{A}$)

$$AP^a : \min_{\substack{\boldsymbol{x}^a \geq 0, \\ \boldsymbol{z}^a \geq 0, \\ \boldsymbol{y}^a \geq 0, \\ \boldsymbol{q}^a \geq 0, \\ \boldsymbol{w}^a.}} \quad \rho^a(\Psi^a)$$

$$\Psi^a(\omega) = Z^a(\omega) + \sum_{c \in \mathcal{C}} (p_c - \mathrm{W}_c(\omega, \boldsymbol{\pi}(\omega))) \cdot w_c^a \qquad \forall \omega \in \Omega, \tag{19}$$

$$\begin{aligned} Z^a(\omega) = & \sum_{i \in \mathcal{I}, k \in \mathcal{K}} \mathrm{xC}_k \cdot x_{i,k}^a + \sum_{i \in \mathcal{I}, k \in \mathcal{K}} \mathrm{oC}_k \cdot z_{i,k}^a \\ & + \sum_{b \in \mathcal{B}} \mathrm{T}(\omega, b) \cdot \sum_{i \in \mathcal{I}, k \in \mathcal{K}} (\mathrm{gC}_k(\omega, b) - \pi_i(\omega, b)) \cdot y_{i,k}^a(\omega, b) \\ & + \sum_{b \in \mathcal{B}} \mathrm{T}(\omega, b) \cdot \sum_{i \in \mathcal{I}, h \in \mathcal{H}} (\pi_i(\omega, b) - \mathrm{r}_h) \cdot (\mathrm{d}_{i,h}^a(\omega, b) - q_{i,h}^a(\omega, b)) \\ & + \sum_{b \in \mathcal{B}} \mathrm{T}(\omega, b) \cdot \sum_{i \in \mathcal{I}, h \in \mathcal{H}} \mathrm{v}_h \cdot q_{i,h}^a(\omega, b) \qquad \forall \omega \in \Omega, \end{aligned} \tag{20}$$

s.t.

$$x_{i,k}^a \leq \mathrm{u}_{i,k}^a \qquad \forall i \in \mathcal{I}, k \in \mathcal{K}, \tag{21}$$

$$z_{i,k}^a \leq x_{i,k}^a + \mathrm{k}_{i,k}^a \qquad \forall i \in \mathcal{I}, k \in \mathcal{K}, \tag{22}$$

$$y_{i,k}^a(\omega, b) \leq \mathrm{m}_k(\omega, b) \cdot z_{i,k}^a \qquad \forall i \in \mathcal{I}, k \in \mathcal{K}, \omega \in \Omega, b \in \mathcal{B}, \tag{23}$$

$$\textstyle\sum_{b \in \mathcal{B}} \mathrm{T}(\omega, b) \cdot y_{i,k}^a(\omega, b) \leq \mathrm{n}_k(\omega) \cdot z_{i,k}^a \qquad \forall i \in \mathcal{I}, k \in \mathcal{K}, \omega \in \Omega, \tag{24}$$

$$q_{i,h}^a(\omega, b) \leq \mathrm{d}_{i,h}^a(\omega, b) \qquad \forall i \in \mathcal{I}, h \in \mathcal{H}, \omega \in \Omega, b \in \mathcal{B}. \tag{25}$$

The objective for each agent, a, is to minimise their own risk-adjusted disbenefit $\rho^a(\Psi^a)$. The net disbenefit $\Psi^a(\omega)$ observed by each agent a in each scenario ω is defined by equation (19). This equation has two components. $Z^a(\omega)$ is the net cost from investing and operating their fleet of generation in scenario ω. The second term is the net payoff from each contract based on the purchase price $\boldsymbol{p}$, and the scenario dependent payoff $\mathbf{W}$. The net effect of each contract depends on the difference between the price and return per unit purchased ($\boldsymbol{p} - \mathbf{W}$) multiplied by the amount w of this contract that is purchased or sold.

The net cost from investing and operating their fleet of generation, $Z^a(\omega)$, observed by each agent, a, in each scenario, ω, is defined by equation (20). This equation includes the costs and revenue streams of all three types of agents (generation, retail, and industrial consumer). In the first line of equation (20), we have the physical capacity investment cost, $\mathbf{xC} \cdot \boldsymbol{x}^a$, and the operation and maintenance cost $\mathbf{oC} \cdot \boldsymbol{z}^a$.

In the second line of equation (20), we have the component of the disbenefit from generation, $(\mathbf{gC} - \boldsymbol{\pi})\boldsymbol{y}$, with $\mathbf{gC}$ giving the marginal cost of generation, $\boldsymbol{\pi}$ the spot market price, and $\boldsymbol{y}$ the output of generation. For each scenario, ω, we then multiply this by the time, $\mathrm{T}(\omega, b)$ spent in each load block, b, to get the total short run disbenefit from each plant (giving $\mathbf{T}(\mathbf{gC} - \boldsymbol{\pi})\boldsymbol{y}$).

In the third term, we define the disbenefit from meeting demand. The per unit cost of meeting demand is given by $\boldsymbol{\pi} - \mathbf{r}$ with the agent having to purchase the electricity directly from the spot market at $\boldsymbol{\pi}$ and given $\mathbf{r}$ by the consumer. The demand met by the retail component of the agent is given by $\mathbf{d} - \boldsymbol{q}$. The exogenous demand of each consumer is given by $\mathbf{d}$, and $\boldsymbol{q}$ is how much the retail company decides to curtail. The overall profit is given by $(\boldsymbol{\pi} - \mathbf{r})(\mathbf{d} - \boldsymbol{q})$. Again, we multiply this by the time spent within each load block giving the total short-run profit, excluding the curtailment penalty given in the next term (giving $\mathbf{T}(\boldsymbol{\pi} - \mathbf{r})(\mathbf{d} - \boldsymbol{q})$).

In the final term, we define the penalty the retail agent must pay for unmet demand, $\boldsymbol{q}$. The penalty is the value of lost load, $\mathbf{v}$, which is much higher than typically observed spot market prices. This penalty is added to the lost revenue from not meeting all of the consumer demand for electricity generation.

In equations (21) through (25), we define the physical constraints on capacity investment, generation, and curtailment. Equation (21) limits the capacity investment $\boldsymbol{x}$ of each plant to a predetermined level $\mathbf{u}$. Equation (22) limits the capacity, $\boldsymbol{z}$ of each generation plant to the sum of expanded and existing capacity $\boldsymbol{x} + \mathbf{k}$, allowing for divestment. Equation (23) limits the power output $\boldsymbol{y}$ of each plant, depending on the capacity investment $\boldsymbol{x}$ and some multiplicative adjustment, $\mathbf{m}$, that depends on the scenario and load block. Equation (24) limits the energy output of a generation plant. Finally, equation (25) limits consumption to be at most the level of demand.

In equation (26) we define the objective function for the system operator. It is a price-taking agent that attempts to maximise the revenue (minimise the disbenefit) they receive from transmission of electricity. The cost to the system operator depends on both the difference between the node prices, $\boldsymbol{\pi}_i - \boldsymbol{\pi}_j$ and the

PROBLEM 3. Independent System Operator objectives and constraints ($\forall \omega \in \Omega, b \in \mathcal{B}$)

$$IP(\omega,b): \min_{\boldsymbol{f}(\omega,b)} \sum_{(i,j)\in\mathcal{L}} (\pi_i(\omega,b) - \pi_j(\omega,b)) \cdot f_{i,j}(\omega,b) + \sum_{i\in\mathcal{I}} \pi_i(\omega,b) \cdot L_i(\omega,b) \tag{26}$$

s.t.

$$\sum_{(i,j)\in\mathcal{L}_e} \mathrm{s}_{i,j,e} \cdot f_{i,j}(\omega,b) = 0 \quad \forall e \in \mathcal{E}, \tag{27}$$

$$L_i(\omega,b) = \sum_{\substack{j:\\((i,j)\cup(j,i))\in\mathcal{L}}} \frac{\mathrm{c}_{i,j}}{2} \cdot (f_{i,j}(\omega,b))^2 \quad \forall i \in \mathcal{I}, \tag{28}$$

$$f_{i,j}(\omega,b) \le \mathrm{f}^+_{i,j} \quad \forall (i,j) \in \mathcal{L}, \tag{29}$$

$$f_{i,j}(\omega,b) \ge \mathrm{f}^-_{i,j} \quad \forall (i,j) \in \mathcal{L}. \tag{30}$$

transmission between the nodes, $\boldsymbol{f}_{i,j}$, giving $(\boldsymbol{\pi}_i - \boldsymbol{\pi}_j)\boldsymbol{f}_{i,j}$. Significant losses occur when we transmit electricity over long distances, thus we add the value of the losses to the objective. The per unit cost accumulated at a node is the spot market price, $\boldsymbol{\pi}_i$. Thus, the total cost from losses at a node is $\boldsymbol{\pi}_i \boldsymbol{L}_i$.

In (27) through (30) we define the transmission constraints. These are the same as the constraints (7) through (11) in the social plan. With free disposal, this problem remains convex as shown in Palma-Benhke et al. (2013).

PROBLEM 4. SM: Spot market equilibrium conditions.

$$\sum_{a\in\mathcal{A},k\in\mathcal{K}} y^a_{i,k}(\omega,b) \tag{31}$$

$$+\sum_{a\in\mathcal{A},h\in\mathcal{H}} \left[q^a_{i,h}(\omega,b)-\mathrm{d}^a_{i,h}(\omega,b)\right] \tag{32}$$

$$+\sum_{\substack{j\in\mathcal{I}:\\(j,i)\in\mathcal{L}}} f_{j,i}(\omega,b) \tag{33}$$

$$-\sum_{\substack{j\in\mathcal{I}:\\(i,j)\in\mathcal{L}}} f_{i,j}(\omega,b) \tag{34}$$

$$-L_i(\omega,b) \qquad \perp \qquad \bar{\mathrm{P}}\geq\pi_i(\omega,b)\geq 0 \qquad\qquad \forall i\in\mathcal{I},\omega\in\Omega,b\in\mathcal{B}. \tag{35}$$

PROBLEM 5. CM: Contract market equilibrium conditions.

$$0\leq-\sum_{a\in\mathcal{A}} w^a_c \perp p_c\geq 0 \qquad\qquad \forall c\in\mathcal{C}. \tag{36}$$

The expressions (31) through (35) is an equilibrium condition that ensures, at each node, supply meets demand at a competitive price. We have free disposal of power within our model, allowing supply to exceed demand at each node. However, when this occurs, the spot market price for electricity at this node will be 0. We also impose an upper bound on the spot market price $\bar{\mathrm{P}}=\max_h(\mathrm{r}_h+\mathrm{v}_h)$.

The complementarity condition (36) is an equilibrium requirement that ensures that the price of each contract is set to a competitive price with the number of contracts of each type sold equal to the number purchased.

The following theorems state that solutions exist for the stated equilibrium problems when agents trade Arrow-Debreu securities or contracts for differences respectively. For both theorems we need the following assumptions.

ASSUMPTION 1. Each agent is endowed with a coherent risk measure with a polyhedral risk set.

ASSUMPTION 2. The intersection of the relative interior of each producer's risk set is non-empty.

ASSUMPTION 3. The set Ω of scenarios is finite.

ASSUMPTION 4. The set $\mathcal{C}$ of contracts is linearly independent.

THEOREM 5. *Under assumptions 1, 2, 3, and 4, the equilibrium for the overall problem that combines problems (AP, IP, SM, CM) exists when agents can trade Arrow-Debreu securities.*

Proof. In Appendix A subsection 7.1 we reformulate the equilibrium problem (AP, IP, SM, CM) to a Nash game $(\mathcal{G}, \mathcal{C}, \mathcal{M}, \mathcal{S}, \mathcal{P})$. Since all optimisation problems are convex, we can show these problems are equivalent by showing that the joint KKT conditions of (AP,IP, SM,CM) are exactly the same as $(\mathcal{G}, \mathcal{C}, \mathcal{M}, \mathcal{S}, \mathcal{P})$. We then show that the game $(\mathcal{G}, \mathcal{S}, \mathcal{M})$ is guaranteed to have an equilibrium, bounding $\boldsymbol{Z}$. Finally in subsection 7.3, we then prove that as long as $\boldsymbol{Z}$ is bounded, we can guarantee existence of equilibrium in the Nash game $(\mathcal{C}, \mathcal{P})$ with Arrow-Debreu contracts. With existence of equilibrium in the problems $(\mathcal{G}, \mathcal{S}, \mathcal{M})$ and $(\mathcal{C}, \mathcal{P})$, we have existence in the overall problem $(\mathcal{G}, \mathcal{C}, \mathcal{M}, \mathcal{S}, \mathcal{P})$. ∎

THEOREM 6. *Under assumptions 2, 3, and 4, the equilibrium for the overall problem that combines problems (AP, IP, SM, CM) exists when agents have coherent risk measures and can trade contracts for differences.*

Proof. Again, we use the reformulated Nash game $(\mathcal{G}, \mathcal{C}, \mathcal{M}, \mathcal{S}, \mathcal{P})$ from Appendix A subsection 7.1. Again, the problems $(\mathcal{G}, \mathcal{S}, \mathcal{M})$ are guaranteed to have an equilibrium, bounding $\boldsymbol{Z}$. Finally in subsection 7.4, with agents able to purchase and sell contracts for differences we prove existence of equilibrium to the Nash game $(\mathcal{C}, \mathcal{P})$ when $\boldsymbol{Z}$ is bounded. The multidimensional case is derived by de Maere dAertrycke and Smeers (2013) who give a proof of bounded contract quantities that generalises the argument presented here. With existence of equilibrium in the problems $(\mathcal{G}, \mathcal{S}, \mathcal{M})$ and $(\mathcal{C}, \mathcal{P})$, we have existence in the overall problem $(\mathcal{G}, \mathcal{C}, \mathcal{M}, \mathcal{S}, \mathcal{P})$. ∎

4. Case study: Expansion of a transmission line

In our case study we assume two nodes S and N, and four agents comprising one retailer, two generators, and the system operator defined as follows.

- The uncertain demand retailer in node N purchases generation on the spot market to sell to consumers whose demand varies over load blocks and is uncertain in terms of the overall amount of demand;
- The peaker plant generator in node N has the opportunity of constructing a plant with known capacity but relatively high running costs;
- The baseload plant generator in node S has the opportunity of constructing a plant with uncertain capacity but lower running costs than the peaker plant.

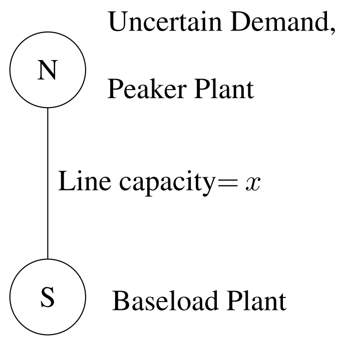

Figure 1 Diagram showing location of generators and consumers of electricity. Throughout this case study, the transmission line is considered lossless with capacity L.

There are two potential sources of uncertainty in our model, as illustrated by scenarios in Figure 2, and summarised in Table 1.

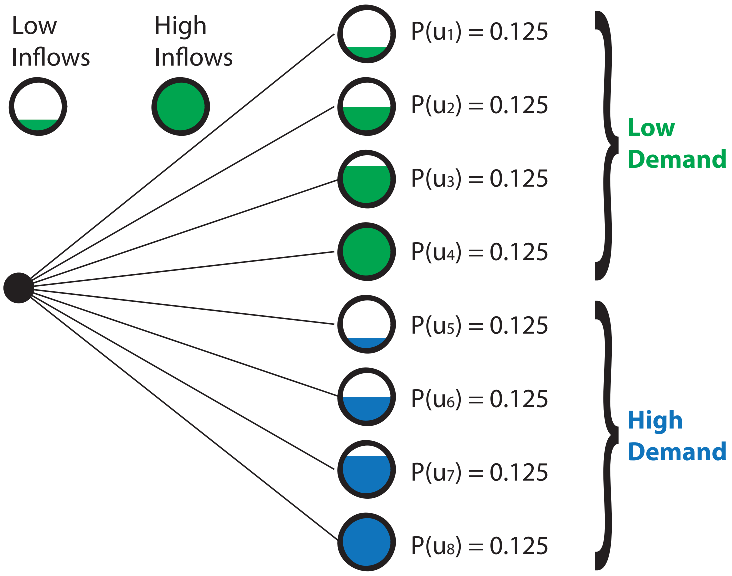

Figure 2 Diagram showing the potential events that can occur within the case study as well as their probability. The sources of randomness in this model are inflows and demand.

Table 1 Summary of parameters across scenarios

Scenario	Probability	m - Baseload plant mult. factor	Demand	
			b_1	b_2
1	0.125	0.5	0.3333	1
2	0.125	1	0.3333	1
3	0.125	1.5	0.3333	1
4	0.125	2	0.3333	1
5	0.125	0.5	0.5	1.5
6	0.125	1	0.5	1.5
7	0.125	1.5	0.5	1.5
8	0.125	2	0.5	1.5

There are 8 scenarios in total, with scenarios 1 to 4, highlighted in green, corresponding to the low demand, and scenarios 5 to 8, highlighted in blue corresponding to high demand. The uncertainty in capacity in the baseload plant is captured through a multiplication factor which changes the actual upper bound on $\boldsymbol{y}$ (generation) for a given $\boldsymbol{z}$ (generation capacity).

The risk measure we will be using in the case study is a convex combination of the Conditional Value at Risk (CVaR), and the expected cost. CVaR is a coherent risk measure, that can be represented by the formula of Rockafellar and Uryasev (2000):

$$\mathrm{CVaR}_{1-\alpha}(\boldsymbol{Z}) = \inf_{\xi} \left[\xi + \frac{1}{\alpha} \mathbb{E}(Z - \xi)_+ \right]. \tag{37}$$

Specifically, our case studies will use a convex combination of this risk measure and the expected disbenefit, so

$$\rho(\boldsymbol{Z}) = \lambda \cdot \mathrm{CVaR}_{1-\alpha}(\boldsymbol{Z}) + (1-\lambda) \cdot \mathbb{E}(\boldsymbol{Z}). \tag{38}$$

We choose the parameters λ and α to be 0.5 and 0.25 respectively. This essentially triples the weight that agents (as well as the social welfare optimising agent) places on the 2 scenarios where they earn the lowest welfare while halving the weighting on the other 6 scenarios.

In node N we have uncertain demand and the peaker plant, and the baseload plant in node S. Thus, there is some source of uncertainty at each node. We connect these two nodes with a transmission line of capacity L that we will change to see how it impacts the equilibrium.

We are modelling agents as price takers with constant marginal costs. Accordingly, if spot market prices are too low, meaning any revenue would not cover the cost of investment, then they will not construct the plant. If prices are set high enough, then agents will construct up to the plant's maximum capacity. There is a point in the middle where agents earn 0 risk-adjusted profit from a plant and construct anywhere from zero up to the plant's maximum capacity depending on the equilibrium of the overall model. In this case study, we set the maximum capacity to infinity. Thus generation agents will not make a risk-adjusted profit from their capacity investment.

In Figure 3, we see that when we initially increase the capacity of the transmission line from zero, in equilibrium there is a trend of increasing baseload capacity and decreasing peaker capacity. This occurs as the baseload plant, previously isolated from the demand, begins to gain access to this demand via the transmission line. As they choose to construct the baseload plant, the baseload plant agent will compete with the peaker plant agent, to make a zero risk-adjusted profit on the spot market.

However, there is some risk faced by each agent, and without a complete market for risk, we cannot guarantee that agents choose socially optimal capacity investment decisions. In fact, as we increase the capacity of the transmission line, we see that social welfare decreases, as shown in the topmost curve in Figure 4.

The total welfare of the competitive solution (the sum of each agent's risk-adjusted welfare as shown near the bottom of Figure 4 is much lower than the social welfare (calculated by applying the risk function to the sum of each agent's welfare) across all transmission capacity levels. This shows a disparity between the scenarios that each agent weighs more heavily due to risk aversion. Observe that these values would be equal if all agents had the same low-profit scenarios.

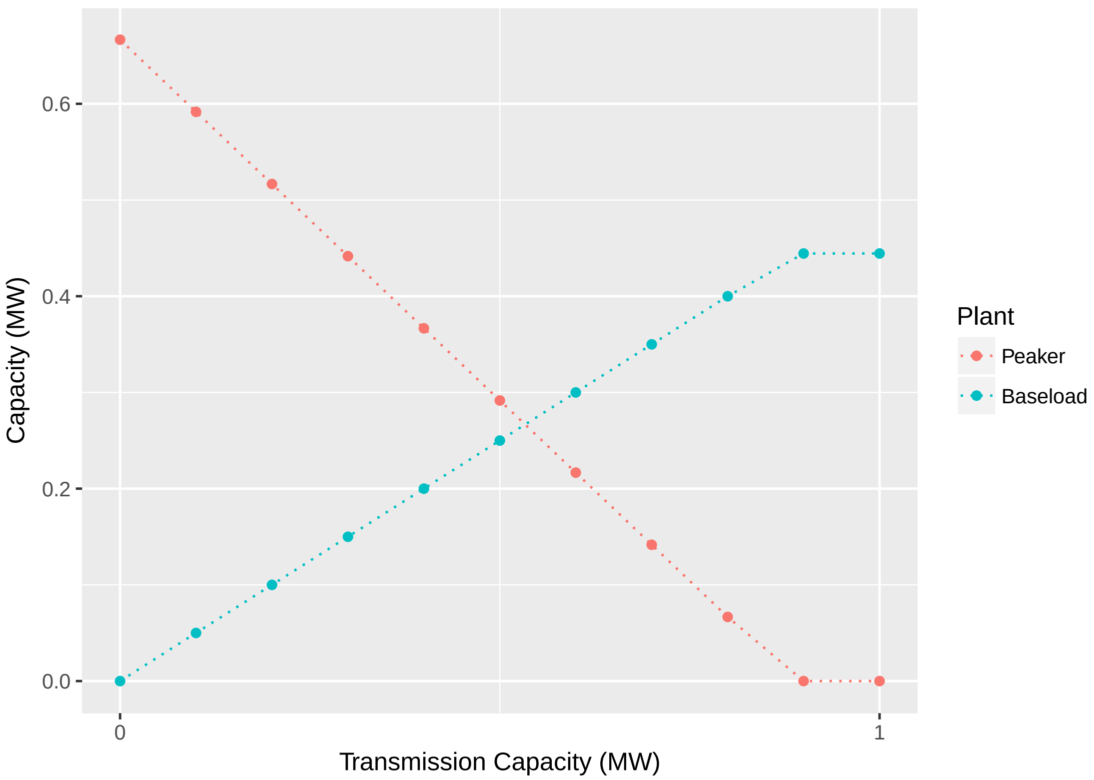

Figure 3 Generation Expansion decisions as a function of line capacity in the no contract model.

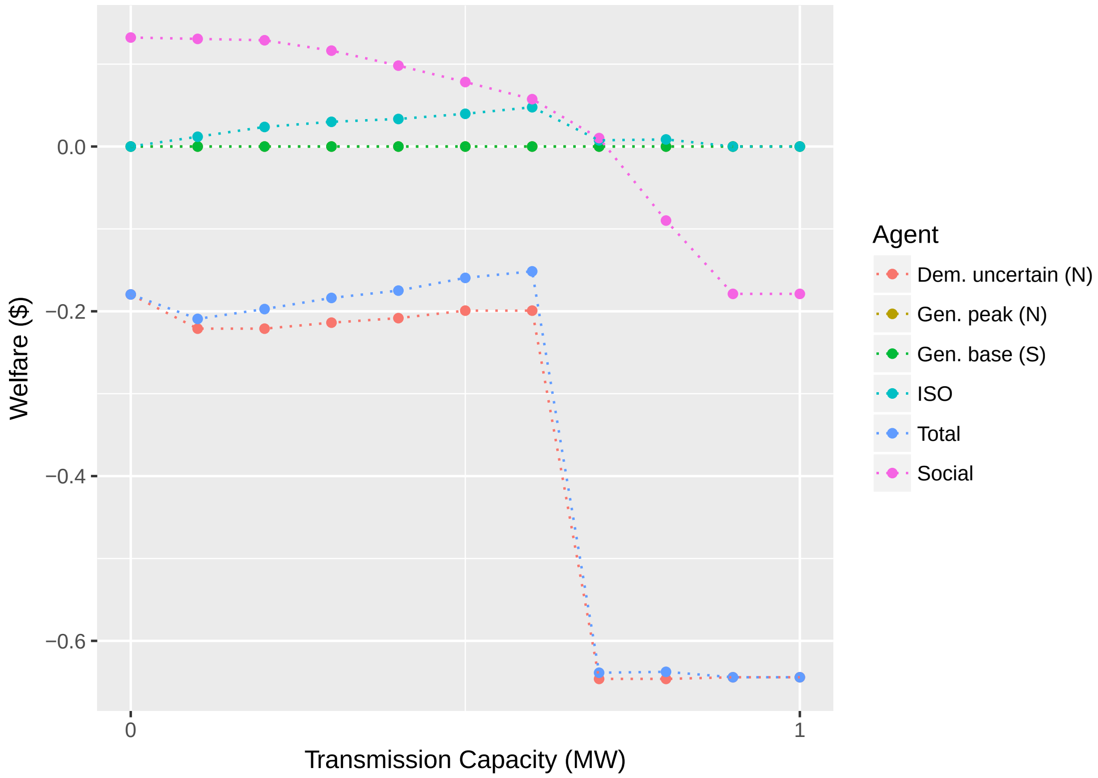

Figure 4 Risk-adjusted welfare of each agent as a function of line capacity in the no contract model.

5. Contracts make transmission beneficial

In the previous example, we see that, when agents are risk-averse, a larger transmission line can exacerbate the problem of under-investment, leading to a reduction in social welfare. Now we introduce contracts and show how they can improve social welfare.

In Figure 5 we can see how a CFD based on the system average spot market price has enabled an improvement in social welfare when there is a larger transmission line. With the ‘Avg Node CFD’ we get close to the socially optimal capacity decisions. We get even closer when we have a CFD at each node, and closer still in the model where agents trade Arrow-Debreu securities. Including the ISO as an agent that can trade Arrow-Debreu securities (‘ADB all’) gives the risk-adjusted system optimal capacity investment decisions for each given transmission capacity, and so total welfare is equal to the optimal social welfare.

In Figure 6 we look at the equilibrium risk-adjusted welfare outcomes for each agent, as well as the risk-adjusted system welfare. Again, introducing contracts substantially improves social welfare and reduces the severity of decreasing total welfare with increased transmission capacity. However, we do see a few cases where total welfare does decrease as we incrementally increase the capacity of the transmission line. This happens in the '2 Node CFD' model where total welfare is decreased, as there is still some difference in what the worst scenarios are for the generators compared with the system. Across all line capacities the contracts have improved the risk-adjusted welfare of the retail agents substantially.

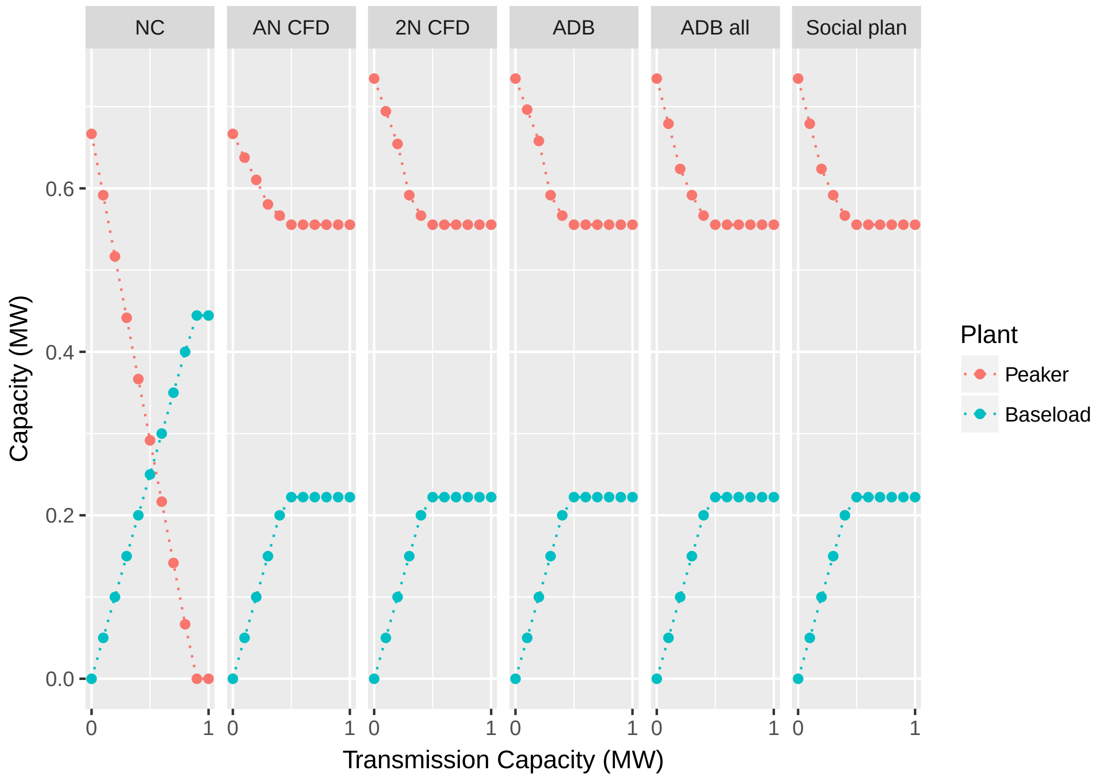

Figure 5 Generation Expansion decisions as a function of line capacity in each model. NC - Default model without contracts available for agents to trade amongst one another, AN CFD - A contract for difference (CFD) which uses the time averaged spot market price to determine the payoff is available to trade amongst agents, 2N CFD - 2 contracts for difference each with their own independent price, with payoff based on the time averaged spot market price, ADB - Agents can trade Arrow-Debreu securities, ADB all participate (equivalent to the social plan) - The ISO is also included as an agent and can trade Arrow-Debreu securities.

In Figures 7,8,9 we highlight the capacity investment decisions and welfare results of a few select transmission capacities. In Figure 7, as we increase the capacity of the transmission line, we see the growing difference between the 'No Contract' model and the models with contracts.

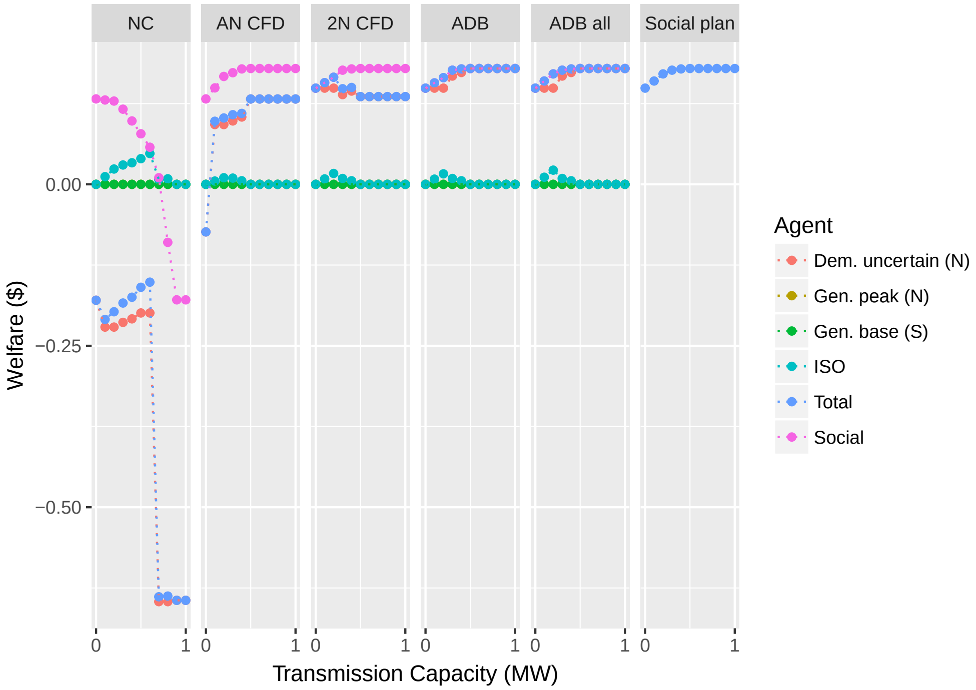

Figure 6 Risk-adjusted welfare of each agent as a function of line capacity in each model.

In Figure 8 we see that, as we include a larger transmission line in our example, there is a nearly consistent trend across all models with contracts that welfare is also improved, in contrast to our non-contract model, where we see a dip in system welfare caused by inefficient capacity investment decisions.

We can explain this discrepancy by looking at Figure 9. Looking at the payoffs in the ‘No Contract’ model for the generators when we increase the transmission line from 0 MW to 0.5 MW, we see that the increased capacity by *gen. base* has caused the revenue earned by *gen. peak* to decrease during shortages of supply (scenarios where the price taker may earn higher profits) and causes them to reduce investment to get back to a zero risk-adjusted profit. The larger transmission line has also caused *gen. peak*’s revenue earned across scenarios to decrease, as the cheaper baseload plant is used to meet much of the demand in scenarios where there is a surplus in supply. Now both generation agents earn higher profits in scenarios where there are shortages (scenarios 1 and 5) which are the worst scenarios for the system as a whole.

In summary, we observe that as we add more contracts to the model, and allow more agents to trade these contracts, we get closer to completing the market. The contracts for differences are sufficient in getting capacity investment decisions that maximise risk-adjusted social welfare for this case study, but the *dem.*

uncertain retail agent did not completely align with the system in terms of worst case scenarios. Thus, contracts for differences are insufficient in completing the risk market in this model.

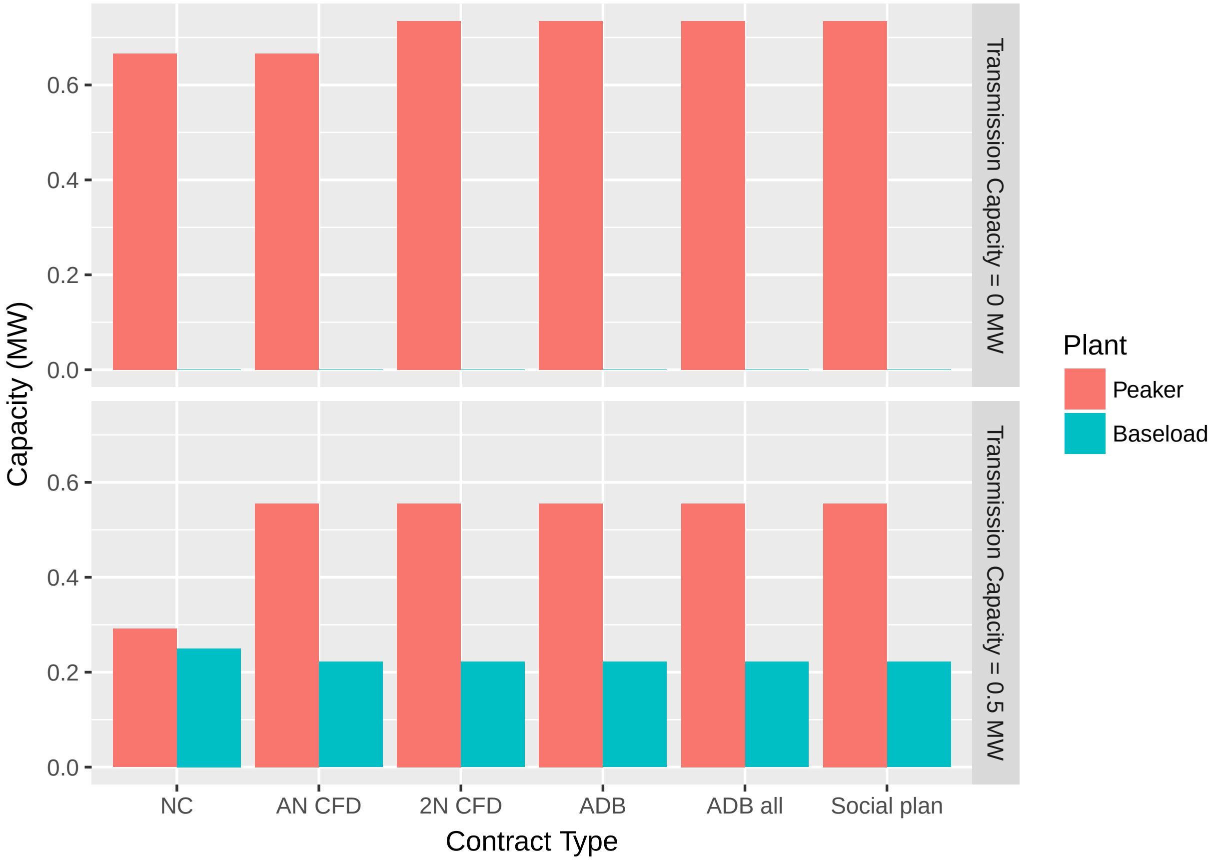

Figure 7 Capacity capacity investment of each plant under specific transmission capacities in each model: 0MW and 200MW.

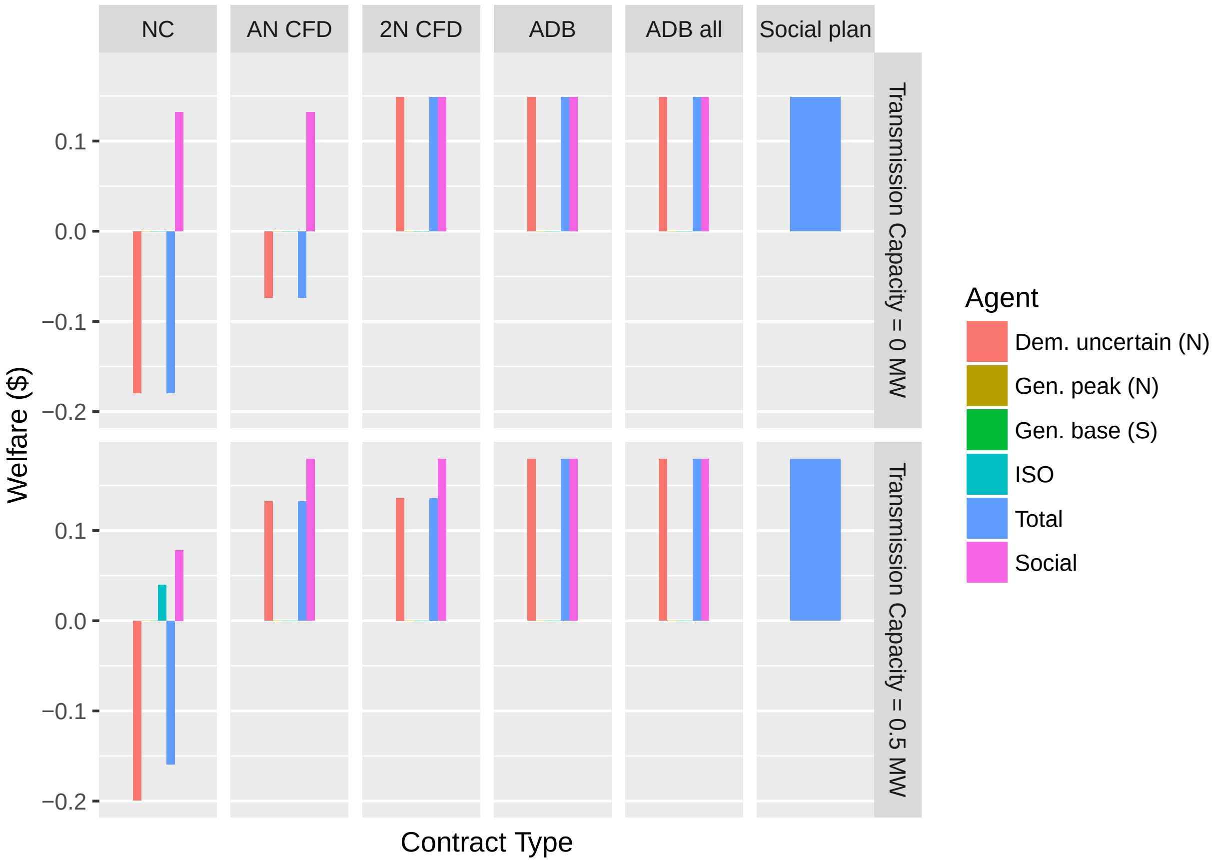

Figure 8 Expected and risk-adjusted welfare for each agent under each model for each of the previously specified transmission capacities.

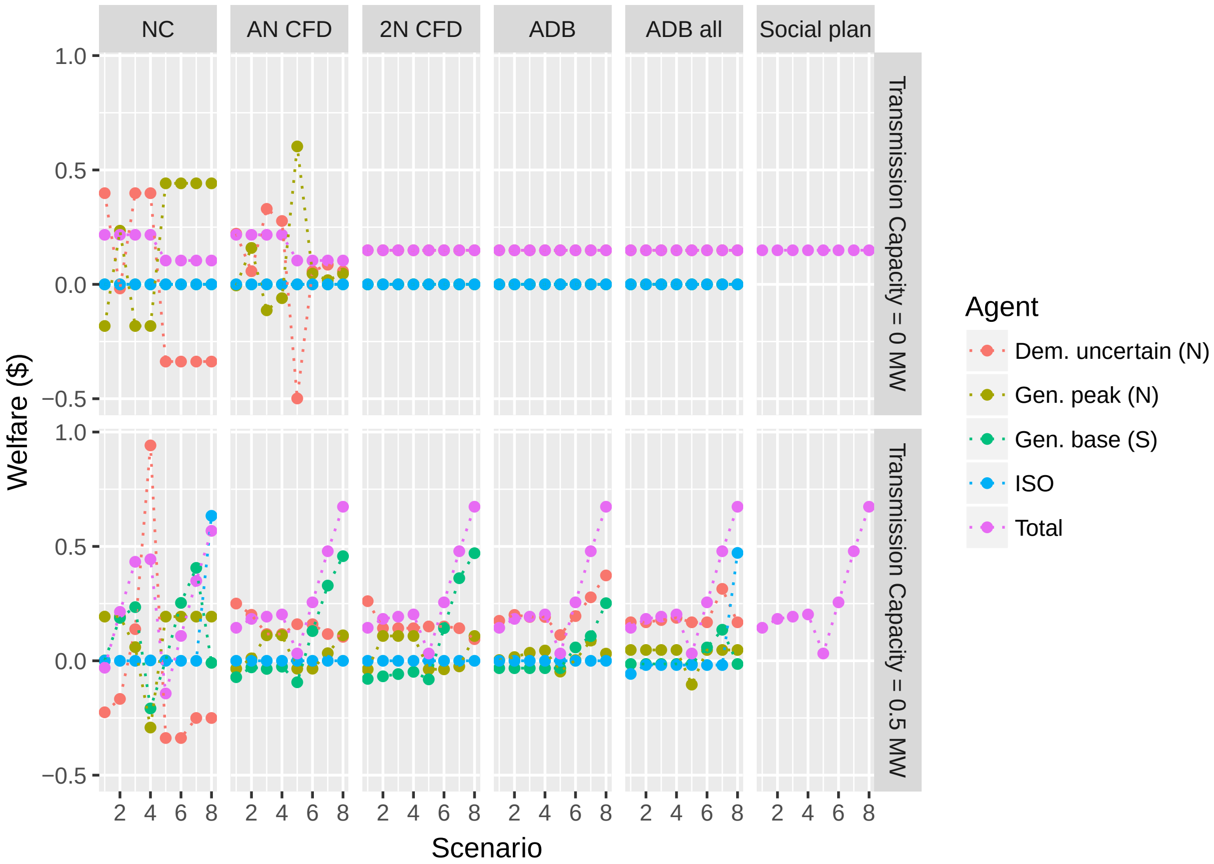

Figure 9 Welfare in each model for each agent under each scenario under the specified transmission capacities.

6. Conclusions

We have created a model with perfectly competitive agents simultaneously making their capacity investment, generation, curtailment, and contract decisions. Under some mild assumptions, we have shown the existence of a competitive equilibrium for models in incomplete markets with Arrow-Debreu securities and contracts-for-differences.

We have solved a case study with uncertainty in supply availability and retail demand. Results show that in some circumstances, as a result of the agents being risk-averse, underinvestment exacerbated by the transmission line can more than cancel out any benefits the transmission line provides within the electricity market. In this setting, introducing contracts for differences improves both individual agent welfare, and system welfare. By aligning the worst-case scenarios for generation agents more towards that of the system as whole, contracts help ensure that having a transmission line (or increasing its capacity) is much more likely to be beneficial to the system as a whole, even when there is uncertainty in how this additional transmission capacity is going to influence spot market prices.

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7. Appendices

7.1. Appendix A: Equilibrium in a simultaneously perfectly competitive spot market and contract market

As previously outlined, we study perfectly competitive players that make simultaneous investment and generation decisions as well as decisions on their hedging positions. Walrasian market clearing conditions for both the energy spot and contract markets complete the description. To prove existence of equilibrium, we can reformulate the equilibrium problem as a non-cooperative game to which we can apply Rosen's theorem Rosen (1965). This theorem requires convexity of the agent problems and compactness of their feasible regions. This requires some care in dealing with contract decisions, which are not explicitly bounded in our formulations.

The steps in the proof are as follows. First, we reformulate the problem in the start of section 3 to split each gentailer agent into two departments, one in charge of contracting and the other in charge of capacity investment and spot sales. The former (i.e. the contract trading department) optimises contracts assuming fixed generation and capacity investment actions, while the latter minimises risk-adjusted disbenefit assuming a fixed quantity of contracts. The KKT conditions derived from each of these formulations when combined are equivalent to the KKT conditions of the original problem. Since each agent problem is convex, and hence equivalent to its KKT conditions, we can show that any equilibrium of the disaggregated formulation is an equilibrium of the original formulation and vice versa.

Next, we apply Rosen's theorem Rosen (1965) to show that an equilibrium exists. This requires convexity, continuity and compactness. It is straightforward to establish the required continuity and convexity results. Compactness, however, requires more effort. The set of capacity investment and production decisions are naturally bounded, but the contract volume and prices are not. We therefore need to establish that the problem of optimising the sum of risk-adjusted welfare, through arranging contract trades, when disbenefits $Z \in \mathcal{Z}$ are given, is bounded. The analysis required to do this depends on the exact form of the contract. We distinguish two forms, Arrow-Debreu securities and contracts for differences, and prove the boundedness of contract trades for these two cases separately.

There are a few assumptions we require to ensure the existence of an equilibrium (we make no claim that this is unique or isolated).

ASSUMPTION 7. Each agent is endowed with a coherent risk measure with a polyhedral risk set.

ASSUMPTION 8. The intersection of the relative interior of each gentailer's risk set is non-empty.

Maere d'Aertrycke, et. al. provide an in-depth analysis of the need for this assumption in de Maere dAertrycke and Smeers (2013) to guarantee the existence of an equilibrium.

ASSUMPTION 9. The set Ω of scenarios is finite.

ASSUMPTION 10. The set $\mathcal{C}$ of contracts is linearly independent.

A set of contracts, $\mathcal{C}$ is linearly dependent if there exists $\boldsymbol{w}^a \neq \mathbf{0}$ such that $\sum_{c\in\mathcal{C}} \mathrm{W}_c(\omega, \boldsymbol{\pi}(\omega)) \cdot w_c^a = 0$. This means we can express one contract as a linear combination of the remaining contracts.

7.2. Problem reformulation

The reformulation of our problem is laid out as follows. We have split each agent into effectively two parts, that can be thought of as the agent's departments. In one part, given the contracts, the agent optimises its generation and retail decisions; this we have labelled $\mathcal{G}^a$. The other part, $\mathcal{C}^a$, may be thought of as the contracts department and is responsible for the volume of contracts purchased. This department takes prices, generation, and retail decisions as given. Splitting the agent this way allows us to address the contract aspect of the model separately from payoffs from their generation and sales actions, which then yields bounds on the contract quantities. We have also restated the Walrasian market-clearing conditions as optimization problems ($\mathcal{M}(i,\omega,b)$ and $\mathcal{P}(\omega,b)$), rather than complementarity conditions. This gives a collection of optimization problems which when solved simultaneously constitute a non-cooperative game which we denote ($\mathcal{G}$, $\mathcal{C}$, $\mathcal{M}$, $\mathcal{S}$, $\mathcal{P}$). The optimization problems are as follows.

For any $(\boldsymbol{w}, \boldsymbol{\pi}, \boldsymbol{p})$ each *gentailer* $a \in \mathcal{A}$ chooses optimal $(\boldsymbol{x}^a, \boldsymbol{z}^a, \boldsymbol{y}^a, \boldsymbol{q}^a)$ to solve

$$\mathcal{G}^a : \min_{\substack{\boldsymbol{x}^a \geq 0, \\ \boldsymbol{z}^a \geq 0, \\ \boldsymbol{y}^a \geq 0, \\ \boldsymbol{q}^a \geq 0}} \rho^a(\Psi^a) \tag{39}$$

$$\Psi^a(\omega) = Z^a(\omega) + \sum_{c \in \mathcal{C}} (p_c - \mathrm{W}_c(\omega, \boldsymbol{\pi}(\omega))) \cdot w_c^a \qquad \forall \omega \in \Omega, \tag{40}$$

$$Z^a(\omega) = \sum_{i \in \mathcal{I}, k \in \mathcal{K}} \mathrm{xC}_k \cdot x_{i,k}^a + \sum_{i \in \mathcal{I}, k \in \mathcal{K}} \mathrm{oC}_k \cdot z_{i,k}^a \tag{41}$$

$$+ \sum_{b \in \mathcal{B}} \mathrm{T}(\omega, b) \left(\sum_{i \in \mathcal{I}, k \in \mathcal{K}} (\mathrm{gC}_k(\omega, b) - \pi_i(\omega, b)) \cdot y_{i,k}^a(\omega, b) \right) \tag{42}$$

$$+ \sum_{b \in \mathcal{B}} \mathrm{T}(\omega, b) \left(\sum_{i \in \mathcal{I}, h \in \mathcal{H}} (\mathrm{r}_h + \mathrm{v}_h - \pi_i(\omega, b)) \cdot q_{i,h}^a(\omega, b) \right) \tag{43}$$

$$+ \sum_{b \in \mathcal{B}} \mathrm{T}(\omega, b) \cdot \left(\sum_{i \in \mathcal{I}, h \in \mathcal{H}} (\pi_i(\omega, b) - \mathrm{r}_h) \cdot \mathrm{d}_{i,h}^a(\omega, b) \right) \qquad \forall \omega \in \Omega, \tag{44}$$

$$s.t. \qquad x_{i,k}^a \leq \mathrm{u}_{i,k}^a \qquad \forall i \in \mathcal{I}, k \in \mathcal{K}, \tag{45}$$

$$z_{i,k}^a \leq x_{i,k}^a + \mathrm{k}_{i,k}^a \qquad \forall i \in \mathcal{I}, k \in \mathcal{K}, \tag{46}$$

$$y_{i,k}^a(\omega, b) \leq \mathrm{m}_k(\omega, b) \cdot z_{i,k}^a \qquad \forall i \in \mathcal{I}, k \in \mathcal{K}, \omega \in \Omega, b \in \mathcal{B}, \tag{47}$$

$$\sum_{b \in \mathcal{B}} \mathrm{T}(\omega, b) \cdot y_{i,k}^a(\omega, b) \leq \mathrm{n}_k(\omega) \cdot z_{i,k}^a \qquad \forall i \in \mathcal{I}, k \in \mathcal{K}, \omega \in \Omega, \tag{48}$$

$$q_{i,h}^a(\omega, b) \leq \mathrm{d}_{i,h}^a(\omega, b) \qquad \forall i \in \mathcal{I}, h \in \mathcal{H}. \tag{49}$$

Given $\pi(\omega, b)$ the *transmission operator* chooses $\boldsymbol{f}(\omega, b)$ to solve

$$\mathcal{S}(\omega, b) : \min_{\mathbf{f}^- \leq \boldsymbol{f}(\omega, b) \leq \mathbf{f}^+} \sum_{(i,j) \in \mathcal{L}} (\pi_i(\omega, b) - \pi_j(\omega, b)) \cdot f_{i,j}(\omega, b) + \sum_{i \in \mathcal{I}} \pi_i(\omega, b) \cdot L_i(\omega, b)$$

$$s.t. \qquad L_i(\omega, b) = \sum_{\substack{j: \\ ((i,j) \cup (j,i)) \in \mathcal{L}}} \frac{\mathrm{c}_{i,j}}{2} \cdot (f_{i,j}(\omega, b))^2 \qquad \forall i \in \mathcal{I},$$

$$\sum_{(i,j) \in \mathcal{L}_e} \mathrm{s}_{i,j,e} \cdot f_{i,j}(\omega, b) = 0 \qquad \forall e \in \mathcal{E}.$$

Given $(y_i(\omega, b), q_i(\omega, b), f_{i,*}(\omega, b), f_{*,i}(\omega, b), L_i(\omega, b))$, $i \in \mathcal{I}, \omega \in \Omega, b \in \mathcal{B}$, the *spot market agent* chooses $\pi_i(\omega, b)$ to solve

$$\mathcal{M}_i(\omega,b): \min_{0\le \pi_i(\omega,b)\le \bar{\mathrm{P}}} \pi_i(\omega,b)\cdot \begin{pmatrix} \sum_{a\in\mathcal{A},k\in\mathcal{K}} y^a_{i,k}(\omega,b) \\ +\sum_{a\in\mathcal{A},k\in\mathcal{K}} q^a_{i,h}(\omega,b) \\ +\sum_{\substack{j\in\mathcal{I}:\\(j,i)\in\mathcal{L}}} f_{j,i}(\omega,b) \\ -\sum_{\substack{j\in\mathcal{I}:\\(i,j)\in\mathcal{L}}} f_{i,j}(\omega,b) \\ -\sum_{a\in\mathcal{A},k\in\mathcal{K}} \mathrm{d}^a_{i,h}(\omega,b) \\ -L_i(\omega,b) \end{pmatrix}.$$

For any $(\boldsymbol{x}^a,\boldsymbol{z}^a,\boldsymbol{y}^a,\boldsymbol{q}^a,\boldsymbol{\pi},\boldsymbol{p})$, each *contract agent* $a\in\mathcal{A}$ chooses $\boldsymbol{w}^a$ to solve

$$\begin{aligned}
\mathcal{C}^a: &\min_{\boldsymbol{w}^a} \rho^a(\Psi^a)\\
&\Psi^a(\omega) = Z^a(\omega) + \sum_{c\in\mathcal{C}} (p_c - \mathrm{W}_c(\omega,\boldsymbol{\pi}(\omega)))\cdot w^a_c \qquad \forall\omega\in\Omega,\\
&Z^a(\omega) = \sum_{i\in\mathcal{I},k\in\mathcal{K}} \mathrm{xC}_k\cdot x^a_{i,k} + \sum_{i\in\mathcal{I},k\in\mathcal{K}} \mathrm{oC}_k\cdot z^a_{i,k}\\
&+\sum_{b\in\mathcal{B}} \mathrm{T}(\omega,b)\cdot\left(\sum_{i\in\mathcal{I},k\in\mathcal{K}} (\mathrm{gC}_k(\omega,b)-\pi_i(\omega,b))\cdot y^a_{i,k}(\omega,b)\right)\\
&+\sum_{b\in\mathcal{B}} \mathrm{T}(\omega,b)\cdot\left(\sum_{i\in\mathcal{I},h\in\mathcal{H}} (\mathrm{r}_h+\mathrm{v}_h-\pi_i(\omega,b))\cdot q^a_{i,h}(\omega,b)\right)\\
&+\sum_{b\in\mathcal{B}} \mathrm{T}(\omega,b)\cdot\left(\sum_{i\in\mathcal{I},h\in\mathcal{H}} (\pi_i(\omega,b)-\mathrm{r}_h)\cdot \mathrm{d}^a_{i,h}(\omega,b)\right) \qquad \forall\omega\in\Omega.
\end{aligned}$$

Given $(\boldsymbol{w})$ the *contract market agent* chooses $\boldsymbol{p}$ to solve

$$\mathcal{P}: \min_{\boldsymbol{p}\ge 0} -\sum_{c\in\mathcal{C}} p_c\cdot\left(\sum_{a\in\mathcal{A}} w^a_c\right)$$

LEMMA 1. *Any Nash equilibrium for (AP,IP,SM,CM) is a Nash equilibrium for* $(\mathcal{G},\mathcal{S},\mathcal{M},\mathcal{C},\mathcal{P})$ *and vice versa.*

Proof. In Appendix B we restate the games (AP, IP, SM, CM) and ($\mathcal{G},\mathcal{S},\mathcal{M},\mathcal{C},\mathcal{P}$) as respective complementarity problems by listing the KKT conditions of each agent problem. Since each agent problem is convex, any solution to the complementarity problem is a Nash equilibrium for the game. The KKT conditions for (AP, IP, SM, CM) and ($\mathcal{G},\mathcal{S},\mathcal{M},\mathcal{C},\mathcal{P}$) are then shown to be equivalent. This yields the result. ∎

We now turn our attention to demonstrating the existence of equilibrium for ($\mathcal{G},\mathcal{S}$, $\mathcal{M},\mathcal{C},\mathcal{P}$). Our results here depend on the form of the contracts that are traded. In general an agent a will buy w_c^a contracts of type c at a price p_c, in order to receive a payoff $\mathrm{W}_c(\omega,\boldsymbol{\pi}(\omega))$ in scenario ω. In an *Arrow-Debreu* contract, the payoff $\mathrm{W}_c(\omega)$ depends only on the exogenous outcome ω, and does not depend on $\boldsymbol{\pi}(\omega)$. In a *contract for differences*, the payoff $\mathrm{W}_c(\boldsymbol{\pi}(\omega)) = \sum_{b\in\mathcal{B}} \mathrm{T}(\omega,b)\pi_i(\omega,b)/(\sum_{b\in\mathcal{B}} \mathrm{T}(\omega,b))$, the time-weighted average price in the node i at which the contract c is settled.

In both cases we can show that if we fix contract levels and treat them as parameters then ($\mathcal{G}$, $\mathcal{C}$, $\mathcal{M}$, $\mathcal{S}$, $\mathcal{P}$) becomes a simpler non-cooperative game ($\mathcal{G}$, $\mathcal{M}$, $\mathcal{S}$), where we ignore all terms containing contract decisions.

LEMMA 2. *The game ($\mathcal{G},\mathcal{S},\mathcal{M}$) has a Nash equilibrium.*

Proof. The problems $\mathcal{G}$, $\mathcal{S}$, and $\mathcal{M}$ have convex continuous objective functions, with linear constraints. Thus, to invoke Rosen (1965), it is sufficient to show that each decision variable is bounded.

This follows from the (continuous) function defining $Z^a(\omega)$, since $\boldsymbol{x}$ satisfies $\mathbf{0} \le \boldsymbol{x} \le \mathbf{u}$ (equations (39) and (45)); $\boldsymbol{z}$ satisfies $\mathbf{0} \le \boldsymbol{z} \le \boldsymbol{x} + \mathbf{k} \le \mathbf{u} + \mathbf{k}$ (equations (39) and (46)); $\boldsymbol{y}$ satisfies $\mathbf{0} \le \boldsymbol{y} \le \mathbf{m}\cdot\boldsymbol{z} \le \mathbf{m}\cdot(\mathbf{u}+\mathbf{k})$ (equations (39) and (47)); $\boldsymbol{q}$ satisfies $\mathbf{0} \le \boldsymbol{q} \le \boldsymbol{d}$ (equations (39) and (49); $\boldsymbol{\pi}$ is regulated to be between $\mathbf{0} \le \boldsymbol{\pi} \le \bar{\mathbf{P}}$; and $\boldsymbol{f}$ satisfies $\mathbf{f}^- \le \boldsymbol{f} \le \mathbf{f}^+$. ∎

Now we show the existence of equilibrium in the game $(\mathcal{C},\mathcal{P})$ where we fix $(\boldsymbol{x},\boldsymbol{z},\boldsymbol{y},\boldsymbol{q},\boldsymbol{\pi},\boldsymbol{f})$. We do this first for Arrow-Debreu contracts in subsection 7.3, and then for contracts for differences in subsection 7.4.

7.3. Existence of equilibrium with Arrow-Debreu contracts

Throughout this subsection we will assume Arrow-Debreu contracts, with payoffs $\mathrm{W}_c(\omega)$ in scenario ω. Repeating equation (1), every coherent risk measure has a dual representation (see theorem 4.16 of Föllmer and Schied (2011))

$$\rho^a(\boldsymbol{Z}) = \max_{\mathbb{Q}\in\mathcal{M}_a} (\mathbb{E}_{\mathbb{Q}}[Z]),$$

where $\mathcal{M}_a$ is a convex set of probability measures. Throughout this section we make assumptions 7, 8, 9, and 10. Thus Ω is finite and $\mathcal{M}_a$ is a polyhedron. Let $\mathcal{E}_a$ denote the extreme points of $\mathcal{M}_a$, so

$$\mathcal{E}_a = \{\mu_k : k = 1, 2, \ldots, |\mathcal{E}_a|\}.$$

Now consider the non-cooperative game $(\mathcal{C}, \mathcal{P})$ that assumes the variables $(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y}, \boldsymbol{q}, \ \boldsymbol{\pi}, \boldsymbol{f})$ are fixed. The decision variables in this game are w_c^a (for each agent $a \in A$) and p_c for the contract market agent, and variables $Z^a(\omega)$ are treated as exogeneous parameters. It is easy to see that any Nash equilibrium for $(\mathcal{C}, \mathcal{P})$ is a solution to

$$\mathcal{W} : \min_{w\in\mathbb{R}^{\mathcal{A}}_{\mathcal{C}}} \sum_{a\in\mathcal{A}} \rho^a(Z^a(\omega) - \sum_{c\in\mathcal{C}} \mathrm{W}_c(\omega) w_c^a) \quad \text{subject to} \quad \sum_{a\in\mathcal{A}} w_c^a \leq 0 \quad [p_c] \quad c \in \mathcal{C},$$

since the Lagrangian for $\mathcal{W}$ decouples it into agent problems

$$\mathcal{W}^a : \min_{\boldsymbol{w}^a\in\mathbb{R}^{\Omega}} \rho^a \left(Z^a(\omega) - \sum_{c\in\mathcal{C}} \mathrm{W}_c(\omega) w_c^a \right) + \sum_{c\in\mathcal{C}} p_c w_c^a$$

and complementarity conditions

$$0 \leq -\sum_{a\in\mathcal{A}} w_c^a \perp p_c \geq 0. \tag{50}$$

This follows because

$$\rho^a(\Psi^a) = \rho^a \left(Z^a(\omega) - \sum_{c\in\mathcal{C}} \mathrm{W}_c(\omega) w_c^a \right) + \sum_{c\in\mathcal{C}} p_c w_c^a, \tag{51}$$

using the translation equivariance property of ρ^a, so $\mathcal{W}^a$ is identical to $\mathcal{C}^a$, and (50) is equivalent to $\mathcal{P}$.

LEMMA 3. *Suppose Ω is finite and each agent's risk set $\mathcal{M}_a$ is a polyhedron. If $Z^a(\omega)$, $a \in \mathcal{A}$, $\omega \in \Omega$ lies in a bounded set, then the optimal solution to $\mathcal{W}$ lies in a bounded set.*

Proof. Recall $\mathcal{E}_a = \{\mu_k : k = 1, 2, \ldots, |\mathcal{E}_a|\}$, the set of extreme points of $\mathcal{M}_a$. The problem $\mathcal{W}$ can be formulated as the linear program

$$\mathcal{WL}: \min_{\theta \in \mathbb{R}^{\mathcal{A}}, w \in \mathbb{R}^{\mathcal{A}}_{\Omega}\cdot} \sum_{a \in \mathcal{A}} \theta^a \tag{52}$$

$$\text{subject to}$$

$$\theta^a + \sum_{\omega \in \Omega} \sum_{c \in \mathcal{C}} \mu_k(\omega) \mathrm{W}_c(\omega) w_c^a \geq \sum_{\omega \in \Omega} \mu_k(\omega) Z^a(\omega), \qquad a \in \mathcal{A}, k = 1, 2, \ldots, |\mathcal{E}_a|,$$

$$-\sum_{a \in \mathcal{A}} w_c^a \geq 0 \qquad c \in \mathcal{C}.$$

The dual of $\mathcal{WL}$ is:

$$\max_{y \geq 0, p \geq 0} \sum_{a \in \mathcal{A}} \sum_{\mu \in \mathcal{M}} \sum_{\omega \in \Omega} \mu_k(\omega) Z^a(\omega) \cdot y_\mu^a \tag{53}$$

$$\text{subject to} \tag{54}$$

$$\sum_{k=1}^{|\mathcal{E}_a|} y_k^a = 1, \qquad a \in \mathcal{A}, \tag{55}$$

$$\sum_{k=1}^{|\mathcal{E}_a|} \sum_{\omega \in \Omega} \mu_k(\omega) \mathrm{W}_c(\omega) \cdot y_k^a - p_c = 0, \qquad a \in \mathcal{A}, c \in \mathcal{C}, \tag{56}$$

Note that the dual problem is feasible and bounded since $0 \leq y_k^a \leq 1$, and $0 \leq p_c \leq \max_{\mu \in \mathcal{M}} \left(\sum_{\omega \in \Omega} Q_{\mu,\omega} \cdot (\mathrm{W}_c(\omega)) \right)$, and so by the duality theorem of linear programming $\mathcal{WL}$ has an optimal solution, that can be taken at an extreme point of its feasible region. If we let B be the corresponding basis matrix, then this optimal solution of $\mathcal{WL}$ can be written as

$$(\bar{\boldsymbol{\theta}}, \bar{\boldsymbol{w}}) = B^{-1} \begin{bmatrix} \sum_{\omega \in \Omega} \mu_1(\omega) Z^a(\omega) \\ \vdots \\ \sum_{\omega \in \Omega} \mu_{|\mathcal{E}_a|}(\omega) Z^a(\omega) \\ 0 \end{bmatrix}.$$

Observe that every basis matrix for $\mathcal{WL}$ has elements that are 1, -1, or $\mu_k(\omega)\mathrm{W}_c(\omega)$, which are fixed parameters. By assumption $Z^a(\omega)$, $a \in \mathcal{A}$, $\omega \in \Omega$ lies in a bounded set, and so

$$\left\{ B^{-1} \begin{bmatrix} \sum_{\omega\in\Omega} \mu_1(\omega) Z^a(\omega) \\ \vdots \\ \sum_{\omega\in\Omega} \mu_{|\mathcal{E}_a|}(\omega) Z^a(\omega) \\ 0 \end{bmatrix} : B \text{ is a basis matrix for } \mathcal{WL},\, a \in \mathcal{A}, \omega \in \Omega \right\}$$

is bounded as required. ∎

LEMMA 4. *$(\mathcal{G}, \mathcal{S}, \mathcal{M}, \mathcal{C}, \mathcal{P})$ has a Nash equilbrium when Arrow-Debreu securities are traded.*

Proof. We know from Lemma 2 that $Z^a(\omega)$, $a \in \mathcal{A}$, $\omega \in \Omega$ lies in a bounded set. Lemma 3 then shows that any Nash equilibrium for $(\mathcal{C}, \mathcal{P})$ lies in a bounded set, and the proof of Lemma 2 then shows that any Nash equilibrium for $(\mathcal{G}, \mathcal{S}, \mathcal{M})$ lies in a bounded set. Note that these bounded sets are dependent only on problem parameters (and not values of decision variables). Thus we can apply Rosen's theorem to show that there exists a Nash equilibrium for $(\mathcal{C}, \mathcal{P})$ and there exists a Nash equilibrium for $(\mathcal{G}, \mathcal{S}, \mathcal{M})$, satisfying the joint KKT conditions of $(\mathcal{G}, \mathcal{S}, \mathcal{M})$ and the joint KKT conditions of $(\mathcal{C}, \mathcal{P})$. Since these are the same as the joint KKT conditions of $(\mathcal{G}, \mathcal{S}, \mathcal{M}, \mathcal{C}, \mathcal{P})$, these solutions yield a Nash equilibrium for $(\mathcal{G}, \mathcal{S}, \mathcal{M}, \mathcal{C}, \mathcal{P})$. ∎

THEOREM 11. *Under assumptions 7, 8, 9, and 10, the equilibrium for the overall problem that combines problems (AP, IP, SM, CM) exists when agents can trade Arrow-Debreu securities.*

Proof. Follows directly from Lemma 1 and Lemma 4. ∎

7.4. Existence of equilibrium in trading contracts for differences

Consider a set of agents $\mathcal{A}$ who trade contracts for differences through a market agent. Each agent $a \in \mathcal{A}$ has a coherent risk measure defining a convex compact risk set $\mathcal{M}^a$, and wants to improve the risk-adjutsted return from their random costs Z^a by trading contracts. If contract c is bought at price p_c and pays out based on the spot price in node i then the payoff for an agent buying the contract is $\sum_{b\in\mathcal{B}} \mathrm{T}(\omega, b)\pi_i(\omega, b) / \sum_{b\in\mathcal{B}} \mathrm{T}(\omega, b) - p_c$.

We want to demonstrate that there exists a contract price p and a contract trade w^a for each agent a, that together clear the market and give a minimum risk-adjusted cost for each agent. To do this we establish an equivalent non-cooperative game $(\mathcal{C}, \mathcal{P})$ between the agents and the market agent. In order to simplify the analysis we restrict attention to a contract in a single node system with one load block. The multidimensional case is derived by de Maere dAertrycke and Smeers (2013) who give a proof of bounded contract quantities that generalises the argument presented here. Given $p \in \mathbb{R}$, each agent a selling w^a contracts at p aims to solve

$$\mathcal{C}^a(\boldsymbol{Z}^a, \boldsymbol{\pi}, p) : \min_{w^a} \left\{ \max_{\mathbb{Q} \in \mathcal{M}^a} [\mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a) - w^a(\mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}] - p)] \right\}.$$

Given $w^a, a \in \mathcal{A}$, the market agent aims to choose $p \geq 0$ to solve

$$\mathcal{P}(w) : \min_{p \geq 0} -p \left(\sum_{a \in \mathcal{A}} w^a \right).$$

A Nash equilibrium of the game $(\mathcal{C}, \mathcal{P})$ is a set of contract quantities $w^a, a \in \mathcal{A}$ and a price p with w^a solving $\mathcal{C}^a(\boldsymbol{Z}^a, \boldsymbol{\pi}, p)$ and p solving $\mathcal{P}(w)$. Observe that $\mathcal{P}(w)$ has a solution only when $\sum_{a \in \mathcal{A}} w^a \leq 0$, and its optimal value is 0.

Let $\overline{\mathcal{M}} := \cap_{a \in \mathcal{A}} \mathcal{M}_a$ represent the intersection of the risk sets. Also, for each agent $a \in \mathcal{A}$ define the set of *admissible contract prices*,

$$\mathrm{P}^a = \{p \in \mathbb{R} | \exists \mathbb{Q} \in \mathcal{M}^a : p = \mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}]\}.$$

(This is called the set of "not too attractive prices" by de Maere dAertrycke and Smeers (2013). An admissible contract price is one at which an agent cannot trade contracts to make an infinite risk-adjusted profit.)

LEMMA 5. P^a *is convex.*

Proof. If $p_1, p_2 \in \mathrm{P}^a$ and $\lambda \in (0,1)$ then $p_i = \mathbb{E}_{\mathbb{Q}_i}[\boldsymbol{\pi}]$, so

$$\begin{aligned}
(1-\lambda)p_1 + \lambda p_2 &= (1-\lambda)\mathbb{E}_{\mathbb{Q}_1}[\boldsymbol{\pi}] + \lambda \mathbb{E}_{\mathbb{Q}_2}[\boldsymbol{\pi}] \\
&= (1-\lambda) \int \pi d\mathbb{Q}_1 + \lambda \int \pi d\mathbb{Q}_2 \\
&= \int \pi d((1-\lambda)\mathbb{Q}_1 + \lambda \mathbb{Q}_2) \\
&= \mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}]
\end{aligned}$$

for some $\mathbb{Q} \in \mathcal{M}^a$, because $\mathcal{M}^a$ is convex. This establishes the result. ∎

LEMMA 6. *If $p \notin \mathrm{P}^a$, then the risk-adjusted disbenefit of agent a is unbounded below.*

Proof. Suppose there exists some M such that that for every w^a

$$\Pi_a = \max_{\mathbb{Q}\in\mathcal{M}^a} \left[\mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a) - w^a(\mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}] - p)\right] > \max_{\mathbb{Q}\in\mathcal{M}^a} \mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a) - M. \tag{57}$$

Since $\mathcal{M}^a$ is convex, $p \notin \mathrm{P}^a$ implies that either $p < \mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}]$ for every $\mathbb{Q} \in \mathcal{M}^a$ or $p > \mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}]$ for every $\mathbb{Q} \in \mathcal{M}^a$. In the former case set

$$w^a = \frac{M}{\min_{\mathbb{Q}'\in\mathcal{M}^a} \mathbb{E}_{\mathbb{Q}'}[\boldsymbol{\pi}] - p}.$$

Then

$$\begin{aligned}
\Pi_a &= \max_{\mathbb{Q}\in\mathcal{M}^a} \left[\mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a) - w^a(\mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}] - p)\right] \\
&\leq \max_{\mathbb{Q}\in\mathcal{M}^a} \left[\mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a) - \left(\frac{M}{\min_{\mathbb{Q}'\in\mathcal{M}^a} \mathbb{E}_{\mathbb{Q}'}[\boldsymbol{\pi}] - p}\right)\left(\min_{\mathbb{Q}'\in\mathcal{M}^a} \mathbb{E}_{\mathbb{Q}'}[\boldsymbol{\pi}] - p\right)\right] \\
&= \max_{\mathbb{Q}\in\mathcal{M}^a} \mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a) - M,
\end{aligned}$$

which contradicts (57).

In the latter case $p > \mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}]$ we set

$$w^a = -\frac{M}{p - \max_{\mathbb{Q}'\in\mathcal{M}^a} \mathbb{E}_{\mathbb{Q}'}[\boldsymbol{\pi}]}$$

so

$$\begin{aligned}
\Pi_a &= \max_{\mathbb{Q}\in\mathcal{M}^a} \left[\mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a) - w^a(\mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}] - p)\right] \\
&\leq \max_{\mathbb{Q}\in\mathcal{M}^a} \left[\mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a) + \left(\frac{M}{p - \max_{\mathbb{Q}'\in\mathcal{M}^a} \mathbb{E}_{\mathbb{Q}'}[\boldsymbol{\pi}]}\right)\left(\max_{\mathbb{Q}'\in\mathcal{M}^a} \mathbb{E}_{\mathbb{Q}'}[\boldsymbol{\pi}] - p\right)\right] \\
&= \max_{\mathbb{Q}\in\mathcal{M}^a} \mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a) - M,
\end{aligned}$$

which contradicts (57). Thus the risk-adjusted disbenefit of agent a is unbounded below when $p \notin \mathrm{P}^a$. ∎

LEMMA 7. *If $p \in \mathrm{int}\mathrm{P}^a$, then the optimal risk-adjusted disbenefit of agent a is bounded.*

Proof. Suppose the risk-adjusted disbenefit for a is unbounded. Then for every $n>0$, there is some $w^a(n)$ such that

$$\max_{\mathbb{Q}\in\mathcal{M}^a}\left[\mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a)-w^a(n)(\mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}]-p)\right]<-n. \tag{58}$$

Given $w^a(n)$ let the maximum of $\mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a)-w^a(n)(\mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}]-p)$ be attained by $\mathbb{Q}_n^*$. Then (58) implies that $w^a(n)$ has the same sign as $\mathbb{E}_{\mathbb{Q}_n^*}[\boldsymbol{\pi}]-p$. There is an infinite subsequence $\mathcal{S}$ with either $w^a(n)>0$ for every n or $w^a(n)<0$ for every $n\in\mathcal{S}$. Without loss of generality assume the former. Then for every $n\in\mathcal{S}$,

$$p<\mathbb{E}_{\mathbb{Q}_n^*}[\boldsymbol{\pi}],\text{ and } w^a(n)>0,$$

and $w^a(n)\to\infty$. Let

$$\mathbb{E}_{\bar{\mathbb{Q}}}[\boldsymbol{\pi}]=\min_{\mathbb{Q}}\mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}].$$

Suppose $\mathbb{E}_{\bar{\mathbb{Q}}}[\boldsymbol{\pi}]<p$. Then for large enough n

$$\mathbb{E}_{\bar{\mathbb{Q}}}(\boldsymbol{Z}^a)-w^a(n)(\mathbb{E}_{\bar{\mathbb{Q}}}[\boldsymbol{\pi}]-p)>\max_{\mathbb{Q}\in\mathcal{M}^a}\mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a)$$

and so

$$\mathbb{E}_{\bar{\mathbb{Q}}}(\boldsymbol{Z}^a)-w^a(n)(\mathbb{E}_{\bar{\mathbb{Q}}}[\boldsymbol{\pi}]-p)\le\mathbb{E}_{\mathbb{Q}_n^*}(\boldsymbol{Z}^a)-w^a(n)(\mathbb{E}_{\mathbb{Q}_n^*}[\boldsymbol{\pi}]-p)$$

yields

$$\max_{\mathbb{Q}\in\mathcal{M}^a}\left[\mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a)-w^a(n)(\mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}]-p)\right]>\max_{\mathbb{Q}\in\mathcal{M}^a}\left[\mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a)\right].$$

This contradicts (58), so $\min_{\mathbb{Q}}\mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}]\ge p$, which is equivalent to $\mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}]-p\ge 0$ for all $\mathbb{Q}\in\mathcal{M}^a$, or $p\notin\text{int}\mathrm{P}^a$. ∎

The preceding lemmas show that the contract trading problem for each agent is bounded if $p\in\text{int}\mathrm{P}^a$ and is unbounded if $p\notin\mathrm{P}^a$. To show that an equilibrium exists we need to bound w^a even when p is on the boundary of P^a. This is possible if we take account of the counterparties in a contract trade. An equilibrium will exist if the sets of admissible contract prices for all agents intersect. This will follow from Assumption 8.

The following lemma establishes that in equilibrium the choice of p will be in P^a, and the contract trading problem for each agent will admit a bounded solution as long as Assumption 8 holds.

LEMMA 8. *If Assumption 8 holds then there exists an equilibrium to* $(\mathcal{C},\mathcal{P})$.

Proof. For $n = 1, 2, \ldots$, consider the noncooperative game $(\mathcal{C}(n), \mathcal{P}(n))$ where agent a solves

$$\mathcal{C}^a(n)(\boldsymbol{Z}^a, \boldsymbol{\pi}, p) : \min_{w^a \in \mathcal{W}^a(n)} \left\{ \max_{\mathbb{Q} \in \mathcal{M}^a} \left[\mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a) - w^a(\mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}] - p) \right] \right\}.$$

and the market agent aims to choose p to solve

$$\mathcal{P}(w) : \min_{p \in \mathcal{P}} -p \cdot \left(\sum_{a \in \mathcal{A}} w^a \right),$$

where $\mathcal{W}^a(n) := \{w^a : |w^a| \leq n\}$, and $\mathcal{P} = [\min_{\omega \in \Omega} \pi_\omega - 1, \max_{\omega \in \Omega} \pi_\omega + 1]$. Given a bounded strategy space for each agent, it follows from Debreu (1952) that $(\mathcal{C}(n), \mathcal{P}(n))$ has at least one equilibrium $(w(n), p(n))$.

If, in this equilibrium, $\sum_{a \in \mathcal{A}} w^a(n) < 0$, then it follows that $p(n) = \min_{\omega \in \Omega} \pi_\omega - 1$. Each agent a then solves

$$\min_{w^a \in \mathcal{W}^a(n)} \left\{ \max_{\mathbb{Q} \in \mathcal{M}^a} \left[\mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a) - w^a(\mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}] - (\min_{\omega \in \Omega} \pi_\omega - 1)) \right] \right\}.$$

Since for every $\mathbb{Q} \in \mathcal{M}^a$, $\mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}] > \min_{\omega \in \Omega} \pi_\omega - 1$, we have $w^a(n) = n$, which contradicts $\sum_{a \in \mathcal{A}} w^a(n) < 0$. Thus

$$\sum_{a \in \mathcal{A}} w^a(n) \geq 0. \tag{59}$$

If in this equilibrium $\sum_{a \in \mathcal{A}} w^a(n) > 0$, then it follows that $p(n) = \max_{\omega \in \Omega} \pi_\omega + 1$. Each agent a then solves

$$\min_{w^a \in \mathcal{W}^a(n)} \left\{ \max_{\mathbb{Q} \in \mathcal{M}^a} \left[\mathbb{E}_{\mathbb{Q}}(\boldsymbol{Z}^a) - w^a(\mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}] - (\max_{\omega \in \Omega} \pi_\omega + 1)) \right] \right\}.$$

Since for every $\mathbb{Q} \in \mathcal{M}^a$, $\mathbb{E}_{\mathbb{Q}}[\boldsymbol{\pi}] < \max_{\omega \in \Omega} \pi_\omega + 1$, we have $w^a(n) = -n$, which contradicts $\sum_{a \in \mathcal{A}} w^a(n) > 0$. Thus

$$\sum_{a \in \mathcal{A}} w^a(n) \leq 0. \tag{60}$$

Combining (59) and (60) these contracts satisfy

$$\sum_{a \in \mathcal{A}} w^a(n) = 0. \tag{61}$$

In order to prove existence of equilibrium we show that there exists a solution $(w(n), p(n))$ with $w^a(n)$ in the interior of $\mathcal{W}^a(n)$ for every a. We assume the contrary and derive a contradiction. So suppose that for every $n > 0$, there is some player $\hat{a}_n$ with $w^{\hat{a}_n}(n) = n$ that solves $\mathcal{C}_n^{\hat{a}_n}(Z^{\hat{a}_n}, p)$ (A similar argument will yield a contradiction if we assume $w^{\hat{a}_n}(n) = -n$). If for every $n > 0$, there is such a player, then there must be a non-empty subset of agents $\hat{\mathcal{A}}_n \subset \mathcal{A}$ with $\sum_{a\in\hat{\mathcal{A}}_n} w^a \leq -n$. The sequence $\{\hat{a}_n, \hat{\mathcal{A}}_n\}$ is finite valued and so it has an infinite subsequence with every element $(\hat{a}_n, \hat{\mathcal{A}}_n) = (\hat{a}, \hat{\mathcal{A}})$ for some $\hat{a}$ and $\hat{\mathcal{A}}$. Without loss of generality, we now trim the sequence to this subsequence and continue using n for the index.

By assumption, for every $n > 0$, $w^{\hat{a}} = n$ minimises

$$\max_{\mathbb{Q}\in\mathcal{M}^{\hat{a}}} \left\{\mathbb{E}_{\mathbb{Q}}[Z^{\hat{a}}] - w^{\hat{a}}(\mathbb{E}_{\mathbb{Q}}[\pi] - p(n))\right\}$$

over $\mathcal{W}^{\hat{a}}(n)$. Choose $\bar{\mathbb{Q}} \in \mathcal{M}^{\hat{a}}$ that satisfies

$$\mathbb{E}_{\bar{\mathbb{Q}}}[\pi] = \min_{\mathbb{Q}\in\mathcal{M}^{\hat{a}}} \mathbb{E}_{\mathbb{Q}}[\pi].$$

Suppose there is some $\varepsilon > 0$ such that for every N, there is some $n > N$ with $p(n) > \mathbb{E}_{\bar{\mathbb{Q}}}[\pi] + \varepsilon$. Then choosing N sufficiently large gives $n > N$ with

$$\begin{aligned}\mathbb{E}_{\bar{\mathbb{Q}}}[Z^{\hat{a}}] - n(\mathbb{E}_{\bar{\mathbb{Q}}}[\pi] - p(n)) &> \mathbb{E}_{\bar{\mathbb{Q}}}[Z^{\hat{a}}] + n\varepsilon \\ &> \max_{\mathbb{Q}\in\mathcal{M}^{\hat{a}}} \mathbb{E}_{\mathbb{Q}}[Z^{\hat{a}}],\end{aligned}$$

yielding

$$\max_{\mathbb{Q}\in\mathcal{M}^{\hat{a}}} \left\{\mathbb{E}_{\mathbb{Q}}[Z^{\hat{a}}] - n(\mathbb{E}_{\mathbb{Q}}[\pi] - p(n))\right\} > \max_{\mathbb{Q}\in\mathcal{M}^{\hat{a}}} \mathbb{E}_{\mathbb{Q}}[Z^{\hat{a}}].$$

It follows that $w^{\hat{a}} = n$ does not minimise

$$\max_{\mathbb{Q}\in\mathcal{M}^{\hat{a}}} \left\{\mathbb{E}_{\mathbb{Q}}[Z^{\hat{a}}] - w^{\hat{a}}(\mathbb{E}_{\mathbb{Q}}[\pi] - p(n))\right\}$$

as assumed. Thus for every $\varepsilon > 0$ there is some N such that $n > N$ implies $p(n) \leq \min_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}[\pi] + \varepsilon$. This means that we can extract a further subsequence of $(w(n), p(n))$ with $\varepsilon_n \to 0$, and

$$\mathbb{E}_{\mathbb{Q}}[\pi] - p(n) \geq -\varepsilon_n \qquad \forall\mathbb{Q} \in \mathcal{M}^{\hat{a}}. \tag{62}$$

Now if $\sum_{a\in\hat{\mathcal{A}}} w^a(n) \leq -n$, then there is an agent $a \in \hat{\mathcal{A}}$ with an optimal choice of contract $w^a(n) \to -\infty$ as $n \to \infty$. For this agent we have

$$\max_{\mathbb{Q}\in\mathcal{M}^a} \{\mathbb{E}_{\mathbb{Q}}[Z^a] - w^a(n)(\mathbb{E}_{\mathbb{Q}}[\pi] - p(n))\} \leq \max_{\mathbb{Q}\in\mathcal{M}^a} \mathbb{E}_{\mathbb{Q}}[Z^a]. \tag{63}$$

Now let

$$\mathbb{E}_{\bar{\mathbb{Q}}}[\pi] = \max_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}[\pi].$$

Suppose there is some $\varepsilon > 0$ such that for every N, there is some $n > N$ with $p(n) < \mathbb{E}_{\bar{\mathbb{Q}}}[\pi] - \varepsilon$. Then (since $w^a(n) \to -\infty$) choosing N sufficiently large gives $n > N$ with

$$\begin{aligned}\mathbb{E}_{\bar{\mathbb{Q}}}[Z^a] - w^a(n)(\mathbb{E}_{\bar{\mathbb{Q}}}[\pi] - p(n)) &> \mathbb{E}_{\bar{\mathbb{Q}}}[Z^a] + |w^a(n)|\,\varepsilon \\ &> \max_{\mathbb{Q}\in\mathcal{M}^{\hat{a}}} \mathbb{E}_{\mathbb{Q}}[Z^a],\end{aligned}$$

yielding

$$\max_{\mathbb{Q}\in\mathcal{M}^{\hat{a}}} \{\mathbb{E}_{\mathbb{Q}}[Z^a] - w^a(n)(\mathbb{E}_{\mathbb{Q}}[\pi] - p(n))\} > \max_{\mathbb{Q}\in\mathcal{M}^{\hat{a}}} \mathbb{E}_{\mathbb{Q}}[Z^a].$$

This contradicts (63) so for every $\varepsilon > 0$ there is some N such that $n > N$ implies $p(n) \geq \max_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}[\pi] - \varepsilon$. This means that we can extract a subsequence of $(w(n), p(n))$ with $\varepsilon_n \to 0$, and

$$\mathbb{E}_{\mathbb{Q}}[\pi] - p(n) \leq \varepsilon_n \qquad \forall \mathbb{Q} \in \mathcal{M}^a. \tag{64}$$

Now $p(n)$ is bounded, so we can extract a convergent subsequence (lying in both the subsequences in (62) and (64)) for which $p(n) \to \bar{p}$, which yields

$$\mathbb{E}_{\mathbb{Q}}[\pi] - \bar{p} \geq 0 \qquad \forall \mathbb{Q} \in \mathcal{M}^{\hat{a}},$$

$$\mathbb{E}_{\mathbb{Q}}[\pi] - \bar{p} \leq 0 \qquad \forall \mathbb{Q} \in \mathcal{M}^a.$$

Thus π defines the normal of a hyperplane that separates $\mathrm{ri}\mathcal{M}^{\hat{a}}$ and $\mathrm{ri}\mathcal{M}^a$. It follows that

$$\mathrm{ri}\mathcal{M}^{\hat{a}} \cap \mathrm{ri}\mathcal{M}^a = \emptyset$$

which contradicts Assumption 8.

It follows that it is not true that for every $n > 0$, there is some player $\hat{a}(n)$ with $w^{\hat{a}(n)}(n) = n$, and no value of $w^{\hat{a}} < n$ solves $\mathcal{C}(n)$. In other words for some n there is an equilibrium for $(\mathcal{C}(n), \mathcal{P}(n))$ in which every player a has $w^a < n$. In this equilibrium the solution w^a satisfies the optimality conditions of $\mathcal{C}$. So (w^a, p) is an equilibrium for $(\mathcal{C}, \mathcal{P})$. ∎

LEMMA 9. *$(\mathcal{G}, \mathcal{S}, \mathcal{M}, \mathcal{C}, \mathcal{P})$ has a Nash equilbrium when contracts for differences are traded.*

Proof. We know from Lemma 2 that $Z^a(\omega)$, $a \in \mathcal{A}$, $\omega \in \Omega$ lies in a bounded set. Lemma 8 then shows that any Nash equilibrium for $(\mathcal{C}, \mathcal{P})$ lies in a bounded set, and the proof of Lemma 2 then shows that any Nash equilibrium for $(\mathcal{G}, \mathcal{S}, \mathcal{M})$ lies in a bounded set. Note that these bounded sets are dependent only on problem parameters (and not values of decision variables). Thus we can apply Rosen's theorem to show that there exists a Nash equilibrium for $(\mathcal{C}, \mathcal{P})$ and there exists a Nash equilibrium for $(\mathcal{G}, \mathcal{S}, \mathcal{M})$, satisfying the joint KKT conditions of $(\mathcal{G}, \mathcal{S}, \mathcal{M})$ and the joint KKT conditions of $(\mathcal{C}, \mathcal{P})$. Since these are the same as the joint KKT conditions of $(\mathcal{G}, \mathcal{S}, \mathcal{M}, \mathcal{C}, \mathcal{P})$, these solutions yield a Nash equilibrium for $(\mathcal{G}, \mathcal{S}, \mathcal{M}, \mathcal{C}, \mathcal{P})$. ∎

It is worth mentioning two degenerate instances for $(\mathcal{C}, \mathcal{P})$. If π_ω is the same across all scenarios then all contracts have zero payoff. An equilibrium exists in which $w^a = 0$, $p = \pi$. A second instance is when $\cap_{a \in \mathcal{A}} \mathcal{M}_a$ is a singleton $\mathbb{P}$, and so has no relative interior. This would occur for example when agents are risk neutral. Then all contracts are traded at $p = \mathbb{E}_{\mathbb{P}}[\boldsymbol{\pi}]$, and have zero payoff. So an equilibrium exists in which $w^a = 0$.

THEOREM 12. *Under assumptions 8, 9, and 10, the equilibrium for the overall problem that combines problems (AP, IP, SM, CM) exists when agents have coherent risk measures and can trade contracts for differences.*

Proof. Follows directly from Lemma 1 and Lemma 9. ∎

7.5. Appendix B.1: Stacked KKT conditions: Model with variational inequalities

Here we take the individual optimisation problems for each of the agents (gentailer, transmission operator, and market side constraints) and state the KKT conditions of the equilibrium model. For readability, we substitute

$$\Psi^a(\omega) = \begin{pmatrix} \sum_{c \in \mathcal{C}} (p_c - (\mathrm{W}_c(\omega, \boldsymbol{\pi}(\omega)))) \cdot w_c^a \\ + \sum_{i \in \mathcal{I}, k \in \mathcal{K}} \mathrm{xC}_k \cdot x_{i,k}^a + \sum_{i \in \mathcal{I}, k \in \mathcal{K}} \mathrm{oC}_k \cdot z_{i,k}^a \\ + \sum_{b \in \mathcal{B}} \mathrm{T}(\omega, b) \cdot \sum_{i \in \mathcal{I}, k \in \mathcal{K}} (\mathrm{gC}_k(\omega, b) - \pi_i(\omega, b)) \cdot y_{i,k}^a(\omega, b) \\ + \sum_{b \in \mathcal{B}} \mathrm{T}(\omega, b) \cdot \sum_{i \in \mathcal{I}, h \in \mathcal{H}} (\pi_i(\omega, b) - \mathrm{r}_h) \cdot (\mathrm{d}_{i,h}^a(\omega, b) - q_{i,h}^a(\omega, b)) \\ + \sum_{b \in \mathcal{B}} \mathrm{T}(\omega, b) \cdot \sum_{i \in \mathcal{I}, h \in \mathcal{H}} \mathrm{v}_h \cdot q_{i,h}^a(\omega, b) \end{pmatrix} \quad \forall a \in \mathcal{A}, \omega \in \Omega. \quad (65)$$

AP: Generation agent problem $(\forall a \in \mathcal{A})$

$$\text{P:} \quad 0 \geq z^a_{i,k} - x^a_{i,k} - \mathrm{k}^a_{i,k} \quad \perp \quad \xi^a_{i,k} \geq 0 \quad \forall i \in \mathcal{I}, k \in \mathcal{K} \tag{66}$$

$$0 \geq y^a_{i,k}(\omega,b) - m_k(\omega,b) \cdot z^a_{i,k} \quad \perp \quad \mu^a_{i,k}(\omega,b) \geq 0 \quad \forall i \in \mathcal{I}, k \in \mathcal{K}, \omega \in \Omega, b \in \mathcal{B} \tag{67}$$

$$0 \geq \sum_{b \in \mathcal{B}} \mathrm{T}(\omega,b) \cdot y^a_{i,k}(\omega,b) - n_k(\omega) \cdot z^a_{i,k} \quad \perp \quad \nu^a_{i,k}(\omega) \geq 0 \quad \forall i \in \mathcal{I}, k \in \mathcal{K}, \omega \in \Omega \tag{68}$$

$$\text{D:} \quad 0 \geq \frac{\partial \rho^a(\Psi^a)}{\partial z^a_{i,k}} - \sum_{\omega \in \Omega, b \in \mathcal{B}} \mathrm{m}_k(\omega,b) \cdot \mu^a_{i,k}(\omega,b) - \sum_{\omega \in \Omega} \mathrm{n}_k(\omega) \cdot \nu^a_{i,k}(\omega) + \xi^a_{i,k} \quad \perp \quad z^a_{i,k} \geq 0 \quad \forall i \in \mathcal{I}, k \in \mathcal{K} \tag{69}$$

$$\frac{\partial \rho^a(\Psi^a)}{\partial x^a_{i,k}} - \xi^a_{i,k} \quad \perp \quad \mathrm{u}^a_{i,k} \geq x^a_{i,k} \geq 0 \quad \forall i \in \mathcal{I}, k \in \mathcal{K} \tag{70}$$

$$0 \geq \frac{\partial \rho^a(\Psi^a)}{\partial y^a_{i,k}(\omega,b)} + \mu^a_{i,k}(\omega,b) + \mathrm{T}(\omega,b) \cdot \nu^a_{i,k}(\omega) \quad \perp \quad y^a_{i,k}(\omega,b) \geq 0 \quad \forall i \in \mathcal{I}, k \in \mathcal{K}, \omega \in \Omega, b \in \mathcal{B} \tag{71}$$

$$\frac{\partial \rho^a(\Psi^a)}{\partial q^a_{i,h}(\omega,b)} \quad \perp \quad \mathrm{d}^a_{i,h}(\omega,b) \geq q^a_{i,h}(\omega,b) \geq 0 \quad \forall i \in \mathcal{I}, h \in \mathcal{H}, \omega \in \Omega, b \in \mathcal{B} \tag{72}$$

$$0 = \frac{\partial \rho^a(\Psi^a)}{\partial w^a_c} \quad \perp \quad w^a_c \quad \forall c \in \mathcal{C} \tag{73}$$

IP: Transmission operator problem

$$P: 0 = \sum_{\substack{j: \\ ((i,j) \cup (j,i)) \in \mathcal{L}}} \frac{\mathrm{c}_{i,j}}{2} \cdot (f_{i,j}(\omega,b))^2 \tag{74}$$

$$-L_i(\omega,b) \quad \perp \quad \phi_i(\omega,b) \quad \forall i \in \mathcal{I}, \omega \in \Omega, b \in \mathcal{B} \tag{75}$$

$$0 = \sum_{(i,j) \in \mathcal{L}_e} \mathrm{s}_{i,j,e} \cdot f_{i,j}(\omega,b) \quad \perp \quad \theta_e(\omega,b) \quad \forall e \in \mathcal{E}, \omega \in \Omega, b \in \mathcal{B} \tag{76}$$

$$D: \quad \pi_i(\omega,b) - \pi_j(\omega,b) \tag{77}$$

$$-\mathrm{c}_{i,j} \cdot \phi_i(\omega,b) \cdot f_{i,j}(\omega,b) \tag{78}$$

$$-\mathrm{c}_{i,j} \cdot \phi_i(\omega,b) \cdot f_{i,j}(\omega,b) \tag{79}$$

$$+ \sum_{\substack{e \in \mathcal{E}: \\ (i,j) \in \mathcal{L}_e}} \mathrm{s}_{i,j,e} \cdot \theta_e(\omega,b) \quad \perp \quad \mathrm{f}^+_{i,j} \geq f_{i,j}(\omega,b) \geq \mathrm{f}^-_{i,j} \quad \forall (i,j) \in \mathcal{L}, \omega \in \Omega, b \in \mathcal{B} \tag{80}$$

$$0 = \qquad \phi_i(\omega,b) + \pi_i(\omega,b) \quad \perp \qquad L_i(\omega,b) \qquad \forall i \in \mathcal{I}, \omega \in \Omega, b \in \mathcal{B} \tag{81}$$

SM: Side constraints - Spot market

$$\sum_{a\in\mathcal{A},k\in\mathcal{K}} y^a_{i,k}(\omega,b) \tag{82}$$

$$+ \sum_{a\in\mathcal{A},h\in\mathcal{H}} \left[q^a_{i,h}(\omega,b) - \mathrm{d}^a_{i,h}(\omega,b)\right] \tag{83}$$

$$+ \sum_{\substack{j\in\mathcal{I}:\\(j,i)\in\mathcal{L}}} f_{j,i}(\omega,b) - \sum_{\substack{j\in\mathcal{I}:\\(i,j)\in\mathcal{L}}} f_{i,j}(\omega,b) - L_i(\omega,b) \quad \perp \qquad \bar{\mathrm{P}} \geq \pi_i(\omega,b) \geq 0 \quad \forall i \in \mathcal{I}, \omega \in \Omega, b \in \mathcal{B} \tag{84}$$

CM: Side constraints - Contract market

$$0 \leq -\sum_{a\in\mathcal{A}} w^a_c \perp p_c \geq 0 \quad \forall c \in \mathcal{C} \tag{85}$$

7.6. Appendix B.2: Stacked KKT conditions: Model with agents representing the market

Here we state the KKT conditions derived from the game comprising the physical investment problem, $\mathcal{G}$, the transmission problem, $\mathcal{S}$, the spot market agent, $\mathcal{M}$, each agent's contract investment problem, $\mathcal{C}$, and the contract market agent, $\mathcal{P}$.

$\mathcal{G}$: Generation agent investment and operation problem $(\forall a \in \mathcal{A})$

$$\text{P:}\quad 0 \geq z^a_{i,k} - x^a_{i,k} - \mathrm{k}^a_{i,k} \quad \perp \quad \xi^a_{i,k} \geq 0 \quad \forall i \in \mathcal{I}, k \in \mathcal{K} \tag{86}$$

$$0 \geq y^a_{i,k}(\omega,b) - m_k(\omega,b)\cdot z^a_{i,k} \quad \perp \quad \mu^a_{i,k}(\omega,b) \geq 0 \quad \forall i \in \mathcal{I}, k \in \mathcal{K}, \omega \in \Omega, b \in \mathcal{B} \tag{87}$$

$$0 \geq \sum_{b\in\mathcal{B}} \mathrm{T}(\omega,b)\cdot y^a_{i,k}(\omega,b) - n_k(\omega)\cdot z^a_{i,k} \quad \perp \quad \nu^a_{i,k}(\omega) \geq 0 \quad \forall i \in \mathcal{I}, k \in \mathcal{K}, \omega \in \Omega \tag{88}$$

$$\text{D:}\quad 0 \geq \frac{\partial \rho^a(\Psi^a)}{\partial z^a_{i,k}} - \sum_{\omega\in\Omega, b\in\mathcal{B}} \mathrm{m}_k(\omega,b)\cdot\mu^a_{i,k}(\omega,b) - \sum_{\omega\in\Omega} \mathrm{n}_k(\omega)\cdot\nu^a_{i,k}(\omega) + \xi^a_{i,k} \quad \perp \quad z^a_{i,k} \geq 0 \quad \forall i \in \mathcal{I}, k \in \mathcal{K} \tag{89}$$

$$\frac{\partial \rho^a(\Psi^a)}{\partial x^a_{i,k}} - \xi^a_{i,k} \quad \perp \quad \mathrm{u}^a_{i,k} \geq x^a_{i,k} \geq 0 \quad \forall i \in \mathcal{I}, k \in \mathcal{K} \tag{90}$$

$$0 \geq \frac{\partial \rho^a(\Psi^a)}{\partial y^a_{i,k}(\omega,b)} + \mu^a_{i,k}(\omega,b) + \mathrm{T}(\omega,b)\cdot\nu^a_{i,k}(\omega) \quad \perp \quad y^a_{i,k}(\omega,b) \geq 0 \quad \forall i \in \mathcal{I}, k \in \mathcal{K}, \omega \in \Omega, b \in \mathcal{B} \tag{91}$$

$$\frac{\partial \rho^a(\Psi^a)}{\partial q^a_{i,h}(\omega,b)} \quad \perp \quad \mathrm{d}^a_{i,h}(\omega,b) \geq q^a_{i,h}(\omega,b) \geq 0 \quad \forall i \in \mathcal{I}, h \in \mathcal{H}, \omega \in \Omega, b \in \mathcal{B} \tag{92}$$

$\mathcal{S}$: Transmission problem

$$P: 0 = \sum_{\substack{j:\\ ((i,j)\cup(j,i))\in\mathcal{L}}} \frac{\mathrm{c}_{i,j}}{2}\cdot(f_{i,j}(\omega,b))^2 \tag{93}$$

$$-L_i(\omega,b) \quad \perp \quad \phi_i(\omega,b) \quad \forall i \in \mathcal{I}, \omega \in \Omega, b \in \mathcal{B} \tag{94}$$

$$0 = \sum_{(i,j)\in\mathcal{L}_e} \mathrm{s}_{i,j,e}\cdot f_{i,j}(\omega,b) \quad \perp \quad \theta_e(\omega,b) \quad \forall e \in \mathcal{E}, \omega \in \Omega, b \in \mathcal{B} \tag{95}$$

$$D: \quad \pi_i(\omega,b) - \pi_j(\omega,b) \tag{96}$$

$$-\mathrm{c}_{i,j}\cdot\phi_i(\omega,b)\cdot f_{i,j}(\omega,b) \tag{97}$$

$$-\mathrm{c}_{i,j} \cdot \phi_i(\omega,b) \cdot f_{i,j}(\omega,b) \tag{98}$$

$$+ \sum_{\substack{e\in\mathcal{E}:\\(i,j)\in\mathcal{L}_e}} \mathrm{s}_{i,j,e} \cdot \theta_e(\omega,b) \quad \perp \quad \mathrm{f}^+_{i,j} \geq f_{i,j}(\omega,b) \geq \mathrm{f}^-_{i,j} \quad \forall (i,j)\in\mathcal{L}, \omega\in\Omega, b\in\mathcal{B} \tag{99}$$

$$0 = \quad \phi_i(\omega,b) + \pi_i(\omega,b) \quad \perp \quad L_i(\omega,b) \quad \forall i\in\mathcal{I}, \omega\in\Omega, b\in\mathcal{B} \tag{100}$$

$\mathcal{M}$: Spot market problem

$$D: \sum_{a\in\mathcal{A},k\in\mathcal{K}} y^a_{i,k}(\omega,b) \tag{101}$$

$$+ \sum_{a\in\mathcal{A},h\in\mathcal{H}} \left[q^a_{i,h}(\omega,b) - \mathrm{d}^a_{i,h}(\omega,b)\right] \tag{102}$$

$$+ \sum_{\substack{j\in\mathcal{I}:\\(j,i)\in\mathcal{L}}} f_{j,i}(\omega,b) - \sum_{\substack{j\in\mathcal{I}:\\(i,j)\in\mathcal{L}}} f_{i,j}(\omega,b) - L_i(\omega,b) \quad \perp \quad \bar{\mathrm{P}} \geq \pi_i(\omega,b) \geq 0 \quad \forall i\in\mathcal{I}, \omega\in\Omega, b\in\mathcal{B} \tag{103}$$

$\mathcal{C}$: Agent contract problem ($\forall a\in\mathcal{A}$)

$$D: 0 = \frac{\partial \rho^a(\Psi^a)}{\partial w^a_c} \perp w^a_c \quad \forall c\in\mathcal{C} \tag{104}$$

$\mathcal{P}$: Contract market agent problem

$$D: 0 \leq -\sum_{a\in\mathcal{A}} w^a_c \perp p_c \geq 0 \quad \forall c\in\mathcal{C} \tag{105}$$