Pro-competitive rationing in multi-unit auctions*

Pär Holmberg†

November 27, 2015

Abstract

In multi-unit auctions, such as auctions of commodities and securities, it is necessary to specify rationing rules to break ties between multiple marginal bids. The standard approach in the literature and in practice is to ration marginal bids proportionally. This paper shows how bidding can be made more competitive - and the auctioneer can increase its surplus - if the rationing rule instead gives increasing priority to bidders with a small volume of marginal bids at clearing prices closer to the reservation price. In comparison to standard rationing, such a rule can for beneficial circumstances have almost the same effect on the competitiveness of bids as a doubling of the number of bidders.

Key words: Divisible-good auctions, multi-unit auctions, rationing rules, bidding format

JEL Classification C72, D44, D45

---

*This paper was initiated when I was visiting the Electricity Policy Research Group (EPRG) at University of Cambridge and I am indebted to David Newbery and Daniel Ralph for very fruitful discussions. A first draft of the paper was circulated on May 7, 2014. I am very grateful for comments on the first and later drafts from Lawrence Ausubel, Alexis Bergès, Peter Cramton, Richard Green, Thomas Greve, Henrik Horn and Michael Pollitt. Nils-Henrik von der Fehr is acknowledged for his comments on a previous paper, which inspired me to analyse rationing rules beyond pro-rata on the margin rationing. I also want to acknowledge comments from participants at a seminar at IFN (May, 2014), a seminar at Stockholm School of Economics (May, 2014), IFN's conference *The Performance of Electricity Markets* in Stockholm (June, 2014), *Fourth Mannheim Energy conference (*May, 2015), 11^{th} *World Congress of the Econometric Society* in Montreal (August, 2015), *European Association for Research in Industrial Economics* in Munich (August, 2015), 10^{th} *Conference on the Economics of Energy and Climate Change* in Toulouse (September, 2015) and the Energy Finance Conference in London (September, 2015). Christina Lönnblad is acknowledged for proof-reading the paper. Financial support by the Torsten Söderberg Foundation, the Energy Agency and the Research Program *The Economics of Electricity Markets* is gratefully acknowledged.

†Research Institute of Industrial Economics (IFN), Stockholm. Associate Researcher, Electricity Policy Research Group (EPRG), University of Cambridge.

1 Introduction

A wide range of products, commodities and assets are traded in divisible-good or multi-unit auctions. For instance, auctions of electricity, treasury bills and emission permits as well as financial exchanges, all allow bids for more than one unit of the traded items. In multi-unit auctions, each bidder submits a stack of bids, where each bid specifies a bid price and a bid quantity such that the bidder is willing to trade the specified bid quantity at the specified bid price or better. Unless by coincidence, at least some of the marginal bids with a bid price equal to the clearing price would have to be partly rejected. It is therefore necessary to specify a rationing rule that breaks ties in multi-unit auctions. Rationing rules are of particular importance for the outcome in auctions where bid prices accumulate at a few price levels, as usually happens in financial exchanges[1], frequent batch auctions[2] and auctions of financial securities. Multi-round auctions (such as clock-auctions[3]) where the auctioneer significantly restricts the number of price levels that a bidder can use is another application where rationing is important. The purpose of this paper is to study how rationing rules can be designed to increase the competition among a set of bidders, to the benefit of the auctioneer.

Most auctions use price priority. This means that all sell bids with a bid price below the clearing price are accepted and all buy bids with a bid price above the clearing price are accepted; only marginal bids are rationed. In single-round auctions, it is standard practice to ration marginal bids *pro-rata*, so that the same percentage of the marginal bid quantity is accepted from each bidder. In exchanges with continuous trading, it is also common to give priority to marginal bids that arrive early at the exchange; this is referred to as *price-time priority*. The IEX[4] exchange uses *price-broker-time* priority: this means that buy and sell orders at the same price from the same broker are matched before giving priority to early bids.[5] Field and Large (2012) empirically observe that, in comparison to price-time priority, pro-rata rationing significantly increases bid quantities in the order book of financial exchanges, but also the cancellation rate of bids. This verifies that the design of the rationing rule influences bidding behaviour in auctions also in practice.

I evaluate rationing rules in uniform-price auctions, where all accepted bids are transacted at the clearing price. Uniform-price auctions are, for example, used

[1] Financial exchanges normally restrict the number of permissible price levels in order to improve liquidity in the market (Harris, 1991; Angel, 1997; Angel, 2012).

[2] A frequent batch auction is a uniform-price sealed-bid double auction conducted at frequent but discrete time intervals. Frequent batch auctions can be used instead of continuous trading in exchanges (Budish et al., 2013).

[3] In a clock auction or Dutch auction, the auctioneer presents a price level and bidders submit a bid quantity that they are willing to trade at that price. The auction starts at the reservation price and the auctioneer's price is then iteratively moved further away from the reservation price until the auction is cleared. Ausubel (2004) outlines a clock auction for multiple objects.

[4] The IEX exchange is a new alternative financial exchange in U.S.. It tries to attract traders by operating according to more transparent rules.

[5] This is to encourage brokers to submit all their bids to the exchange, rather than matching them internally first.

in most wholesale electricity markets, in U.S. treasury auctions and in frequent batch auctions. I focus on the procurement auction, where the auctioneer buys items, but the results are analogous for sales auctions. In my model, a parameter $\mu_j \geq 0$ indicates the extent to which the rationing rule gives disproportionate priority to bidders with a large volume of marginal bids at a clearing price P_j. Obviously, a procurer benefits if bidders offer many items at low prices, thus a procurer would prefer a rationing rule that encourages bidders to specify large bid quantities at low bid prices, which corresponds to a large μ_j. However, bid stacks that result in large volumes of marginal bids when the clearing price is high should be discouraged by the auctioneer, as they will lead to less quantity being offered at low bid prices. Thus μ_j should be small for high clearing prices. Hence, disproportionality of the rationing rule should change with the clearing price, such that μ_j is lower for clearing prices closer to the reservation price.

I assume that each bidder submits a stack of $v+1$ sell bids with different bid prices and that the auctioneer wants to maintain the same pro-competitive effect at each bid price. In this case, I show that an optimal use of disproportionate rationing on the margin in an auction with N symmetric bidders gives the auctioneer approximately the same procurement cost as an auction with pro rata on the margin rationing and $\left(1+\frac{1}{v}\right)(N-1)+1 > N$ symmetric bidders with the same aggregate production cost. Thus, changing to the optimal rationing rule from pro-rata on the margin almost corresponds to a doubling of the number of bidders when each bidder submits a stack with two bid prices. The effect is smaller, the larger the number of bids per supplier.

It is optimal to let disproportionality of the rationing rule, the parameter μ_j, depend on the clearing price. Still, non-optimal rationing rules where μ_j is independent of the clearing price can also have a pro-competitive effect. This is for example the case when price increments (the difference between bid prices in the bid stack of a bidder) are non-constant. I show that a rationing rule that gives disproportionate priority to bidders with a small volume of marginal bids (e.g. $\mu_j = 0$) at every clearing price would boost competition if price increments of bidders are larger towards the reservation price. This would be the case if the bidding format requires large price increments near the reservation price and allow for smaller price increments towards the clearing price. Such bidding formats are sometimes used to speed up multi-round auctions, such as in Canadian spectrum auctions.

I use Nash equilibria of a static game to predict the bidding behaviour for different rationing rules. Analogous to Back and Zender (1993), Kremer and Nyborg (2004) and the literature on Supply Function Equilibria (SFE), I assume that costs are common knowledge among sellers, but the auctioneer is imperfectly informed of the costs. The auctioneer's demand is uncertain as in the SFE model. This uncertainty could also represent an uncertain amount of non-competitive bids (Kremer and Nyborg, 2004; Wang and Zender, 2002).[6] The SFE model is often

[6] Non-competitive bids just specify a bid quantity and are accepted irrespective of the clearing price. Non-competitive bids are for example allowed in U.S. treasury auctions. In financial exchanges, non-competitive bids are referred to as market orders. In wholesale electricity markets,

used to evaluate the design of wholesale electricity markets.[7] Analogous equilibria of demand functions from bidders in sales auctions have been used to evaluate treasury auctions in the U.S. (Wang and Zender, 2002).

In this paper, a stepped supply function is used to represent the bid stack of each bidder. Similar to Holmberg et al. (2013), I use a discrete version of Klemperer and Meyer's (1989) Supply Function Equilibrium (SFE) concept to analyse Nash equilibria of stepped supply functions. But I generalize Holmberg et al.'s (2013) model to allow for disproportionate rationing on the margin and non-constant price increments. I establish existence of symmetric Nash equilibria for simple cases with constant marginal costs, uniformly distributed demand shocks and two bids per supplier. My results for multiple bids per supplier, increasing marginal costs and non-uniform demand shocks follow from an approximation of the first-order condition for symmetric suppliers.

As in practice, rationing is on the margin in my analysis. However, Kremer and Nyborg (2004) show that an auctioneer could make equilibrium bids even more competitive if it is prepared to sidestep price priority and ration also infra-marginal bids. The spread rationing rule (SRR) and the concentrate rationing rule (CRR) examined by Saez et al. (2007) may also result in rationing of infra-marginal bids. Gresik (2001) proposes a new rule, ζ-rationing, where marginal bids (when possible) are rationed in proportion to the total amount that a bidder wants to trade at the marginal price. McAdams (2000) explores the extent to which rationing rules may provide the auctioneer with a tool for deterring collusive bidding. In order to ensure the existence of Nash equilibria in theoretical models of auctions, such as in papers by Deneckere and Kovenock (1996), Fabra et al. (2006), Simon and Zame (1990), and Jackson and Swinkels (1999), it is sometimes convenient to consider type-dependent rationing rules, for example where priority is given to the most efficient marginal bids, such as marginal sell bids with the lowest marginal cost. However, such rationing rules are difficult to apply in practice, where bidders' true costs/values are normally not observed by the auctioneer. The present paper is the first to use a rationing rule that depends on the clearing price, and that explores the advantages with such a rule.

Section 2 describes the setting of the game. The analysis is carried out in Section 3. Section 4 discusses some extensions that may be of practical relevance. Section 5 concludes the paper. All proofs are derived in the Appendix.

---

renewable production and production that cannot be switched off is often offered at the lowest possible price, which corresponds to a non-competitive bid.

[7] In electricity markets, technology characteristics and fuel prices are transparent and producers make offers before the demand for electricity has been realized (Anderson and Hu, 2008; Green and Newbery, 1992; Holmberg and Newbery, 2009). Observed offers match the first-order condition of a stepped SFE model so well that the theory cannot be rejected (Wolak, 2007). The continuous SFE model is less precise. In practice, it can only make accurate predictions of bids from large firms, whose submitted supply functions have many steps (Hortaçsu and Puller, 2008; Sioshansi and Oren, 2007).

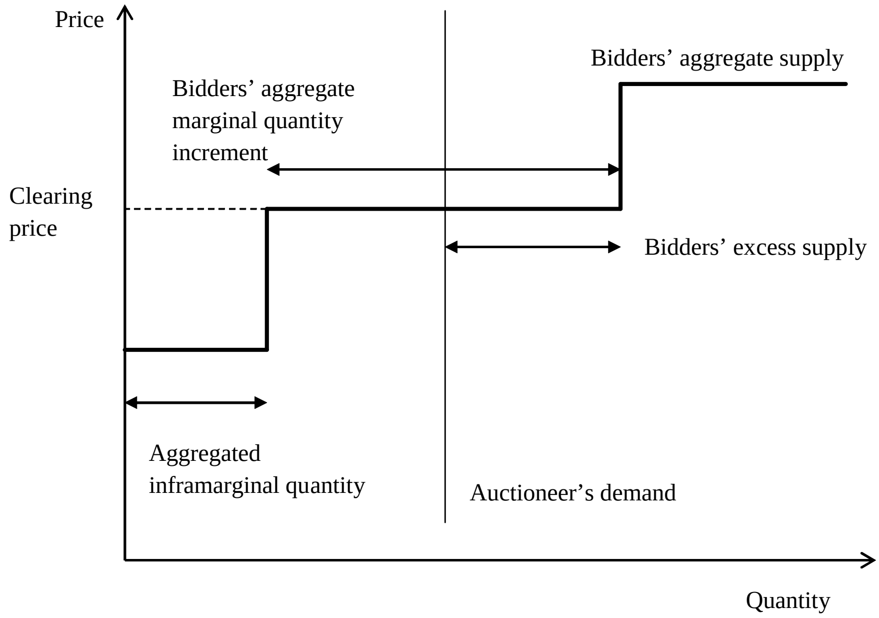

Figure 1: Clearing of and excess supply in the procurement auction.

2 Model

Consider a uniform-price procurement auction, so that all accepted bids are paid the Market Clearing Price (MCP). A stepped supply function is used to represent the bid stack of each bidder. As illustrated by Figure 1, the market is cleared at the lowest price where aggregated supply is larger than the auctioneer's demand. Any excess supply at the MCP is rationed on the margin. I calculate a pure strategy Nash equilibrium of a one-shot game, where each risk-neutral supplier chooses a step supply function to maximize its expected profit.

Similar to Holmberg et al. (2013) there are M permissible price levels, P_j, $j \in \{1, \ldots, M\}$, with the price tick $\Delta P_j = P_j - P_{j-1} > 0$. The minimum quantity increment is zero, i.e. quantities can be continuously varied. The difference to Holmberg et al. (2013) is that I now allow for non-constant tick-sizes and non-pro rata rationing on the margin. I let $r = \frac{\Delta P_j}{\Delta P_{j+1}}$, where it is assumed that r is a bounded positive constant.

Producer $i \in \{1, \ldots, N\}$ submits a supply vector $\boldsymbol{S}^i = \left\{S_j^i\right\}_{j=1}^M$consisting of the non-negative maximum quantities it is willing to produce at each permissible price level. The quantity increment $\Delta S_j^i = S_j^i - S_{j-1}^i$ is non-negative (the supply must be non-decreasing in the price). Let $\boldsymbol{S} = \left\{\boldsymbol{S}^i\right\}_{i=1}^N$ and denote competitors' collective offered quantity at price P_j as S_j^{-i} and total market supply at P_j as S_j. The cost function of supplier i, $C_i\left(S^i\right)$, is a smooth, increasing and convex function up to the capacity constraint k_i.[8] Let k be the total production capacity in the market. The auctioneer is imperfectly informed, but costs are common

[8]Production capacities in my procurement setting corresponds to purchase constraints in sales auctions. As an example, the U.S. Treasury auction has a 35% rule, which prevents anyone from

knowledge among suppliers. Klemperer and Meyer's (1989) continuous model is used as a benchmark. The set of individual smooth supply functions in the continuous model is given by $\{q_i(p)\}_{i=1}^N$.

The auctioneer's demand is perfectly inelastic up to the reservation price P_M. Demand is uncertain and given by the shock ε. The shock has a continuous probability density, $g(\varepsilon)$, with $\overline{g} \geq g(\varepsilon) \geq \underline{g} > 0$ on the support $[\underline{\varepsilon}, \overline{\varepsilon}]$. MCP is the lowest price at which the offered supply is (strictly) larger than the stochastic demand shock. Thus, the equilibrium price as a function of the demand shock, $P(\varepsilon)$, is right continuous, and the MCP equals P_j if $\varepsilon \in [S_{j-1}, S_j)$. Given chosen step supply functions, the market clearing price can be calculated for each demand shock in the interval $[\underline{\varepsilon}, \overline{\varepsilon}]$. The lowest and highest prices that are realized are denoted by P_L and P_H, respectively, where $1 \leq L < H \leq M$. I let $s(\varepsilon)$ and $s_i(\varepsilon)$ be total accepted supply and supplier i's accepted supply at ε, respectively.

2.1 The rationing rule

I consider a new class of rules with rationing on the margin and where disproportionality of the rationing rule depends on the clearing price. The rules are such that any bid accepted for some demand shock ε_0 is also accepted for any $\varepsilon > \varepsilon_0$. This means that a bidder's acceptance is monotonic with respect to the demand shock, so that $\frac{ds_i(\varepsilon)}{d\varepsilon} \geq 0$. For a given set of supply schedules, the outcome of the auction is the same (irrespective of the sharing rule) when there is no excess supply at MCP, i.e. $S_{j-1} = \varepsilon$. In this case, we have:

$$s_i(S_{j-1}) \equiv S_{j-1}^i. \tag{1}$$

The rationing rule determines how to accept bids when $S_{j-1} < \varepsilon < S_j$. In this paper, the parameter μ_j determines the non-linearity of the sharing rule at the clearing price P_j, i.e. the extent to which large quantity increments at this clearing price are given priority to small increments. The increment of producer i's accepted supply Δs_i for a shock increment $\Delta\varepsilon$ is determined by its volume of marginal bids that were not accepted at demand ε, i.e. $S_j^i - s_i(\varepsilon)$, how it relates to the total volume of marginal bids that were not accepted at ε, and the non-linearity of the rationing rule, μ_j.

$$\frac{ds_i(\varepsilon)}{d\varepsilon} = \frac{\left(S_j^i - s_i(\varepsilon)\right)^{\mu_j}}{\sum_{k=1}^N \left(S_j^k - s_k(\varepsilon)\right)^{\mu_j}} \text{ if } \varepsilon \in \left(S_{j-1}, S_j\right). \tag{2}$$

I consider $\mu_j \geq 0$, so that the rationing rule results in monotonic acceptance (in absolute terms) in the sense that a larger quantity increment at the marginal price will (weakly) increase the accepted volume from marginal bids of the supplier. Similarly, the rationing rule gives monotonic rejection (in absolute terms), i.e. a larger quantity increment at the marginal price will also (weakly) increase the rejected volume from marginal bids of a supplier. For $\mu_j = 1$, we get pro rata on

buying more than 35% of the auctioneer's supply. This is to avoid a situation where a single bidder can corner the market.

the margin rationing, where any additional demand $\Delta\varepsilon$ is allocated in proportion to a supplier's unmet supply at the clearing price, $S_j^i - s_i(\varepsilon)$.[9] It follows from (2) that with $\mu_j > 1$, disproportionate priority is given to producers with large unmet supply at the clearing price. When $\mu_j \to \infty$, $\Delta\varepsilon$ is shared equally among suppliers with the largest unmet supply at the clearing price, while suppliers with less unmet supply at P_j get no share of $\Delta\varepsilon$. We say that this rule gives maximum priority to large quantity increments at P_j (subject to rejection being monotonic for the rationing rule). The parameter range $0 \leq \mu_j < 1$ gives more priority to small quantity increments. In particular, $\mu_j = 0$ gives maximum priority to small quantity increments at P_j (subject to acceptance being monotonic for the rationing rule). In this case, all suppliers with unmet supply at the clearing price get the same share of any additional marginal demand increment $\Delta\varepsilon$. Note that

$$\sum_{i=1}^{N} \frac{ds_i(\varepsilon)}{d\varepsilon} \equiv 1, \tag{3}$$

i.e. the marginal increase in total accepted supply always equals the marginal shock increment, regardless of the rationing rule.

Together with the initial condition in (1), a system of differential equations of the type in (2), with one equation per bidder, can be used to calculate the accepted quantity for each supplier as a function of the demand shock for any given set of monotonic step supply functions.[10] From the supply $s_i(\varepsilon)$ allocated to each supplier, it is straightforward to calculate the supplier's expected profit:

$$E(\pi_i) = \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} [P(\varepsilon) s_i(\varepsilon) - C_i(s_i(\varepsilon))] g(\varepsilon)\, d\varepsilon. \tag{4}$$

3 Analysis

In the following subsection I derive a first-order condition for optimal bids when rationing is disproportionate on the margin and non-linearity of the rationing rule depends on the clearing price. Then, I will analyse a simple case with two permissible price levels, where existence of a Nash equilibrium is ensured for uniform-demand shocks and constant marginal costs. The third subsection of the analysis section presents approximate results for cases with many permissible price levels, increasing marginal costs and non-uniform demand shocks.

3.1 The first-order condition

Optimal bids of a supplier can be determined from the following first-order condition.

[9] Lemma 4 in the Appendix formally establishes that this corresponds to pro-rata on the margin rationing.

[10] Lemma 5 in the Appendix formally establishes that there exists a unique allocation for any given set of non-decreasing supply schedules.

Lemma 1 *The first-order condition for a uniform-price auction with N symmetric suppliers is given by:*

$$\left.\frac{\partial E(\pi_i)}{\partial S_j^i}\right|_{S_j^i=S_j^k} = -\Delta P_{j+1} S_j^i g\left(S_j\right)$$
$$+\frac{(N-1)\Delta S_j}{N}\int_0^1 \left[P_j - C_i'\left(\overline{S}_j\left(u\right)/N\right)\right]\left(1-u^{\mu_j}\right) g\left(\overline{S}_j\left(u\right)\right) du \tag{5}$$
$$+\frac{(N-1)\Delta S_{j+1}}{N}\int_0^1 \left[P_{j+1} - C_i'(\overline{S}_{j+1}\left(u\right)/N)\right] u^{\mu_{j+1}} g\left(\overline{S}_{j+1}\left(u\right)\right) du = 0,$$

where $k \neq i$ *and* $\overline{S}_j\left(u\right) := uS_{j-1} + \left(1-u\right)S_j$.

The first-order condition can be intuitively interpreted as follows. When calculating $\partial E\left(\pi_i\right)/\partial S_j^i$, supply is increased at P_j while holding the supply at all other price levels constant. This implies that the bid price of one (infinitesimally small) unit of quantity is decreased from P_{j+1} to P_j. This decreases the MCP for the event when the unit is price-setting, i.e. when $\varepsilon = S_j$. This event brings a negative contribution to the expected profit, which corresponds to the first term in the first-order condition (5). This term corresponds to the price effect; the term is negative as a bid price was decreased. Due to the rationing mechanism, decreasing the price for one unit of quantity (weakly) increases the accepted supply for demand outcomes $\varepsilon \in \left[S_{j-1}, S_{j+1}\right)$. This gives a positive contribution to the expected profit; the two integrals in (5). The first integral covers $\varepsilon \in \left[S_{j-1}, S_j\right)$ when the MCP is P_j, and the second for $\varepsilon \in \left[S_{j-1}, S_j\right)$ when the MCP is P_{j+1}. The first integral corresponds to the loss associated with the quantity effect at price P_j and the second integral corresponds to the loss associated with the quantity effect at price P_{j+1}. The two integral terms are positive since a bid price was decreased. There are two reasons why supplier i's loss associated with the quantity effect at P_j would dominate the loss associated with the quantity effect at P_{j+1}. First, if the market is more likely to clear at P_j than at P_{j+1}. The other reason is that supplier i has higher average mark-ups at P_j than at P_{j+1}. We also note the following from Lemma 1:

Remark 1 *For given supply schedules* $\mathbf{S}$*, the loss associated with supplier i's quantity effect when increasing the bid price for some units of output from* P_j *to* P_{j+1} *becomes larger if*

1. the rationing rule gives increased priority to large quantity increments at P_j compared to P_{j+1}, i.e. μ_j increases and/or μ_{j+1} decreases.

2. supplier i's loss associated with the quantity effect at P_j dominates the loss associated with the quantity effect at P_{j+1}, the same rationing rule is used at P_j and P_{j+1}, and the rationing rule gives increased priority to large quantity increments, i.e. $\mu_j = \mu_{j+1}$ increases.

3. supplier i's loss associated with the quantity effect at P_{j+1} dominates the loss associated with the quantity effect at P_j, the same rationing rule is used at P_j and P_{j+1}, and the rationing rule gives increased priority to small quantity increments, i.e. $\mu_j = \mu_{j+1}$ decreases.

We notice from the first-order condition in Lemma 1 that the price effect does not depend on the rationing rule, so bidding becomes more competitive for rationing rules that strengthen the quantity effect. Thus the first point indicates that an optimal rationing rule would have a μ_j that decreases for higher clearing prices, or equivalently, increases for lower clearing prices. Even if it is normally not optimal to have a μ_j that is independent of the clearing price, the two last points indicate circumstances where such a rule can have a pro-competitive effect. The following subsections explore Remark 1 in more detail.

3.2 Two price levels

To illustrate the effect of disproportionate rationing on equilibrium bids, we first analyse a simple case with only two admissible price levels, P_1 and P_2. In order to ensure existence of Nash equilibria, we make the following restrictive assumption:

Assumption 1. The uniform-price auction has two price levels, P_1 and P_2. The suppliers are symmetric, each supplier has production capacity k_i and a constant marginal cost $c \leq P_1 < P_2$, such that $(N-1)(P_2-c) \leq N\Delta P_2$. Demand is uniformly distributed on $[0,k]$. We set $S_0^i = 0$.

The inequality $(N-1)(P_2-c) \leq N\Delta P_2$ is used to avoid that Nash equilibria become too competitive, so that all production capacity is offered at P_1 for the most pro-competitive rationing rules. However, one implication of this inequality is also that mark-ups for bids at P_1 must be sufficiently small. We can reformulate the inequality as follows:

$$(N-1)(P_1-c) \leq \Delta P_2. \tag{6}$$

$P_2 > c$ is the highest possible price, so irrespective of competitors' bids, it is the best response for each supplier to offer its entire capacity k_i at P_2, i.e. $S_2^i = k_i$. Thus, market performance is determined by S_1^i. A higher S_1^i means that bids are more competitive, i.e. the average mark-ups are lower. We get the following result:

Lemma 2 *Under Assumption 1, the solution to the first-order condition in Lemma 1 is:*

$$S_1^i = \frac{(N-1)\,k_i\,(P_2-c)}{\left((\mu_2+1)\,\Delta P_2 + (N-1)\,(P_2-c) - \frac{(N-1)(P_1-c)(1+\mu_2)\mu_1}{(1+\mu_1)}\right)}. \tag{7}$$

As expected from Remark 1, we have from Lemma 2 and the inequality in (6) that S_1^i increases when μ_2 decreases and/or when μ_1 increases. We can verify that the following first-order solutions are Nash equilibria.

Proposition 1 *Under Assumption 1, we can establish Nash equilibria for the following cases*

1. A rationing rule that gives maximum priority to large quantity increments at P_1 ($\mu_1 = \infty$) and maximum priority to small quantity increments at P_2 ($\mu_2 = 0$) results in the most competitive first-order solution. The symmetric Nash equilibrium for this case is:

$$S_1^i = \frac{(N-1)\, k_i\, (P_2 - c)}{N \Delta P_2}. \tag{8}$$

2. Auction competitiveness is also improved, but to a smaller extent, when maximum priority is given to small quantity increments at both P_1 and P_2 ($\mu_2 = \mu_1 = 0$). The Nash equilibrium for this case is:

$$S_1^i = \frac{(N-1)\, k_i\, (P_2 - c)}{\Delta P_2 + (N-1)\,(P_2 - c)}. \tag{9}$$

3. The Nash equilibrium for pro rata on the margin rationing ($\mu_2 = \mu_1 = 1$) is:

$$S_1^i = \frac{(N-1)\, k_i (P_2 - c)}{(N+1)\, \Delta P_2}. \tag{10}$$

   In this case, supplier i's loss associated with the quantity effect at P_2 dominates the loss associated with the quantity effect at P_1.

The first result is consistent with the first point of Remark 1. It is optimal to have a large μ_j at the lowest clearing price and a small μ_j at the highest clearing price. The third result establishes that supplier i's loss associated with the quantity effect at P_2 dominates the loss associated with the quantity effect at P_1 for a standard rationing rule. This is a consequence of (6), which ensures that mark-ups of bids at P_1 are sufficiently small. Thus the second result is consistent with the third point of Remark 1. In the special case when $P_1 = c$, the loss associated with the quantity effect at P_1 is zero, so that it is only the price level P_2 that contributes to this loss. In this special case, giving maximum priority to small quantity increments at both P_1 and P_2 ($\mu_2 = \mu_1 = 0$) will have the same effect as the optimal rationing rule, i.e. (8) and (9) give the same result.

We can multiply the first and third result in Proposition 1 by N to get expressions for total market supply at P_1. By using the fact that $k = Nk_i$, we can deduce the following:

Corollary 1 *Under Assumption 1, a uniform-price auction with N symmetric suppliers and optimal rationing on the margin gives the auctioneer the same total procurement cost as a uniform-price auction with pro rata on the margin rationing and $2N - 1$ symmetric suppliers with the same total production cost (the same marginal cost c and total production capacity k).*

Proposition 1 and Corollary 1 are illustrated by the four cases in Figure 2.

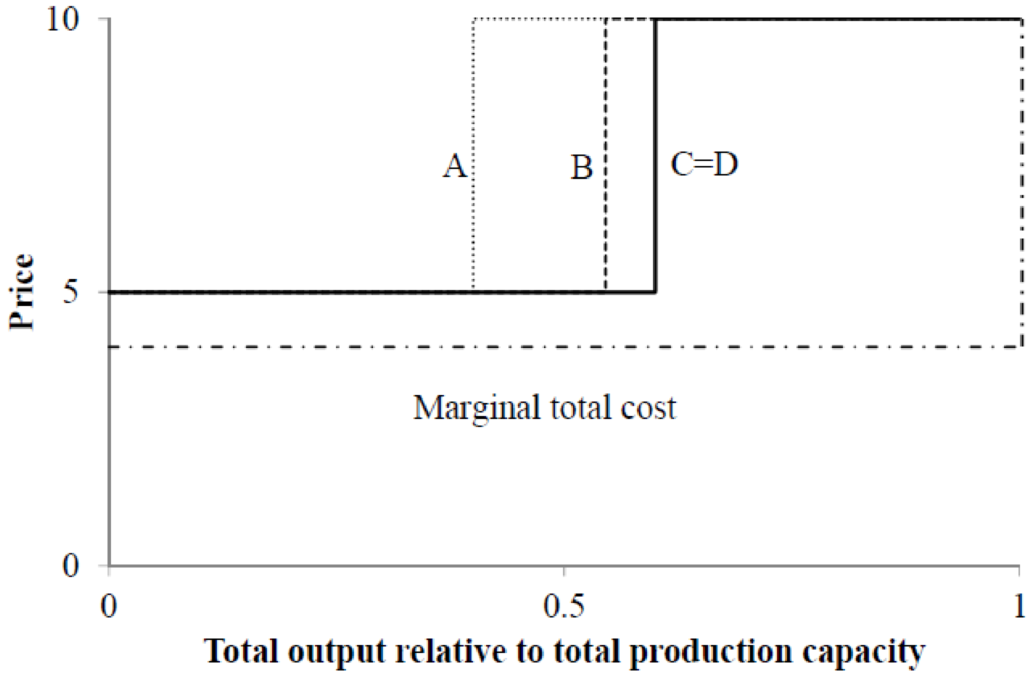

Figure 2: Aggregate stepped supply function equilibria when $c = 4$, $P_1 = 5$ and $P_2 = 10$ for four different cases: A) $N = 2$, $\mu_1 = \mu_2 = 1$ (pro-rata on the margin rationing), B) $N = 2$, $\mu_1 = \mu_2 = 0$ (maximum priority to short marginal quantity increments at both prices), C) $N = 2$, $\mu_1 = \infty$, $\mu_2 = 0$ (optimal rationing) and D) $N = 3$, $\mu_1 = \mu_2 = 1$ (pro-rata on the margin rationing for 3 firms).

3.3 Many price levels

In this section, we analyse the case with non-uniform demand shocks, marginal costs that are increasing and supply functions that have many steps, so that the difference equation in Lemma 1 can be approximated by a differential equation. A difference equation is said to be consistent with a differential equation, if the difference equation converges to the said differential equation as the number of steps in the supply schedules increases towards infinity (LeVeque, 2007;Holmberg et al., 2013).

Lemma 3 *For N symmetric suppliers, the discrete first-order condition in Lemma 1 is consistent with the continuous differential equation*

$$-q_i(P_j) + [P_j - C_i'(q_i(P_j))]\left(\frac{1}{(\mu_{j+1}+1)} + \frac{\mu_j r}{(\mu_j+1)}\right)(N-1)\,q_i'(P_j) = 0 \quad (11)$$

if $P_j > C_i'(q_i(P_j))$ *and* $\mu_j > 0$.

In the special case when tick-sizes are constant, i.e. $r = 1$, and rationing is proportionate on the margin, i.e. $\mu_j = 1$, (11) can be simplified to

$$-q_i(P_j) + [P_j - C_i'(q_i(P_j))](N-1)\,q_i'(P_j) = 0, \quad (12)$$

which is the differential equation of continuous supply function equilibria for symmetric suppliers with inelastic demand (Rudkevich, 1998; Anderson and Philpott, 2002; Holmberg, 2008). This confirms the consistency result in Holmberg et al. (2013) for pro rata on the margin rationing and constant tick-sizes. A comparison of (11) and (12) implies that for constant tick-sizes ($r = 1$) and disproportionate rationing on the margin, competitiveness (the number of competitors, $N-1$) is approximately boosted by the factor

$$\lambda = \frac{1}{(\mu_{j+1}+1)} + \frac{\mu_j}{(\mu_j+1)} \quad (13)$$

relative to the case with pro rata on the margin rationing. As in the case with two price levels, we note that it is beneficial for competition to use rationing parameters such that $\mu_j > \mu_{j+1}$. However, with more price levels, there will be smaller changes in μ_j from one price level to the next and a lower pro-competitive effect, if one wants to maintain the same effect on competition at each price level. We can write (13) in the following form:

$$\mu_j = \frac{1}{1 + \frac{1}{\mu_{j+1}+1} - \lambda} - 1.$$

By setting the competition boosting factor λ to a constant and $\mu_H = 0$ (the rationing parameter at the highest realized price), we can iteratively solve for μ_j for sequentially smaller j, until a non-negative solution of μ_j no longer exists. In this way, we can approximately determine for how many steps in a supply function we can maintain λ at the desired level. The results are summarized in Table 1.

We can multiply the differential equation in (11) by N, so that we get an equation for total supply, and then note the following from Table 1.

Table 1: The competition boosting factor λ and the number of steps in a supply function, for which the factor can be maintained.

No. of steps	λ
1	2
2	1.4
3	1.3
4	1.2
6	1.15
9	1.1
19	1.05
49	1.02
99	1.01
199	1.005
499	1.002
999	1.001

Remark 2 *A uniform-price auction with optimal rationing on the margin and N symmetric suppliers with v steps in each supply function has approximately the same total procurement cost as a uniform-price auction with pro rata on the margin rationing and $(1 + 1/v)\ (N - 1) + 1$ symmetric suppliers with the same total production costs and v steps in each supply function.*

We also note the following from Lemma 3:

Remark 3 *If the rationing rule is the same for each price level, $\mu_j = \mu_{j+1} = \mu$, but tick-sizes are non-constant, then*

1. If tick-sizes decrease towards the reservation price ($r > 1$), then the competition boosting factor $\lambda = \frac{1}{(\mu+1)} + \frac{\mu r}{(\mu+1)}$ increases when the rationing rule gives increased disproportionate priority to large quantity increments at all prices ($\mu \uparrow$).

2. If tick-sizes increase towards the reservation price ($r < 1$), then the competition boosting factor $\lambda = \frac{1}{(\mu+1)} + \frac{\mu r}{(\mu+1)}$ increases when the rationing rule gives increased disproportionate priority to small quantity increments at all prices ($\mu \downarrow$).

The intuition behind this result is that smaller tick-sizes towards the reservation price tend to also decrease a supplier's chosen quantity increments in that direction, so that supplier i's loss associated with the quantity effect at P_j tends to dominate the loss associated with the quantity effect at P_{j+1}. The opposite is true if tick-sizes are instead larger towards the reservation price. Thus the results above are consistent with point 2 and 3 in Remark 1.

David et al. (2007) and Li and Kuo (2011;2013) show that non-constant tick-sizes can improve market competitiveness for single-round auctions with single objects. In multi-unit auctions with a standard rationing rule ($\mu = 1$), the competition boosting factor λ is equal to one irrespective of whether tick-sizes are constant or not. However, note that the first-order condition in Lemma 3 is just an approximation, so this does not rule out that there are circumstances where non-constant tick-sizes have a small pro-competitive effect also for the standard rationing rule.

4 Extensions of the auction design

In the analysed model, each rationing parameter has been tied to a price level, but this may not be optimal in practice. In practice, the bidding format often restricts the number of steps in supply schedules and/or bidders do not always use all allowed steps, because the additional effort required of a supplier to submit another step may not be negligible (Kastl, 2011). In such cases, it should be sufficient to boost competition at bid prices that are used by the supplier, so that a higher boosting factor can be maintained at those fewer prices. In practice, it may therefore be beneficial to have individual rationing parameters for suppliers, μ_j^i, where a supplier's parameter could, for example, depend on the step number in its supply function. The auctioneer may also want to weight supplier's unmet supply, in order to avoid that the disproportionate rationing rule favours small or large suppliers, or to optimize rationing for asymmetric bidders. As an example, a supplier's weight ω_i could be inversely proportional to its production capacity or maximum offered supply S_H^i. Thus (2), could be generalized as follows

$$\frac{ds_i(\varepsilon)}{d\varepsilon} = \frac{\left(\omega_i\left(S_j^i - s_i(\varepsilon)\right)\right)^{\mu_j^i}}{\sum_{k=1}^N \left(\omega_k\left(S_j^k - s_k(\varepsilon)\right)\right)^{\mu_j^k}}.$$

5 Conclusions

For an auctioneer, it is beneficial if bidders increase quantity increments at prices far from the reservation price and if bidders decrease their quantity increments near the reservation price. It is shown that such a pro-competitive effect on bids can be achieved with rationing rules that prioritize large marginal quantity increments at clearing prices far from the reservation price and then gives increased priority to small marginal quantity increments at price levels closer to the reservation price.

I establish existence of Nash equilibria and derive precise results for the simple case with uniformly distributed demand shocks, constant marginal costs that are common knowledge among symmetric suppliers and two price levels. In this case, I show that the optimal use of disproportionate rationing on the margin for a uniform-price auction with N symmetric suppliers gives the auctioneer the same procurement cost as a uniform-price auction with pro rata on the margin rationing and $2N - 1$ symmetric suppliers with the same total production cost.

Results for non-uniform demand shocks, increasing marginal costs and multiple price levels are approximate. The pro-competitive effect is smaller for supply schedules with more steps that involve multiple price levels. For supply functions with v steps, a uniform-price auction with N symmetric suppliers and an optimal use of disproportionate rationing on the margin at each step roughly gives the auctioneer the same procurement cost as a uniform-price auction with pro rata on the margin rationing and $\left(1+\frac{1}{v}\right)(N-1)+1>N$ symmetric suppliers with the same total production cost.

However, even if supply functions have many steps, the auctioneer can still substantially boost competition locally by using disproportionate rationing on the margin at a few price levels, where the auctioneer expects the auction to clear or where the auctioneer is mostly concerned with market competitiveness. Forward prices, prices in when-issued markets or clearing prices of previous auctions can be used to predict the clearing price of an auction.

The paper also identifies situations where the competitiveness of the auction can be improved if the same rationing rule is used at all price levels. It is also shown how the bidding format, such as the tick-sizes that decrease towards the clearing price, can be tailored to create such situations. Such bidding formats are sometimes used in multi-round auctions, such as spectrum auctions.

The bidding format and parts of the auction software that receives and manages bids can be kept unchanged when implementing a pro-competitive rationing rule, so it should be straightforward to implement it in practice.

As shown by von der Fehr and Harbord (1993), uniform-price auctions with small-tick sizes and production/purchase constraints would normally have a price instability, where small cost changes have a large impact on bid prices or where traders use randomized strategies in equilibrium. This exaggerates price volatility, which is a nuisance. Holmberg et al. (2013) show that this problem disappears when tick-sizes are sufficiently large relative to the lot size (the smallest allowed quantity increment). This condition is satisfied in my setting where the lot-size is infinitesimally small, so that bid quantities can be continuously varied.

Avoiding price-instability is one advantage with having a sufficiently large tick-size. Other advantages with having a positive tick-size is that it improves liquidity, simplifies the trading environment for human traders, reduces the cost of band-width and data storage in computers, and reduces the time needed in negotiations (Harris, 1991; Angel, 1997; Angel, 2012). But there can also be disadvantages. In my model, a positive tick-size does not introduce any welfare losses, as firms are symmetric and the demand side is perfectly inelastic. In a financial exchange, however, the tick-size introduces a transaction cost, which has a negative impact on welfare. A more competitive market due to an improved rationing rule would lower welfare losses.

6 References

Anderson, E. J. and X. Hu (2008). 'Finding Supply Function Equilibria with Asymmetric Firms', *Operations Research* **56**(3), pp. 697-711.

Anderson, E.J. and A. B. Philpott (2002). 'Using supply functions for offering generation into an electricity market', *Operations Research* **50** (3), pp. 477-489.

Angel, J. J. (1997). 'Tick size, share prices, and stock splits', *The Journal of Finance* **52**(2), pp. 655-681.

Angel, J. J. (2012). 'Tick size regulation: costs, benefits and risks'. Report, Economic Impact Assessment EIA7, Government Office for Science, United Kingdom.

Ausubel, L. (2004). 'An Efficient Ascending-Bid Auction for Multiple Objects', *American Economic Review* **94**(5), pp. 1452-1475.

Back, K., and Zender, J. F. (1993). 'Auctions of divisible goods: on the rationale for the treasury experiment', *Review of Financial Studies* **6**(4), pp. 733-764.

Budish, E., P. Cramton and J. Shim (2013), 'The High-Frequency Trading Arms Race: Frequent Batch Auctions as a Market Design Response', Working Paper, Booth School of Business, University of Chicago.

David, E., Rogers, A., Jennings, N.R., Schiff, J., Kraus, S. and Rothkopf, M.H. (2007). 'Optimal Design of English Auctions with Discrete Bid Levels', *ACM Transactions on Internet Technology* **7**(2), Article 12.

Deneckere, R. and D. Kovenock (1996). 'Bertrand-Edgeworth duopoly with unit cost asymmetry', *Economic Theory* **8**, pp. 1-25.

Fabra, N., von der Fehr, N-H. M., Harbord, D. (2006). 'Designing Electricity Auctions', *RAND Journal of Economics* **37**(1), pp. 23-46.

von der Fehr, N-H. M., Harbord, D., 1993. 'Spot Market Competition in the UK Electricity Industry', *Economic Journal* **103**(418), pp. 531-46.

Field, J. and J. Large (2012). 'Pro-Rata Matching in One-Tick Markets'. Mimeo.

Green, R.J. and D.M. Newbery (1992). 'Competition in the British Electricity Spot Market', *Journal of Political Economy* **100** (5), pp. 929-53.

Gresik, T. A. (2001). 'Rationing rules and European Central Bank auctions', *Journal of International Money and Finance* **20**(6), pp. 793-808.

Harris, L. (1991). 'Stock price clustering and discreteness', *Review of Financial Studies* **4**(3), pp. 389-415.

Holmberg, P. 2008. 'Unique supply function equilibrium with capacity constraints', *Energy Economics* **30**, 148–172.

Holmberg, P., D. Newbery (2010). 'The supply function equilibrium and its policy implications for wholesale electricity auctions', *Utilities Policy* **18**(4), pp. 209–226.

Holmberg, P., D. Newbery, D. Ralph (2013), 'Supply Function Equilibria: Step functions and continuous representations', *Journal of Economic Theory* **148**(4), pp. 1509–1551.

Hortaçsu, A. and S. Puller (2008). 'Understanding Strategic Bidding in Multi-Unit Auctions: A Case Study of the Texas Electricity Spot Market', *Rand Journal of Economics* **39** (1), pp. 86-114.

Jackson, M. and Swinkels, J., (1999). 'Existence of equilibrium in auctions and discontinuous Bayesian games: Endogenous and incentive compatible sharing rules'. Mimeo.

Kastl, J. (2011). 'Discrete Bids and Empirical Inference in Divisible Good Auctions', *Review of Economic Studies* **78**, pp. 978-1014.

Kastl, J., 2012. 'On the properties of equilibria in private value divisible good auctions with constrained bidding', *Journal of Mathematical Economics* **48**(6), pp. 339-352.

Klemperer, P. D. and M.A. Meyer, (1989). 'Supply Function Equilibria in Oligopoly under Uncertainty', *Econometrica* **57** (6), pp. 1243-1277.

Kremer, I and K.G. Nyborg (2004). 'Divisible Good Auctions: The Role of Allocation Rules', *RAND Journal of Economics* **35**, pp. 147–159.

Lehmann, B.N. and Modest, D.M. (1994). 'Trading and Liquidity on the Tokyo Stock Exchange: A Bird's Eye View', The Journal of Finance **49**(3), pp. 951–984.

LeVeque, R., 2007. *Finite Difference Methods for Ordinary and Partial Differential Equations*, Philadelphia: Society for Industrial and Applied Mathematics.

Li, Z. and Kuo, C-C. (2011). 'Revenue-maximizing Dutch auctions with discrete bid levels', *European Journal of Operational Research* **215**(3), pp. 721-729.

Li, Z. and Kuo, C-C. (2013). 'Design of discrete Dutch auctions with an uncertain number of bidders', *Annals of Operations Research* **211**(1), pp. 255-272.

McAdams, D. (2000). 'Collusive-seeming equilibria" in the uniform-price auction'. Mimeo.

Reny, P. (1999). 'On the existence of pure and mixed strategy Nash equilibria in discontinuous games', *Econometrica* **67**, pp. 1029–1056.

Rudkevich, A., M. Duckworth and R. Rosen (1998). 'Modelling electricity pricing in a deregulated generation industry: The potential for oligopoly pricing in poolco', *The Energy Journal* **19** (3), pp. 19-48, 1998.

Saez, Y., D. Quintana, P. Isasi and A. Mochon (2007). 'Effects of a rationing rule on the Ausubel auction: A genetic algorithm implementation', *Computational Intelligence* **23** (2), pp. 221–235.

Simon, L., Zame, W. (1990). 'Discontinuous games and endogenous sharing rules', *Econometrica* **58**, 861–872.

Sioshansi, R. and S. Oren (2007). 'How Good are Supply Function Equilibrium Models: An Empirical Analysis of the ERCOT Balancing Market', *Journal of Regulatory Economics* **31** (1), pp. 1-35.

Wang, J. J. D., J.F. Zender (2002). 'Auctioning divisible goods', *Economic Theory* **19**, pp. 673–705.

Wilson, R. (1979). 'Auctions of Shares', *Quarterly Journal of Economics* **93**(4), pp. 675-689.

Wolak, F.A. (2007). 'Quantifying the Supply-Side Benefits from Forward Contracting in Wholesale Electricity Markets', *Journal of Applied Econometrics* **22**, pp. 1179-1209.

Appendix

First, we verify that the special case when $\mu_j = 1$ corresponds to pro rata on the margin rationing at the price level P_j.

Lemma 4 *The auction has pro rata on the margin rationing at the price level P_j when $\mu_j = 1$.*

Proof. We can use the identities $\sum_{k=1}^{N} s_k(\varepsilon) \equiv \varepsilon$ and $\sum_{k=1}^{N} S_j^k \equiv S_j$ to simplify and then solve (2) when $\mu_j = 1$:

$$\frac{ds_i(\varepsilon)}{d\varepsilon} = \frac{S_j^i - s_i(\varepsilon)}{S_j - \varepsilon}$$
$$\frac{\frac{ds_i(\varepsilon)}{d\varepsilon}}{S_j - \varepsilon} + \frac{s_i(\varepsilon)}{(S_j - \varepsilon)^2} = \frac{S_j^i}{(S_j - \varepsilon)^2}.$$

It now follows from the product rule and integration that:

$$\frac{d}{d\varepsilon}\left(\frac{s_i(\varepsilon)}{S_j - \varepsilon}\right) = \frac{S_j^i}{(S_j - \varepsilon)^2}$$
$$\frac{s_i(\varepsilon)}{S_j - \varepsilon} - \frac{s_i(S_{j-1})}{S_j - S_{j-1}} = \frac{S_j^i}{S_j - \varepsilon} - \frac{S_j^i}{S_j - S_{j-1}}.$$

It now follows from (1) that:

$$s_i(\varepsilon) = S_j^i - \frac{\Delta S_j^i (S_j - \varepsilon)}{\Delta S_j} = S_{j-1}^i + \frac{\Delta S_j^i (\varepsilon - S_{j-1})}{\Delta S_j},$$

which is identical to the accepted supply of a supplier in a uniform-price auction with pro rata on the margin rationing (Holmberg et al., 2013) when demand is inelastic. ■

The following statement ensures that there is a unique allocation under disproportionate rationing. Note that rationing is never required at price levels where no supplier has a quantity increment.

Lemma 5 *For a given set of non-decreasing stepped supply functions* $\mathbf{S}$*, such that $S_j^k > S_{j-1}^k$ for at least one supplier $k \in \{1, \ldots, N\}$, there exists a unique rationing allocation at price P_j, defined by the initial value problem (1) and (2). This unique solution satisfies $s_i(\varepsilon) \leq S_j^i = s_i(S_j)$ and $s_i'(\varepsilon) \geq 0$ for $\varepsilon \in [S_{j-1}, S_j)$ and $\forall i \in \{1, \ldots, N\}$.*

Proof. We have $S_j^i \geq S_{j-1}^i = s_i(S_{j-1})$. Thus, it follows from (2) that $s_i'(\varepsilon) \geq 0$ when $s_i(\varepsilon) < S_j^i$ and that $s_i'(\varepsilon) = 0$ when $s_i(\varepsilon) = S_j^i$, as long as there is some supplier $k \in \{1, \ldots, N\}$ with $s_k(\varepsilon) < S_j^k$. There must be at least one such supplier for $\varepsilon \in [S_{j-1}, S_j)$, otherwise we would get the contradiction that $S_j \leq s(\varepsilon) = \varepsilon$ for some $\varepsilon \in [S_{j-1}, S_j)$. We also note that the right-hand side of (2) is Lipschitz continuous in the interval $[S_{j-1}, \varepsilon^*]$ for any $\varepsilon^* \in [S_{j-1}, S_j)$, so it follows from the Picard–Lindelöf theorem that the initial value problem has a unique solution in the interval $[S_{j-1}, S_j)$. ■

A.1 First-order conditions

From the properties of the sharing rule, it is now possible to derive a first-order condition for the optimal supply schedule of a supplier.

Lemma 6 *The first-order condition for supplier i's optimal output at price P_j is:*

$$\frac{\partial E(\pi_i)}{\partial S_j^i} = -\Delta P_{j+1} S_j^i g(S_j)$$
$$+ \int_{S_{j-1}}^{S_j} [P_j - C_i'(s_i(\varepsilon))] \frac{\partial s_i(\varepsilon)}{\partial S_j^i} g(\varepsilon)\, d\varepsilon + \int_{S_j}^{S_{j+1}} [P_{j+1} - C_i'(s_i(\varepsilon))] \frac{\partial s_i(\varepsilon)}{\partial S_j^i} g(\varepsilon)\, d\varepsilon = 0. \tag{14}$$

Proof. The accepted supply of supplier i only depends on S_j^i for $\varepsilon \in [S_{j-1}, S_j)$ when the clearing price is P_j and for outcomes $\varepsilon \in [S_j, S_{j+1})$ when the clearing price is P_{j+1}. The contribution to the expected profit from outcomes $\varepsilon \in [S_{j-1}, S_j)$ is given by:

$$E_j^i = \int_{S_{j-1}}^{S_j} [P_j s_i(\varepsilon) - C_i(s_i(\varepsilon))] g(\varepsilon)\, d\varepsilon,$$

so

$$\frac{\partial E_j^i}{\partial S_j^i} = [P_j s_i(S_j) - C_i(s_i(S_j))] g(S_j) + \int_{S_{j-1}}^{S_j} [P_j - C_i'(s_i(\varepsilon))] \frac{\partial s_i(\varepsilon)}{\partial S_j^i} g(\varepsilon)\, d\varepsilon. \tag{15}$$

The contribution to the expected profit from outcomes $\varepsilon \in [S_j, S_{j+1})$ is given by:

$$E_{j+1}^i = \int_{S_j}^{S_{j+1}} [P_{j+1} s_i(\varepsilon) - C_i(s_i(\varepsilon))] g(\varepsilon)\, d\varepsilon,$$

so

$$\frac{\partial E_{j+1}^i}{\partial S_j^i} = -[P_{j+1} s_i(S_j) - C_i(s_i(S_j))] g(S_j) + \int_{S_j}^{S_{j+1}} [P_{j+1} - C_i'(s_i(\varepsilon))] \frac{\partial s_i(\varepsilon)}{\partial S_j^i} g(\varepsilon)\, d\varepsilon. \tag{16}$$

Summing the contributions from (15) and (16) establishes the result in (14). ■

In this paper, I will focus on characterizing symmetric Nash equilibria. Thus, I want to find the optimal response of a supplier i when its $N-1$ competitors submit identical bids. It follows from (14) that the optimal stepped supply function is to a large extent determined by how supplier i's accepted supply $s_i(\varepsilon)$ depends on its supply function. The following Lemma specifies this dependence when the supplier's $N-1$ competitors submit identical bids.

Lemma 7 *For N symmetric producers we have that :*

$$\left.\frac{\partial s_i(\varepsilon)}{\partial S_j^i}\right|_{S_j^i=\frac{S_j}{N}} = \begin{cases} \frac{(N-1)}{N}\left(1-\frac{(S_j-\varepsilon)^{\mu_j}}{(\Delta S_j)^{\mu_j}}\right) & \text{if } \varepsilon\in[S_{j-1},S_j) \\ \frac{(N-1)(S_{j+1}-\varepsilon)^{\mu_j+1}}{N(\Delta S_{j+1})^{\mu_j+1}} & \text{if } \varepsilon\in[S_j,S_{j+1}) \\ 0 & \text{otherwise} \end{cases}$$

Proof. For fixed S_k^i $\forall k\neq j$, increasing S_j^i will increase producer i's quantity increment at the price p_j and decrease its quantity increment at the price p_{j+1}. The quantity increments and the offered supply at all other price levels will remain unchanged. Thus, a change in S_j^i will only influence the accepted supply for outcomes $\varepsilon\in[S_{j-1},S_j)$ when the clearing price is p_j and outcomes $\varepsilon\in[S_j,S_{j+1})$ when the clearing price is p_{j+1}. Let $u_{ki}(\varepsilon)=\frac{\partial s_k(\varepsilon)}{\partial S_j^i}$ and first consider $\varepsilon\in(S_{j-1},S_j)$. It follows from (2) that

$$\begin{aligned} u_{ii}'(\varepsilon) = {} & \frac{\mu_j(1-u_{ii}(\varepsilon))\left(S_j^i-s_i(\varepsilon)\right)^{\mu_j-1}}{\sum_{k=1}^N\left(S_j^k-s_k(\varepsilon)\right)^{\mu_j}} \\ & -\frac{\mu_j\left(S_j^i-s_i(\varepsilon)\right)^{\mu_j}(1-u_{ii}(\varepsilon))\left(S_j^i-s_i(\varepsilon)\right)^{\mu_j-1}}{\left[\sum_{k=1}^N\left(S_j^k-s_k(\varepsilon)\right)^{\mu_j}\right]^2} \\ & -\frac{\mu_j\left(S_j^i-s_i(\varepsilon)\right)^{\mu_j}\sum_{k\neq i}^N(-u_{ki}(\varepsilon))\left(S_j^k-s_k(\varepsilon)\right)^{\mu_j-1}}{\left[\sum_{k=1}^N\left(S_j^k-s_k(\varepsilon)\right)^{\mu_j}\right]^2}. \end{aligned}$$

Symmetry, i.e. $S_j^i=S_j^k$, yields

$$u_{ii}'(\varepsilon)=\frac{\mu_j(1-u_{ii}(\varepsilon))}{N\left(S_j^i-s_i(\varepsilon)\right)}-\frac{\mu_j(1-u_{ii}(\varepsilon))}{N^2\left(S_j^i-s_i(\varepsilon)\right)}+\frac{\mu_j\sum_{k\neq i}^N u_{ki}(\varepsilon)}{N^2\left(S_j^i-s_i(\varepsilon)\right)}. \tag{17}$$

Notice that $\sum_{k=1}^N s_k(\varepsilon)\equiv\varepsilon$ and accordingly $\sum_{k=1}^N u_{ki}(\varepsilon)\equiv 0$. Thus, we can write (17) as follows:

$$\begin{aligned} u_{ii}'(\varepsilon) &= \frac{\mu_j(1-u_{ii}(\varepsilon))}{N\left(S_j^i-s_i(\varepsilon)\right)}-\frac{\mu_j}{N^2\left(S_j^i-s_i(\varepsilon)\right)} \\ &= \frac{\mu_j((N-1)/N-u_{ii}(\varepsilon))}{S_j-\varepsilon}, \end{aligned}$$

where $S_j=NS_j^i$. Hence,

$$(S_j-\varepsilon)\,u_{ii}'(\varepsilon)+\mu_j u_{ii}(\varepsilon)=\mu_j(N-1)/N.$$

We solve this differential equation by means of an integrating factor. Multiplying all terms by $\frac{1}{(S_j-\varepsilon)^{\mu_j+1}}$ yields:

$$\frac{u_{ii}'(\varepsilon)}{(S_j-\varepsilon)^{\mu_j}}+\frac{\mu_j u_{ii}(\varepsilon)}{(S_j-\varepsilon)^{\mu_j+1}}=\frac{\mu_j(N-1)/N}{(S_j-\varepsilon)^{\mu_j+1}}.$$

By means of the product rule, we get

$$\frac{d}{d\varepsilon}\frac{u_{ii}(\varepsilon)}{(S_j-\varepsilon)^{\mu_j}}=\frac{d}{d\varepsilon}\frac{(N-1)/N}{(S_j-\varepsilon)^{\mu_j}},$$

so that

$$\frac{u_{ii}(\varepsilon)}{(S_j-\varepsilon)^{\mu_j}}-\frac{u_{ii}(S_{j-1})}{(S_j-S_{j-1})^{\mu_j}}=\frac{(N-1)/N}{(S_j-\varepsilon)^{\mu_j}}-\frac{(N-1)/N}{(S_j-S_{j-1})^{\mu_j}}.$$

We have $u_{ii}(S_{j-1})=0$, so

$$\frac{\partial s_i(\varepsilon)}{\partial S_j^i}=u_{ii}(\varepsilon)=\frac{(N-1)}{N}\left(1-\frac{(S_j-\varepsilon)^{\mu_j}}{(\Delta S_j)^{\mu_j}}\right)\text{ if }\varepsilon\in(S_{j-1},S_j).$$

Now, we will repeat the same procedure for the interval $\varepsilon\in(S_j,S_{j+1})$ when the price is p_{j+1}. Again, let $u_{ki}(\varepsilon)=\frac{\partial s_k(\varepsilon)}{\partial S_j^i}$. In this interval, we have (compare with (2))

$$s_i'(\varepsilon)=\frac{\left(S_{j+1}^i-s_i(\varepsilon)\right)^{\mu_{j+1}}}{\sum_{k=1}^N\left(S_{j+1}^k-s_k(\varepsilon)\right)^{\mu_{j+1}}}.$$

Thus

$$u_{ii}'(\varepsilon)=-\frac{\mu_{j+1}u_{ii}(\varepsilon)\left(S_{j+1}^i-s_i(\varepsilon)\right)^{\mu_{j+1}-1}}{\sum_{k=1}^N\left(S_{j+1}^k-s_k(\varepsilon)\right)^{\mu_{j+1}}}+\frac{\left(S_{j+1}^i-s_i(\varepsilon)\right)^{\mu_{j+1}}\sum_{k=1}^N u_{ki}(\varepsilon)\mu_{j+1}\left(S_{j+1}^k-s_k(\varepsilon)\right)^{\mu_{j+1}-1}}{\left[\sum_{k=1}^N\left(S_{j+1}^k-s_k(\varepsilon)\right)^{\mu_{j+1}}\right]^2}.$$

Symmetry implies that

$$u_{ii}'(\varepsilon)=-\frac{\mu_{j+1}u_{ii}(\varepsilon)}{N\left(S_{j+1}^i-s_i(\varepsilon)\right)}+\frac{\mu_{j+1}\sum_{k=1}^N u_{ki}(\varepsilon)}{N^2\left(S_{j+1}^i-s_i(\varepsilon)\right)}.$$

As before, $\sum_{k=1}^N s_k(\varepsilon)\equiv\varepsilon$ implies that $\sum_{k=1}^N u_{ki}(\varepsilon)\equiv 0$, so

$$u_{ii}'(\varepsilon)=\frac{-\mu_{j+1}u_{ii}(\varepsilon)}{S_{j+1}-\varepsilon},$$

where $S_{j+1}=NS_{j+1}^i$. Hence,

$$(S_{j+1}-\varepsilon)u_{ii}'(\varepsilon)+\mu_{j+1}u_{ii}(\varepsilon)=0.$$

As above, we solve this differential equation by means of an integrating factor. Multiplying all terms by $\frac{1}{(S_{j+1}-\varepsilon)^{\mu_{j+1}+1}}$ yields:

$$\frac{u_{ii}'(\varepsilon)}{(S_{j+1}-\varepsilon)^{\mu_{j+1}}}+\frac{\mu_{j+1}u_{ii}(\varepsilon)}{(S_{j+1}-\varepsilon)^{\mu_{j+1}+1}}=0.$$

Thus, it follows from the product rule that

$$\frac{d}{d\varepsilon}\frac{u_{ii}(\varepsilon)}{(S_{j+1}-\varepsilon)^{\mu_{j+1}}}=0,$$

so that

$$\frac{u_{ii}(\varepsilon)}{(S_{j+1}-\varepsilon)^{\mu_{j+1}}}=\frac{u_{ii}(S_j)}{(S_{j+1}-S_j)^{\mu_{j+1}}}, \tag{18}$$

where $u_{ii}(S_j)$ can be determined from the relation

$$1=\frac{dS_j^i}{dS_j^i}=\frac{ds_i(S_j)}{dS_j^i}=u_{ii}(S_j)+s_i'(S_j)\frac{dS_j}{dS_j^i}.$$

We have $s_i'(S_j)=s_i'(\varepsilon)=\frac{1}{N}$ due to symmetry and $\frac{dS_j}{dS_j^i}=1$, so

$$u_{ii}(S_j)=1-\frac{1}{N}=\frac{N-1}{N}.$$

Now, it follows from (18) that

$$\frac{\partial s_i(\varepsilon)}{\partial S_j^i}=u_{ii}(\varepsilon)=\frac{(N-1)(S_{j+1}-\varepsilon)^{\mu_{j+1}}}{N(\Delta S_{j+1})^{\mu_{j+1}}}\text{ if }\varepsilon\in(S_j,S_{j+1}).$$

Finally, we note that $\frac{\partial s_i(\varepsilon)}{\partial S_j^i}$ is continuous at the points $\varepsilon=S_j$ and $\varepsilon=S_{j+1}$. ■

We can now conclude the following from Lemma 6 and Lemma 7 above.

Corollary 2 *The first-order condition of a market with N symmetric suppliers is given by:*

$$\begin{aligned}\frac{\partial E(\pi_i)}{\partial S_j^i}&=-\Delta P_{j+1}S_j^i g(S_j)\\&+\frac{(N-1)}{N}\int_{S_{j-1}}^{S_j}[P_j-C_i'(s_i(\varepsilon))]\left(1-\frac{(S_j-\varepsilon)^{\mu_j}}{(\Delta S_j)^{\mu_j}}\right)g(\varepsilon)\,d\varepsilon\\&+\frac{(N-1)}{N(\Delta S_{j+1})^{\mu_{j+1}}}\int_{S_j}^{S_{j+1}}[P_{j+1}-C_i'(s_i(\varepsilon))](S_{j+1}-\varepsilon)^{\mu_{j+1}}g(\varepsilon)\,d\varepsilon=0.\end{aligned} \tag{19}$$

We are now able to prove the first-order condition presented in the main text.

Proof. (Lemma 1) This follows from Corollary 2 in Appendix and the substitutions $u=\frac{S_j-\varepsilon}{\Delta S_j}$ and $u=\frac{S_{j+1}-\varepsilon}{\Delta S_{j+1}}$, respectively. ■

The first-order condition can be solved as follows.

Proof. (Lemma 2) We have

$$\int_0^1(1-u^{\mu_j})\,du=\left[u-\frac{u^{\mu_j+1}}{\mu_j+1}\right]_0^1=\frac{\mu_j}{\mu_j+1}$$

and

$$\int_0^1 u^{\mu_{j+1}} du = \left[\frac{u^{\mu_{j+1}+1}}{\mu_{j+1}+1}\right]_0^1 = \frac{1}{\mu_{j+1}+1},$$

so it follows from Lemma 1 and Assumption 1 that:

$$\Delta P_2 S_1^i = \frac{(N-1)(P_1-c)\mu_1 \Delta S_1^i}{(\mu_1+1)} + \frac{(N-1)(P_2-c)\Delta S_2^i}{(\mu_2+1)}$$

$$\Delta P_2 S_1^i = \frac{(N-1)(P_1-c)\mu_1 S_1^i}{(\mu_1+1)} + \frac{(N-1)(P_2-c)(k_i-S_1^i)}{(\mu_2+1)}$$

$$S_1^i = \frac{(N-1)(P_2-c)k_i}{(\mu_2+1)\left(\Delta P_2 - \frac{(N-1)(P_1-c)\mu_1}{(\mu_1+1)} + \frac{(N-1)(P_2-c)}{(\mu_2+1)}\right)}.$$

■

A.2 Second-order conditions

For extreme cases when $\mu_j = 0$ or $\mu_j = \infty$, the acceptance sensitivity with respect to quantity increments, i.e. $\frac{\partial s_i(\varepsilon)}{\partial S_j^i}$, can also be determined at asymmetric points, where $S_j^i \neq S_j^k$.

Lemma 8 *If $\mu_j = 0$ and competitors have identical supply functions, S_j^k, then:*

$$\frac{\partial s_i(\varepsilon)}{\partial S_j^i} = \begin{cases} 0 & \text{if } \Delta S_j^i > \Delta S_j^k \text{ and } \varepsilon \in (S_{j-1}, S_j) \\ 0 & \text{if } \Delta S_j^i < \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1}, S_{j-1} + N\Delta S_j^i\right) \\ 1 & \text{if } \Delta S_j^i < \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1} + N\Delta S_j^i, S_j\right) \end{cases}$$

and

$$\frac{\partial s_i(\varepsilon)}{\partial S_{j-1}^i} = \begin{cases} \frac{N-1}{N} & \text{if } \Delta S_j^i > \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1}, S_{j-1} + N\Delta S_j^k\right) \\ 0 & \text{if } \Delta S_j^i > \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1} + N\Delta S_j^k, S_j\right) \\ \frac{N-1}{N} & \text{if } \Delta S_j^i < \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1}, S_{j-1} + N\Delta S_j^i\right) \\ 0 & \text{if } \Delta S_j^i < \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1} + N\Delta S_j^i, S_j\right) \end{cases}$$

Proof. It follows from (2) that for $\mu_j = 0$ and $\Delta S_j^i > \Delta S_j^k$, all producers get the same accepted quantity from marginal bids for $\varepsilon \in \left(S_{j-1}, S_{j-1} + N\Delta S_j^k\right)$, while competitors' accepted quantity of marginal bids is constant in the interval $\left(S_{j-1} + N\Delta S_j^k, S_j\right)$. Thus

$$s_i(\varepsilon) = \begin{cases} S_{j-1}^i + \frac{\varepsilon - S_{j-1}}{N} & \text{if } \varepsilon \in \left(S_{j-1}, S_{j-1} + N\Delta S_j^k\right) \\ S_{j-1}^i + \Delta S_j^k + \varepsilon - S_{j-1} - N\Delta S_j^k & \text{if } \varepsilon \in \left(S_{j-1} + N\Delta S_j^k, S_j\right). \end{cases}$$

For $\mu_j = 0$ and $\Delta S_j^i < \Delta S_j^k$, all producers get the same accepted quantity of marginal bids for $\varepsilon \in \left(S_{j-1}, S_{j-1} + N\Delta S_j^i\right)$, while supplier i's accepted quantity from marginal bids is constant in the interval $\left(S_{j-1} + N\Delta S_j^i, S_j\right)$. Thus

$$s_i(\varepsilon) = \begin{cases} S_{j-1}^i + \frac{\varepsilon - S_{j-1}}{N} & \text{if } \varepsilon \in \left(S_{j-1}, S_{j-1} + N\Delta S_j^i\right) \\ S_j^i & \text{if } \varepsilon \in \left(S_{j-1} + N\Delta S_j^i, S_j\right). \end{cases}$$

The statement follows from differentiation of the above expressions with respect to S_{j-1}^i and S_j^i. ■

Lemma 9 *If $\mu_j = \infty$ and competitors have identical supply functions, S_j^k, then:*

$$\frac{\partial s_i(\varepsilon)}{\partial S_j^i} = \begin{cases} 0 & \text{if } \Delta S_j^i > \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1}, S_{j-1} + \Delta S_j^i - \Delta S_j^k\right) \\ \frac{N-1}{N} & \text{if } \Delta S_j^i > \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1} + \Delta S_j^i - \Delta S_j^k, S_j\right) \\ 0 & \text{if } \Delta S_j^i < \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1}, S_{j-1} + (N-1)\left(\Delta S_j^k - \Delta S_j^i\right)\right) \\ \frac{N-1}{N} & \text{if } \Delta S_j^i < \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1} + (N-1)\left(\Delta S_j^k - \Delta S_j^i\right), S_j\right) \end{cases}$$

and

$$\frac{\partial s_i(\varepsilon)}{\partial S_{j-1}^i} = \begin{cases} 0 & \text{if } \Delta S_j^i > \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1}, S_{j-1} + \Delta S_j^i - \Delta S_j^k\right) \\ 0 & \text{if } \Delta S_j^i > \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1} + \Delta S_j^i - \Delta S_j^k, S_j\right) \\ 1 & \text{if } \Delta S_j^i < \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1}, S_{j-1} + (N-1)\left(\Delta S_j^k - \Delta S_j^i\right)\right) \\ 0 & \text{if } \Delta S_j^i < \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1} + (N-1)\left(\Delta S_j^k - \Delta S_j^i\right), S_j\right). \end{cases}$$

Proof. It follows from (2) that for $\mu_j = \infty$ and $\Delta S_j^i > \Delta S_j^k$ marginal bids are only accepted from supplier i, as long as its unmet supply at P_j, $S_j^i - s_i(\varepsilon)$, is larger than for each other supplier. Thus

$$s_i(\varepsilon) = \begin{cases} S_{j-1}^i + \varepsilon - S_{j-1} & \text{if } \varepsilon \in \left(S_{j-1}, S_{j-1} + \Delta S_j^i - \Delta S_j^k\right) \\ S_j^i - \Delta S_j^k + \frac{\varepsilon - S_{j-1} - \Delta S_j^i + \Delta S_j^k}{N} & \text{if } \varepsilon \in \left(S_{j-1} + \Delta S_j^i - \Delta S_j^k, S_j\right). \end{cases}$$

If instead $\mu_j = \infty$ and $\Delta S_j^i < \Delta S_j^k$, then marginal bids are only accepted from competitors of supplier i, as long as each competitor's unmet supply at P_j, $S_j^k - s_k(\varepsilon)$, is larger than for supplier i.

$$s_i(\varepsilon) = \begin{cases} S_{j-1}^i & \text{if } \varepsilon \in \left(S_{j-1}, S_{j-1} + (N-1)\left(\Delta S_j^k - \Delta S_j^i\right)\right) \\ S_{j-1}^i + \frac{\varepsilon - S_{j-1} - (N-1)\left(\Delta S_j^k - \Delta S_j^i\right)}{N} & \text{if } \varepsilon \in \left(S_{j-1} + (N-1)\left(\Delta S_j^k - \Delta S_j^i\right), S_j\right). \end{cases}$$

The statement follows from differentiation of the above expressions with respect to S_{j-1}^i and S_j^i. ■

We are now able to establish the Nash equilibria stated in the main text.

Proof. (Proposition 1) It follows from (6) and Lemma 2 that the first-order solution of S_1^i increases when μ_2 decreases and that S_1^i increases when μ_1 increases. Thus, competitiveness is maximized when $\mu_1 = \infty$ and $\mu_2 = 0$. (8) - (10) follows from Lemma 2.

In the next step, we want to prove that the first-order solution in (8) constitutes an NE. It follows from Lemma 6, Lemma 8 and Lemma 9 in Appendix that if $\mu_1 = \infty$ and $\mu_2 = 0$, and competitors have an identical supply, $S_1^k = \frac{(N-1)k_i(P_2-c)}{N\Delta P_2}$, then:

$$\begin{gathered} \frac{\partial E(\pi_i)}{\partial S_1^i} = -\Delta P_2 S_1^i g \\ +\frac{N-1}{N}(P_1 - c)\, g \min\left(\Delta S_1 - \Delta S_1^i + \Delta S_1^k, \Delta S_1 - (N-1)\left(\Delta S_1^k - \Delta S_1^i\right)\right) \\ +\frac{N-1}{N}\min\left(N\Delta S_2^i, N\Delta S_2^k\right)(P_2 - c)\, g \\ = -\Delta P_2 S_1^i g + (N-1)(P_1 - c)\, g \min\left(S_1^k, S_1^i\right) \\ +(N-1)\min\left(k_i - S_1^i, k_i - S_1^k\right)(P_2 - c)\, g. \end{gathered} \tag{20}$$

We note that $\frac{\partial E(\pi_i)}{\partial S_1^i}$ is piece-wise linear in S_1^i with a break point at $S_1^i = S_1^k$, where $\frac{\partial E(\pi_i)}{\partial S_1^i} = 0$. Moreover, $\frac{\partial E(\pi_i)}{\partial S_1^i} = (N-1)\left(k_i - S_1^k\right)(P_2 - c)\, g \geq 0$ for $S_1^i = 0$ and it follows from (6) that $\frac{\partial E(\pi_i)}{\partial S_1^i} = -\Delta P_2 k_i g + (N-1)(P_1 - c)\, g S_1^k \leq 0$ for $S_1^i = k_i$. Hence, we can conclude that $\frac{\partial^2 E(\pi_i)}{\partial \left(S_1^i\right)^2} \leq 0$. Thus, $S_1^i = S_1^k$ is the best response to $S_1^k = \frac{(N-1)k_i(P_2-c)}{N\Delta P_2}$, which verifies that (8) constitutes a Nash equilibrium if $\mu_1 = \infty$ and $\mu_2 = 0$.

In the next step, we want to prove that the first-order solution in (9) constitutes an NE. It follows from Lemma 6 and Lemma 8 in Appendix that if $\mu_1 = \mu_2 = 0$, and competitors have an identical supply, $S_1^k = \frac{(N-1)(P_2-c)\;\, k_i}{\Delta P_2 + (N-1)(P_2-c)}$, then:

$$\begin{gathered} \frac{\partial E(\pi_i)}{\partial S_1^i} = -\Delta P_2 S_1^i g \\ + (N-1)\max\left(0, \Delta S_1^k - \Delta S_1^i\right)(P_1 - c)\, g \\ + (N-1)\min\left(\Delta S_2^i, \Delta S_2^k\right)(P_2 - c)\, g \\ = -\Delta P_2 S_1^i g \\ + (N-1)\max\left(0, S_1^k - S_1^i\right)(P_1 - c)\, g \\ + (N-1)\min\left(k_i - S_1^i, k_i - S_1^k\right)(P_2 - c)\, g. \end{gathered} \tag{21}$$

We have $\frac{\partial E(\pi_i)}{\partial S_1^i} = (N-1)\, S_1^k (P_1 - c)\, g + (N-1)\left(k_i - S_1^k\right)(P_2 - c)\, g \geq 0$ for $S_1^i = 0$ and $\frac{\partial E(\pi_i)}{\partial S_1^i} = -\Delta P_2 k_i g \leq 0$ for $S_1^i = k_i$. $\frac{\partial E(\pi_i)}{\partial S_1^i}$ is piece-wise linear in S_1^i with a break point at $S_1^i = S_1^k$, where $\frac{\partial E(\pi_i)}{\partial S_1^i} = 0$, so we can now conclude that $\frac{\partial^2 E(\pi_i)}{\partial \left(S_1^i\right)^2} \leq 0$. Thus $S_1^i = S_1^k$ is the best response to $S_1^k = \frac{(N-1)(P_2-c)\;\, k_i}{\Delta P_2 + (N-1)(P_2-c)}$, which verifies that (9) constitutes a Nash equilibrium for $\mu_1 = \mu_2 = 0$.

It follows from Holmberg et al. (2013) that (10) constitutes a Nash equilibrium. Finally, the following argument shows that supplier i's loss associated with the quantity effect at P_2 dominates the loss associated with the quantity effect at P_1 for pro rata on the margin rationing. It follows from Assumption 1 and (6) that

$$\begin{aligned} \Delta S_1^i (P_1 - c) = S_1^i (P_1 - c) &= \frac{(N-1)(P_2 - c)\, k_i}{(N+1)\,\Delta P_2}(P_1 - c) \\ &= \frac{(N-1)(P_1 - c)}{(N+1)\,\Delta P_2}(P_2 - c)\, k_i \\ &\leq \frac{\Delta P_2}{(N+1)\,\Delta P_2}(P_2 - c)\, k_i \\ &\leq \frac{(N+1)\,\Delta P_2 - (N-1)(P_2 - c)}{(N+1)\,\Delta P_2}(P_2 - c)\, k_i \\ &= \left(k_i - S_1^i\right)(P_2 - c) = \Delta S_2^i (P_2 - c), \end{aligned}$$

when $S_1^i = \frac{(N-1)k_i(P_2-c)}{(N+1)\Delta P_2}$. ■

A.3 Approximate first-order condition for multiple price levels

The following lemma is useful when we want to analyse the convergence properties of the first-order condition as the number of steps per supply function increases.

Lemma 10 *We can make the following statements for the first-order condition in Corollary 2 when $P_j - C_i'\left(S_j^i\right) > 0$ and $\mu_j > 0$ for all price levels:*

1. *The difference $S_{j+1}^i - S_j^i$ is of the order ΔP_{j+1}.*
2. *The discrete first-order condition in Corollary 2 can be approximated by:*

$$\frac{\partial E(\pi_i)}{\partial S_j^i} = -\Delta P_{j+1} S_j^i g(S_j) + \frac{(N-1)}{N}\left[P_j - C_i'(S_j^i)\right] g(S_j)\left(\frac{\mu_j \Delta S_j}{(\mu_j+1)} + \frac{\Delta S_{j+1}}{(\mu_{j+1}+1)}\right) \tag{22}$$

$$+ O\left((\Delta P_{j+1})^2\right). \tag{23}$$

Proof. The sum

$$\begin{aligned} I :=& \tfrac{(N-1)}{N}\int_{S_{j-1}}^{S_j}\left[P_j - C_i'(s_i(\varepsilon))\right]\left(1-\tfrac{(S_j-\varepsilon)^{\mu_j}}{(\Delta S_j)^{\mu_j}}\right) g(\varepsilon)\, d\varepsilon \\ &+ \tfrac{(N-1)}{N(\Delta S_{j+1})^{\mu_{j+1}}}\int_{S_j}^{S_{j+1}}\left[P_{j+1} - C_i'(s_i(\varepsilon))\right](S_{j+1}-\varepsilon)^{\mu_{j+1}} g(\varepsilon)\, d\varepsilon \end{aligned} \tag{24}$$

must be of the order ΔP_{j+1}, otherwise the first-order condition in Corollary 2 in Appendix cannot be satisfied for small ΔP_{j+1}. Supply schedules are symmetric and non-decreasing. Moreover, $P_{j+1} - C_i'\left(S_{j+1}^i\right) > 0$, $\mu_j > 0$, $N \geq 2$, and $g(\varepsilon) > 0$, so it follows that we must have:

$$I \geq \frac{(N-1)\left[P_j - C_i'(S_j^i)\right]\underline{g}}{N}\int_{S_{j-1}}^{S_j}\left(1-\frac{(S_j-\varepsilon)^{\mu_j}}{(\Delta S_j)^{\mu_j}}\right) d\varepsilon \geq 0. \tag{25}$$

We have that I is of the order ΔP_{j+1} and $\Delta S_j \geq \Delta S_j^i \geq 0$, so the above inequality implies that $\Delta S_j$and ΔS_j^i must both be of the order ΔP_{j+1} or, equivalently, of the order ΔP_j, as $r = \frac{\Delta P_j}{\Delta P_{j+1}}$ is bounded.

In the next step, we want to derive the Taylor expansions of the first-order conditions. Using Taylor expansions and the above result, the first-order condition in Corollary 2 can be written:

$$\begin{aligned} \tfrac{\partial E(\pi_i)}{\partial S_j^i} =& -\Delta P_{j+1} S_j^i g(S_j) \\ &+ \tfrac{(N-1)}{N}\int_{S_{j-1}}^{S_j}\left[P_j - C_i'(S_j^i) + O(\Delta P_j)\right]\left(1-\tfrac{(S_j-\varepsilon)^{\mu_j}}{(\Delta S_j)^{\mu_j}}\right)\left[g(S_j) + O(\Delta P_j)\right] d\varepsilon \\ &+ \tfrac{(N-1)}{N}\int_{S_j}^{S_{j+1}}\left[P_{j+1} - C_i'(S_{j+1}^i) + O(\Delta P_{j+1})\right]\left(\tfrac{S_{j+1}-\varepsilon}{\Delta S_{j+1}}\right)^{\mu_{j+1}}\left[g(S_{j+1}) + O(\Delta P_{j+1})\right] d\varepsilon \end{aligned}$$

Hence, as $\Delta S_j$and ΔS_j^i are of the order ΔP_{j+1}:

$$\frac{\partial E(\pi_i)}{\partial S_j^i} = -\Delta P_{j+1} S_j^i g\left(S_j\right)$$
$$+\frac{(N-1)}{N}\left[P_j - C_i'(S_j^i)\right] g\left(S_j\right) \int\limits_{S_{j-1}}^{S_j} \left(1 - \frac{(S_j-\varepsilon)^{\mu_j}}{(\Delta S_j)^{\mu_j}}\right) d\varepsilon$$
$$+\frac{(N-1)}{N}\left[P_{j+1} - C_i'(S_{j+1}^i)\right] g\left(S_{j+1}\right) \int\limits_{S_j}^{S_{j+1}} \frac{(S_{j+1}-\varepsilon)^{\mu_{j+1}}}{(\Delta S_{j+1})^{\mu_{j+1}}} d\varepsilon \tag{26}$$
$$+O\left((\Delta P_{j+1})^2\right).$$

It can be shown that

$$\int_{S_{j-1}}^{S_j} \left(1 - \frac{(S_j-\varepsilon)^{\mu_j}}{(\Delta S_j)^{\mu_j}}\right) d\varepsilon = \frac{\mu_j \Delta S_j}{(\mu_j+1)}$$
$$\int_{S_j}^{S_{j+1}} \frac{(S_{j+1}-\varepsilon)^{\mu_{j+1}}}{(\Delta S_{j+1})^{\mu_{j+1}}} d\varepsilon = \frac{\Delta S_{j+1}}{(\mu_{j+1}+1)}.$$

Using these results and that $\Delta S_j$and ΔS_j^i are of the order ΔP_{j+1}, the Taylor expansion in (26) can be simplified to:

$$\frac{\partial E\left(\pi_i\right)}{\partial S_j^i} = -\Delta P_{j+1} S_j^i g\left(S_j\right) + \frac{(N-1)}{N}\left[P_j - C_i'(S_j^i)\right] g\left(S_j\right)\left(\frac{\mu_j \Delta S_j}{(\mu_j+1)} + \frac{\Delta S_{j+1}}{(\mu_{j+1}+1)}\right)$$
$$+ O\left((\Delta P_{j+1})^2\right). \tag{27}$$

■

We are now able to prove the following consistency statement in the main text.

Proof. (Lemma 3) We use the Taylor approximation in Lemma 10 to approximate the difference equation in Lemma 1:

$$-\Delta P_{j+1} S_j^i g\left(S_j\right) + \left(P_j - C_i'\left(S_j^i\right)\right) g\left(S_j\right) \frac{(N-1)}{N}\left[\frac{\Delta S_{j+1}}{\mu_{j+1}+1} + \frac{\mu_j \Delta S_j}{\mu_j+1}\right] + O\left((\Delta P_{j+1})^2\right) = 0.$$

We have assumed that g is bounded away from zero. Thus

$$-\Delta P_{j+1} S_j^i + \left(P_j - C_i'\left(S_j^i\right)\right) \frac{(N-1)}{N}\left[\frac{\Delta S_{j+1}}{\mu_{j+1}+1} + \frac{\mu_j \Delta S_j}{\mu_j+1}\right] + O\left((\Delta P_{j+1})^2\right) = 0. \tag{28}$$

Symmetry implies that

$$-\Delta P_{j+1} S_j^i + \left(P_j - C_i'\left(S_j^i\right)\right)(N-1)\left[\frac{\Delta S_{j+1}^i}{\mu_{j+1}+1} + \frac{\mu_j \Delta S_j^i}{\mu_j+1}\right] + O\left((\Delta P_{j+1})^2\right) = 0.$$

Thus

$$-S_j^i + \left(P_j - C_i'\left(S_j^i\right)\right)(N-1)\frac{\left[\frac{\Delta S_{j+1}^i}{\mu_{j+1}+1} + \frac{\mu_j \Delta S_j^i}{\mu_j+1}\right]}{\Delta P_{j+1}} + O\left(\Delta P_{j+1}\right) = 0,$$

so with $\Delta P_j = r\Delta P_{j+1}$

$$-S_j^i + \left(P_j - C_i'\left(S_j^i\right)\right)(N-1)\left(\frac{\Delta S_{j+1}^i}{(\mu_{j+1}+1)\Delta P_{j+1}} + \frac{\mu_j r \Delta S_j^i}{(\mu_j+1)\Delta P_j}\right) + O\left(\Delta P_{j+1}\right) = 0.$$

Hence,

$$\frac{1}{\left(\frac{1}{(\mu_{j+1}+1)}+\frac{\mu_j r}{(\mu_j+1)}\right)}\left(\frac{\Delta S_{j+1}^i}{(\mu_{j+1}+1)\,\Delta P_{j+1}}+\frac{\mu_j r \Delta S_j^i}{(\mu_j+1)\,\Delta P_j}\right)$$
$$=\frac{S_j^i}{\left(\frac{1}{(\mu_{j+1}+1)}+\frac{\mu_j r}{(\mu_j+1)}\right)(N-1)\left(P_j - C_i{}'\left(S_j^i\right)\right)}+O\left(\Delta P_{j+1}\right).$$

If S_j^i are replaced by samples of the continuous supply function $q_i(p)$ at price P_j, then the left-hand side becomes an estimate of $q_i'(P_j)$ and the right-hand side converges to:

$$\frac{q_i(P_j)}{(N-1)\left(\frac{1}{(\mu_{j+1}+1)}+\frac{\mu_j r}{(\mu_j+1)}\right)(P_j - C_i{}'(q_i(P_j)))}$$

when $q_i'(P_j)$ is bounded. Thus, the first-order condition in Lemma 1 is consistent with the ordinary differential equation in (11) when $P_j > C_i'(q_i(P_j))$ and $\mu_j > 0$.
■