Two Players Game Theory with Information: Introducing the Witsenhausen Intrinsic Model

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Outline of the presentation

Witsenhausen intrinsic model and game theory with information

Nash equilibrium with information

Witsenhausen intrinsic model and principal-agent models

Games solvable by dynamic programming

Open questions

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Witsenhausen intrinsic model with Nature and two players

We lay out

- basic sets
  - decision sets
  - states of Nature
  - history set

  and their σ-fields
- objective functions
- beliefs
- information σ-fields, admissible strategies and predecessors

Nature's moves and players decisions

- Let Ω be a measurable set equipped with a σ-field $\mathcal{F}$ which represents all uncertainties:
  any $\omega \in \Omega$ is called a state of Nature
- The player a makes one decision $u_a \in \mathbb{U}_a$
  where the decision set $\mathbb{U}_a$ is equipped with a σ-field $\mathcal{U}_a$
- The player b makes one decision $u_b \in \mathbb{U}_b$
  where the decision set $\mathbb{U}_b$ is equipped with a σ-field $\mathcal{U}_b$

History space

The history space is the product space

$$\mathbb{H} = \mathbb{U}_a \times \mathbb{U}_b \times \Omega$$

equipped with the product history field

$$\mathcal{H} = \mathcal{U}_a \otimes \mathcal{U}_b \otimes \mathcal{F}$$

Criteria (costs or payoffs)

- For any history $h = (u_a, u_b, \omega) \in \mathbb{H}$,
  player a and player b undergo costs or payoffs
- Costs or payoffs are materialized under the form of (measurable) objective functions or criteria

$$j_a : \mathbb{U}_a \times \mathbb{U}_b \times \Omega \to \mathbb{R}$$

$$j_b : \mathbb{U}_a \times \mathbb{U}_b \times \Omega \to \mathbb{R}$$

Beliefs (and risk attitudes)

- All sources of randomness are collected in the set Ω
- Player a and player b express beliefs and risk attitudes towards events in $\mathcal{F}$
- We denote real-valued random variables on $(\Omega, \mathcal{F})$ by

$$\mathbb{L}(\Omega, \mathcal{F}) = \{\mathbf{X} : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \ , \ \ \mathbf{X}^{-1}(\mathcal{B}_{\mathbb{R}}) \subset \mathcal{F}\}$$

Belief

A belief of the player $c \in \{a, b\}$ is
a probability distribution $\mathbb{P}_c$ over $(\Omega, \mathcal{F})$

- We denote by

$$\mathbb{L}(\Omega, \mathcal{F}, \mathbb{P}_c) = \{\mathbf{X} \in \mathbb{L}(\Omega, \mathcal{F}) \ , \ \ \mathbb{E}_{\mathbb{P}_c}\big(|\mathbf{X}|\big) < +\infty\}$$

the space of $\mathbb{P}_c$-integrable random variables,
where $\mathbb{E}_{\mathbb{P}_c}$ denotes the mathematical expectation

Information and predecessors

Information

- When making a decision,
  player a and player b can make use of information,
  materialized under the form of σ-fields
- The information field $\mathcal{I}_a$ of the player a
  is a subfield of the history field $\mathcal{H}$

$$\mathcal{I}_a \subset \mathcal{U}_a \otimes \mathcal{U}_b \otimes \mathcal{F}$$

- The information field $\mathcal{I}_b$ of the player b
  is a subfield of the history field $\mathcal{H}$

$$\mathcal{I}_b \subset \mathcal{U}_a \otimes \mathcal{U}_b \otimes \mathcal{F}$$

Absence of "self-information"

- The information fields $\mathcal{I}_a$ and $\mathcal{I}_b$ display the absence of "self-information" when

$$\mathcal{I}_a \subset \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \otimes \mathcal{F}$$

$$\mathcal{I}_b \subset \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$$

- In what follows, we always assume absence of "self-information" (otherwise, we would be led to paradoxes)

Classical information patterns in game theory

Two players: the principal P (leader) and the agent A (follower)

- Moral hazard (the insurance company cannot observe if the insured plays with matches at home)

$$\mathcal{I}_P \subset \{\emptyset, \mathbb{U}_A\} \otimes \{\emptyset, \mathbb{U}_P\} \otimes \mathcal{F}$$

- Stackelberg leadership model

$$\mathcal{I}_A \subset \{\emptyset, \mathbb{U}_A\} \otimes \mathcal{U}_P \otimes \mathcal{F} \,, \;\; \mathcal{I}_P \subset \{\emptyset, \mathbb{U}_A\} \otimes \{\emptyset, \mathbb{U}_P\} \otimes \mathcal{F}$$

- Adverse selection (the insurance company cannot observe if the insured has good health)

$$\{\emptyset, \mathbb{U}_A\} \otimes \{\emptyset, \mathbb{U}_P\} \otimes \mathcal{F} \subset \mathcal{I}_A \,, \;\; \mathcal{I}_P \subset \mathcal{U}_A \otimes \{\emptyset, \mathbb{U}_P\} \otimes \{\emptyset, \Omega\}$$

- Signaling

$$\{\emptyset, \mathbb{U}_A\} \otimes \{\emptyset, \mathbb{U}_P\} \otimes \mathcal{F} \subset \mathcal{I}_A \,, \;\; \mathcal{I}_P = \mathcal{U}_A \otimes \{\emptyset, \mathbb{U}_P\} \otimes \{\emptyset, \Omega\}$$

Cylindric subfields

- Information only carried by the moves of Nature

$$\mathcal{H}_{\emptyset} = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$$

- Information only carried by the moves of Nature and by the decisions of agent a

$$\mathcal{H}_{\{a\}} = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$$

- Information only carried by the moves of Nature and by the decisions of agent b

$$\mathcal{H}_{\{b\}} = \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \otimes \mathcal{F}$$

- Information carried by the moves of Nature and by the decisions of agents a and b

$$\mathcal{H}_{\{a,b\}} = \mathcal{U}_a \otimes \mathcal{U}_b \otimes \mathcal{F} = \mathcal{H}$$

Definition of predecessor, excluding Nature

Consider a subset B of $\{a, b\}$ — $B \in \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ — and define

$$\mathcal{H}_B = \prod_{c \in B} \mathcal{U}_c \otimes \prod_{c \notin B} \{\emptyset, \mathbb{U}_c\} \otimes \mathcal{F}$$

Predecessor

For any player $c \in \{a, b\}$, we define $\langle c \rangle_{\mathfrak{P}}$
as the intersection of all subsets B of $\{a, b\}$ such that $\mathcal{I}_c \subset \mathcal{H}_B$

$$\langle c \rangle_{\mathfrak{P}} = \bigcap_{B,\ \mathcal{I}_c \subset \mathcal{H}_B} B$$

When non empty, an element of $\langle c \rangle_{\mathfrak{P}}$ is called a predecessor of c

- Nature has no predecessor: Nature plays before the players (but is not necessarily revealed to the players)
- As an illustration, absence of “self-information” is equivalent to $c \notin \langle c \rangle_{\mathfrak{P}}$, for any $c \in \{a, b\}$

Sequential and non-sequential information patterns

- Sequential patterns
  - When $\langle a \rangle_{\mathfrak{P}} = \emptyset$ and $\langle b \rangle_{\mathfrak{P}} = \emptyset$, player a and player b both play first (static team)
  - When $\langle a \rangle_{\mathfrak{P}} = \emptyset$ and $\langle b \rangle_{\mathfrak{P}} = \{a\}$, player a plays first, player b plays second
  - When $\langle a \rangle_{\mathfrak{P}} = \{b\}$ and $\langle b \rangle_{\mathfrak{P}} = \emptyset$, player b plays first, player a plays second
- Non-sequential pattern
  - When $\langle a \rangle_{\mathfrak{P}} = \{b\}$ and $\langle b \rangle_{\mathfrak{P}} = \{a\}$, player a and player b
    - can be in a deadlock (non causal system)
    - or can be first and second players depending on Nature's move (causal system)

Strategies and admissible strategies

Pure strategies

- A (pure) strategy of the player a is a measurable mapping

$$\lambda_a : \mathbb{U}_a \times \mathbb{U}_b \times \Omega \to \mathbb{U}_a \;, \;\; \lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{H}$$

and the set of strategies of agent a is

$$\Lambda_a = \left\{ \lambda_a : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_a, \mathcal{U}_a) \;\middle|\; \lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{H} \right\}$$

- A (pure) strategy of the player b is a measurable mapping

$$\lambda_b : \mathbb{U}_a \times \mathbb{U}_b \times \Omega \to \mathbb{U}_b \;, \;\; \lambda_b^{-1}(\mathcal{U}_b) \subset \mathcal{H}$$

and the set of strategies of agent b is

$$\Lambda_a = \left\{ \lambda_a : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_a, \mathcal{U}_a) \;\middle|\; \lambda_b^{-1}(\mathcal{U}_a) \subset \mathcal{H} \right\}$$

- We denote the set of strategies of all agents by

$$\Lambda_{\mathbb{A}} = \Lambda_a \times \Lambda_b$$

Mixed strategies

- A mixed strategy (or randomized strategy) for agent a is an element of $\Delta(\Lambda_a)$, the set of probability distributions over the set of strategies of agent a
- A mixed strategy (or randomized strategy) for agent b is an element of $\Delta(\Lambda_b)$, the set of probability distributions over the set of strategies of agent b
- We denote the set of mixed strategies of all agents by

$$\Delta(\Lambda_a) \times \Delta(\Lambda_b)$$

We introduce admissible strategies to account for the interplay between decision and information

- Information is the fuel of strategies

Admissible strategy

An admissible strategy of the player $c \in \{a, b\}$ is a mapping

$$\lambda_c : \mathbb{U}_a \times \mathbb{U}_b \times \Omega \to \mathbb{U}_c \text{ such that } \lambda_c^{-1}(\mathcal{U}_c) \subset \mathcal{I}_c$$

- The set of admissible strategies of the player $c \in \{a, b\}$ is

$$\Lambda_c^{ad} = \{\lambda_c \mid \mathbb{U}_a \times \mathbb{U}_b \times \Omega \to \mathbb{U}_c \ , \ \lambda_c^{-1}(\mathcal{U}_c) \subset \mathcal{I}_c\}$$

- The set of admissible strategies is

$$\Lambda^{ad} = \Lambda_a^{ad} \times \Lambda_b^{ad}$$

- The set of mixed admissible strategies is

$$\Delta\big(\Lambda_a^{ad}\big) \times \Delta\big(\Lambda_b^{ad}\big)$$

Absence of "self-information" and structure of admissible strategies

- The information fields $\mathcal{I}_a$ and $\mathcal{I}_b$ display the absence of "self-information" when

$$\mathcal{I}_a \subset \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \otimes \mathcal{F} \iff a \notin \langle a \rangle_{\mathfrak{P}}$$

$$\mathcal{I}_b \subset \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F} \iff b \notin \langle b \rangle_{\mathfrak{P}}$$

- When σ-fields include singletons and we exclude "self-information", then, for any admissible strategy λ_c of the player $c \in \{a, b\}$, we have that the expression $\lambda_c(u_a, u_b, \omega)$ does not depend on u_c:

$$\lambda_a(\cancel{u_a}, u_b, \omega) = \widetilde{\lambda}_a(u_b, \omega) \; , \quad \lambda_b(u_a, \cancel{u_b}, \omega) = \widetilde{\lambda_b}(u_a, \omega)$$

Sequential patterns and structure of admissible strategies

- When $\langle a \rangle_{\mathfrak{P}} = \emptyset$ and $\langle b \rangle_{\mathfrak{P}} = \emptyset$

$$\lambda_a(\cancel{u_a}, \cancel{u_b}, \omega) = \widetilde{\lambda}_a(\omega)\ , \ \ \lambda_b(\cancel{u_a}, \cancel{u_b}, \omega) = \widetilde{\lambda_b}(\omega)$$

- When $\langle a \rangle_{\mathfrak{P}} = \emptyset$ and $\langle b \rangle_{\mathfrak{P}} = \{a\}$

$$\lambda_a(\cancel{u_a}, \cancel{u_b}, \omega) = \widetilde{\lambda}_a(\omega)\ , \ \ \lambda_b(u_a, \cancel{u_b}, \omega) = \widetilde{\lambda_b}(u_a, \omega)$$

- When $\langle a \rangle_{\mathfrak{P}} = \{b\}$ and $\langle b \rangle_{\mathfrak{P}} = \emptyset$

$$\lambda_a(\cancel{u_a}, u_b, \omega) = \widetilde{\lambda}_a(u_b, \omega)\ , \ \ \lambda_b(\cancel{u_a}, \cancel{u_b}, \omega) = \widetilde{\lambda_b}(\omega)$$

Non-sequential information patterns and structure of admissible strategies

When $\langle a \rangle_{\mathfrak{P}} = \{b\}$ and $\langle b \rangle_{\mathfrak{P}} = \{a\}$, player a and player b

- can be in a deadlock

$$\lambda_a(\not{u_a}, u_b, \omega) = \widetilde{\lambda_a}(u_b, \omega) \; , \;\; \lambda_b(u_a, \not{u_b}, \omega) = \widetilde{\lambda_b}(u_a, \omega)$$

- or can be first and second players depending on Nature's move
  - when Nature's move is ω^+, player a plays first, player b plays second

$$\lambda_a(\not{u_a}, \not{u_b}, \omega^+) = \widetilde{\lambda_a}(\omega^+) \; , \;\; \lambda_b(u_a, \not{u_b}, \omega^+) = \widetilde{\lambda_b}(u_a, \omega^+)$$

  - when Nature's move is ω^-, player b plays first, player a plays second

$$\lambda_a(\not{u_a}, u_b, \omega^-) = \widetilde{\lambda_a}(u_b, \omega^-) \; , \;\; \lambda_b(\not{u_a}, \not{u_b}, \omega^-) = \widetilde{\lambda_b}(\omega^-)$$

Solvability property

The information fields $\mathcal{I}_a$ and $\mathcal{I}_b$ display the solvability property when,

- for any couple $(\lambda_a, \lambda_b) \in \Lambda_a^{ad} \times \Lambda_b^{ad}$ of admissible strategies and any state of Nature $\omega \in \Omega$,
- there exists one, and only one, couple $(u_a, u_b) \in \mathbb{U}_a \times \mathbb{U}_b$ of decisions such that

$$u_a = \lambda_a(u_a, u_b, \omega)$$

$$u_b = \lambda_b(u_a, u_b, \omega)$$

Solvability property and solution map

Solution map

In case of solvability, we can define $S_{(\lambda_a,\lambda_b)}(\omega)$, for any $\omega \in \Omega$, by

$$S_{(\lambda_a,\lambda_b)}(\omega) = (u_a, u_b, \omega) \iff \begin{cases} u_a = \lambda_a(u_a, u_b, \omega) \\ u_b = \lambda_b(u_a, u_b, \omega) \end{cases}$$

Hence, we obtain a mapping called the solution map

$$S_{(\lambda_a,\lambda_b)} : \Omega \to \mathbb{U}_a \times \mathbb{U}_b \times \Omega$$

- The solvability property holds true in the sequential cases
- The graph of $S_{(\lambda_a,\lambda_b)}$ belongs to $\mathcal{I}_a \vee \mathcal{U}_a \vee \mathcal{I}_b \vee \mathcal{U}_b$.

Co-cycle property of the solution map (I)

- We suppose that $\langle a \rangle_{\mathfrak{P}} = \{b\}$ and $\langle b \rangle_{\mathfrak{P}} = \emptyset$,
  that is, player b plays first, player a plays second
- We consider a couple $(\lambda_a, \lambda_b) \in \Lambda_a^{ad} \times \Lambda_b^{ad}$ of admissible strategies

Co-cycle property of the solution map

We have that

- the strategy λ_b can be identified with $\lambda_b : \Omega \to \mathbb{U}_b$ and the partial solution map $S_{\lambda_b} : \Omega \to \mathbb{U}_b \times \Omega$ is such that $S_{\lambda_b}(\omega) = (\lambda_b(\omega), \omega)$
- the strategy λ_a can be identified with $\lambda_a : \mathbb{U}_b \times \Omega \to \mathbb{U}_a$
- the solution map has the following co-cycle property

$$S_{(\lambda_a,\lambda_b)} = (\lambda_a \circ S_{\lambda_b}, S_{\lambda_b}) : \Omega \to \mathbb{U}_a \times \big(\mathbb{U}_b \times \Omega\big)$$

$$S_{(\lambda_a,\lambda_b)}(\omega) = \Big(\lambda_a\Big(\lambda_b(\omega), \omega\Big), \lambda_b(\omega), \omega\Big) \; , \quad \forall \omega \in \Omega$$

Co-cycle property of the solution map (II)

The co-cycle property

$$S_{(\lambda_a,\lambda_b)} = (\lambda_a \circ S_{\lambda_b}, S_{\lambda_b})$$

is equivalent to

$$S_{(\lambda_a,\lambda_b)}(\omega) = (u_a, u_b, \omega) \iff \begin{cases} (u_b, \omega) & = S_{\lambda_b}(\omega) \\ u_a & = \lambda_a(u_b, \omega) \end{cases}$$

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Criteria composed with solution map

- Costs or payoffs are

$$j_a : \mathbb{U}_a \times \mathbb{U}_b \times \Omega \to \mathbb{R}$$

$$j_b : \mathbb{U}_a \times \mathbb{U}_b \times \Omega \to \mathbb{R}$$

- Solution map is

$$S_{(\lambda_a,\lambda_b)} : \Omega \to \mathbb{U}_a \times \mathbb{U}_b \times \Omega$$

- The composition of criteria with the solution map provides random variables

$$j_a \circ S_{(\lambda_a,\lambda_b)} : \Omega \to \mathbb{R}$$

$$j_b \circ S_{(\lambda_a,\lambda_b)} : \Omega \to \mathbb{R}$$

Pure Bayesian Nash equilibrium

We suppose that player a has belief $\mathbb{P}_a$ and player b has belief $\mathbb{P}_b$

Bayesian Nash equilibrium

We say that the couple $(\lambda_a^\star, \lambda_b^\star) \in \Lambda_a^{ad} \times \Lambda_b^{ad}$ of admissible strategies is a Bayesian Nash equilibrium if (in case of payoffs)

$$\mathbb{E}_{\mathbb{P}_a}\left[j_a \circ S_{(\lambda_a^\star, \lambda_b^\star)}\right] \geq \mathbb{E}_{\mathbb{P}_a}\left[j_a \circ S_{(\lambda_a, \lambda_b^\star)}\right] , \quad \forall \lambda_a \in \Lambda_a^{ad}$$

$$\mathbb{E}_{\mathbb{P}_b}\left[j_b \circ S_{(\lambda_a^\star, \lambda_b^\star)}\right] \geq \mathbb{E}_{\mathbb{P}_b}\left[j_b \circ S_{(\lambda_a^\star, \lambda_b)}\right] , \quad \forall \lambda_b \in \Lambda_b^{ad}$$

Mixed Bayesian Nash equilibrium

We say that the couple of mixed admissible strategies

$$(\mu_a^\star, \mu_b^\star) \in \Delta\left(\Lambda_a^{ad}\right) \times \Delta\left(\Lambda_b^{ad}\right)$$

is a Bayesian Nash equilibrium if (in case of payoffs)

$$\int_{\Lambda_a^{ad}\times\Lambda_b^{ad}} \mu_a^\star(d\lambda_a) \otimes \mu_b^\star(d\lambda_b)\mathbb{E}_{\mathbb{P}_a}\left[j_a \circ S_{(\lambda_a,\lambda_b)}\right] \geq$$

$$\int_{\Lambda_a^{ad}\times\Lambda_b^{ad}} \mu_a(d\lambda_a) \otimes \mu_b^\star(d\lambda_b)\mathbb{E}_{\mathbb{P}_a}\left[j_a \circ S_{(\lambda_a,\lambda_b)}\right] \ , \ \forall \mu_a \in \Delta\left(\Lambda_a^{ad}\right)$$

$$\int_{\Lambda_a^{ad}\times\Lambda_b^{ad}} \mu_a^\star(d\lambda_a) \otimes \mu_b^\star(d\lambda_b)\mathbb{E}_{\mathbb{P}_b}\left[j_b \circ S_{(\lambda_a,\lambda_b)}\right] \geq$$

$$\int_{\Lambda_a^{ad}\times\Lambda_b^{ad}} \mu_a^\star(d\lambda_a) \otimes \mu_b(d\lambda_b)\mathbb{E}_{\mathbb{P}_b}\left[j_b \circ S_{(\lambda_a^\star,\lambda_b)}\right] \ , \ \forall \mu_a \in \Delta\left(\Lambda_b^{ad}\right)$$

Outline of the presentation

Principal-agent models with two players

- A branch of Economics studies so-called principal-agent models
- Principal-agent models display a general information structure, which can be transparently expressed thanks to Witsenhausen intrinsic model
- The model exhibits two players
  - the principal P (leader), makes decisions $u_P \in \mathbb{U}_P$, where the set of decisions is equipped with a σ-field $\mathcal{U}_P$
  - the agent A (follower) makes decisions $u_A \in \mathbb{U}_A$, where the set of decisions is equipped with a σ-field $\mathcal{U}_A$
- and Nature, corresponding to private information (or type) of the agent A
  - Nature selects $\omega \in \Omega$, where Ω is equipped with a σ-field $\mathcal{F}$

Here is the most general information structure of principal-agent models

$$\mathcal{I}_P \subset \mathcal{U}_A \otimes \{\emptyset, \mathbb{U}_P\} \otimes \mathcal{F}$$

$$\mathcal{I}_A \subset \{\emptyset, \mathbb{U}_A\} \otimes \mathcal{U}_P \otimes \mathcal{F}$$

- By these expressions of the information fields
  - $\mathcal{I}_P$ of the principal P (leader)
  - $\mathcal{I}_A$ of the agent A (follower)
- we have excluded self-information, that is, we suppose that the information of a player cannot be influenced by his actions

Classical information patterns in game theory

Now, we will make the information structure more specific

- Stackelberg leadership model
- Moral hazard
- Adverse selection
- Signaling

Stackelberg leadership model

- In the Stackelberg leadership model of game theory,
- the follower A may partly observe the action of the leader P

$$\mathcal{I}_A \subset \{\emptyset, \mathbb{U}_A\} \otimes \mathcal{U}_P \otimes \mathcal{F}$$

- whereas the leader P observes at most the state of Nature

$$\mathcal{I}_P \subset \{\emptyset, \mathbb{U}_A\} \otimes \{\emptyset, \mathbb{U}_P\} \otimes \mathcal{F}$$

- As a consequence, the system is sequential
  - with the principal P as first player (leader)
  - and the agent A as second player (follower)
- Stackelberg games can be solved by bi-level optimization, for some information structures, like when

$$\mathcal{I}_P \vee \{\emptyset, \mathbb{U}_A\} \otimes \mathcal{U}_P \otimes \{\emptyset, \Omega\} \subset \mathcal{I}_A$$

Moral hazard

- An insurance company (the principal P) cannot observe the efforts of the insured (the agent A) to avoid risky behavior
- The firm faces the hazard that insured persons behave "immorally" (playing with matches at home)
- Moral hazard (hidden action) occurs when the decisions of the agent A are hidden to the principal P

$$\mathcal{I}_P \subset \{\emptyset, \mathbb{U}_A\} \otimes \{\emptyset, \mathbb{U}_P\} \otimes \mathcal{F}$$

- In case of moral hazard, the system is sequential with the principal as first player, (which does not preclude to choose the agent as first player in some special cases, as in a static team situation)
- Moral hazard games can be solved by bi-level optimization, for some information structures

Adverse selection

- In the absence of observable information on potential customers (the agent A), an insurance company (the principal P) offers a unique price for a contract hence screens and selects the "bad" ones
- Adverse selection occurs when
  - the agent A knows the state of nature (his type, or private information)

    $$\{\emptyset, \mathbb{U}_A\} \otimes \{\emptyset, \mathbb{U}_P\} \otimes \mathfrak{F} \subset \mathfrak{I}_A$$

    (the agent A can possibly observe the principal P action)
  - but the principal P does not know the state of nature

    $$\mathfrak{I}_P \subset \mathfrak{U}_A \otimes \{\emptyset, \mathbb{U}_P\} \otimes \{\emptyset, \Omega\}$$

    (the principal P can possibly observe the agent A action)
- In case of adverse selection, the system may or may not be sequential

Signaling

- In biology, a peacock signals its "good genes" (genotype) by its lavish tail (phenotype)
- In economics, a worker signals his working ability (productivity) by his educational level (diplomas)
- There is room for signaling
  - when the agent A knows the state of nature (his type)

$$\{\emptyset, \mathbb{U}_A\} \otimes \{\emptyset, \mathbb{U}_P\} \otimes \mathfrak{F} \subset \mathfrak{I}_A$$

    (the agent A can possibly observe the principal P action)
  - whereas the principal P does not know the state of nature, but the principal P observes the agent A action

$$\mathfrak{I}_P = \mathfrak{U}_A \otimes \{\emptyset, \mathbb{U}_P\} \otimes \{\emptyset, \Omega\}$$

as the agent A may reveal his type
by his decision which is observable by the principal P

Signaling

- The system is sequential (with the agent as first player) when

$$\mathcal{I}_A = \{\emptyset, \mathbb{U}_A\} \otimes \{\emptyset, \mathbb{U}_P\} \otimes \mathcal{F}$$

- The system is non causal when

$$\{\emptyset, \mathbb{U}_A\} \otimes \{\emptyset, \mathbb{U}_P\} \otimes \mathcal{F} \subsetneq \mathcal{I}_A \subset \{\emptyset, \mathbb{U}_A\} \otimes \mathcal{U}_P \otimes \mathcal{F}$$

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Stackelberg leadership model

- In the Stackelberg leadership model of game theory, we consider a leader P (principal) and a follower A (agent)
- We suppose that $\langle P \rangle_{\mathfrak{P}} = \emptyset$, that is, leader P plays first,

$$\mathcal{I}_P \subset \{\emptyset, \mathbb{U}_A\} \otimes \{\emptyset, \mathbb{U}_P\} \otimes \mathcal{F}$$

- and that $\langle A \rangle_{\mathfrak{P}} = \{P\}$, that is, follower A plays second

$$\mathcal{I}_A \subset \{\emptyset, \mathbb{U}_A\} \otimes \mathcal{U}_P \otimes \mathcal{F}$$

and

$$\mathcal{I}_A \bigcap \{\emptyset, \mathbb{U}_A\} \otimes \mathcal{U}_P \otimes \{\emptyset, \Omega\} \neq \emptyset$$

We work on a reduced history space

- As both information fields — $\mathcal{I}_P \subset \{\emptyset, \mathbb{U}_A\} \otimes \{\emptyset, \mathbb{U}_P\} \otimes \mathcal{F}$ and $\mathcal{I}_A \subset \{\emptyset, \mathbb{U}_A\} \otimes \mathcal{U}_P \otimes \mathcal{F}$ — do not depend on $\mathcal{U}_A$, we introduce
- the reduced history space $\widetilde{\mathbb{H}}$
  equipped with the reduced history field $\widetilde{\mathcal{H}}$

$$\widetilde{\mathbb{H}} = \mathbb{U}_P \times \Omega \,, \quad \widetilde{\mathcal{H}} = \mathcal{U}_P \otimes \mathcal{F}$$

- and the reduced information fields $\widetilde{\mathcal{I}}_P$ and $\widetilde{\mathcal{I}}_A$ defined by

$$\begin{aligned} \mathcal{I}_P = & \quad \{\emptyset, \mathbb{U}_A\} \otimes \widetilde{\mathcal{I}}_P \quad \text{with } \widetilde{\mathcal{I}}_P \subset \{\emptyset, \mathbb{U}_P\} \otimes \mathcal{F} \subset \widetilde{\mathcal{H}} \\ \mathcal{I}_A = & \quad \{\emptyset, \mathbb{U}_A\} \otimes \widetilde{\mathcal{I}}_A \quad \text{with } \widetilde{\mathcal{I}}_A \subset \mathcal{U}_P \otimes \mathcal{F} = \widetilde{\mathcal{H}} \end{aligned}$$

Here is what become the admissible strategies on the reduced history space

We consider a couple $(\lambda_A, \lambda_P) \in \Lambda_A^{ad} \times \Lambda_P^{ad}$ of admissible strategies

- As $\mathcal{I}_P = \{\emptyset, \mathbb{U}_A\} \otimes \widetilde{\mathcal{I}}_P$ with $\widetilde{\mathcal{I}}_P \subset \{\emptyset, \mathbb{U}_P\} \otimes \mathcal{F}$, the strategy λ_P of the leader P can be identified with

$$\widetilde{\lambda}_P : \Omega \to \mathbb{U}_P$$

- As $\mathcal{I}_A = \{\emptyset, \mathbb{U}_A\} \otimes \widetilde{\mathcal{I}}_A$, with $\widetilde{\mathcal{I}}_A \subset \mathcal{U}_P \otimes \mathcal{F}$, the strategy λ_A of the follower A can be identified with

$$\widetilde{\lambda}_A : \mathbb{U}_P \times \Omega \to \mathbb{U}_A$$

Therefore, we can work with reduced admissible strategies

$$(\widetilde{\lambda}_A, \widetilde{\lambda}_P) \in \widetilde{\Lambda}_A^{ad} \times \widetilde{\Lambda}_P^{ad}$$

Here is what becomes the solution map on the reduced history space

- By sequentiality, the solution map $S_{(\lambda_A,\lambda_P)}$ satisfies the co-cycle property

$$S_{(\lambda_A,\lambda_P)} = (\lambda_A \circ S_{\lambda_P}, S_{\lambda_P}) = (\lambda_A, \mathrm{Id}_{\mathbb{U}_P\times\Omega}) \circ S_{\lambda_P}$$

- If we introduce a reduced solution map $S_{\widetilde{\lambda}_P} = \big(\widetilde{\lambda}_P, \mathrm{Id}_\Omega\big)$

$$\Omega \xrightarrow{S_{\widetilde{\lambda}_P}} \mathbb{U}_P \times \Omega \ , \ \ \omega \mapsto \big(\widetilde{\lambda}_P(\omega), \omega\big) \ ,$$

we can now write $S_{(\lambda_A,\lambda_P)} = \big(\widetilde{\lambda}_A, \mathrm{Id}_{\mathbb{U}_P\times\Omega}\big) \circ S_{\widetilde{\lambda}_P}$, that is,

$$S_{(\lambda_A,\lambda_P)} : \Omega \xrightarrow{S_{\widetilde{\lambda}_P}} \mathbb{U}_P \times \Omega \xrightarrow{(\widetilde{\lambda}_A, \mathrm{Id}_{\mathbb{U}_P\times\Omega})} \mathbb{U}_A \times \mathbb{U}_P \times \Omega$$

that is,

$$S_{(\lambda_A,\lambda_P)} : \omega \mapsto \big(\widetilde{\lambda}_P(\omega), \omega\big) \mapsto \Big(\widetilde{\lambda}_A\big(\widetilde{\lambda}_P(\omega), \omega\big), \widetilde{\lambda}_P(\omega), \omega\Big)$$

Strategy independence of conditional expectation (SICE)

Assumption SICE

There exists a probability $\mathbb{Q}$ on $\widetilde{\mathbb{H}} = \mathbb{U}_P \times \Omega$ such that

$$\mathbb{P}_A \circ S_{\widetilde{\lambda}_P}^{-1} = T_{\widetilde{\lambda}_P} \mathbb{Q} \text{ with } \mathbb{E}_{\mathbb{Q}}\left[T_{\widetilde{\lambda}_P} \mid \widetilde{\mathfrak{I}}_A \right] > 0 \,, \quad \forall \widetilde{\lambda}_P \in \widetilde{\Lambda}_P^{ad}$$

and that

$$\mathbb{E}_{\mathbb{Q}}\left[j_A(u_A, \cdot) \mid \widetilde{\mathfrak{I}}_A \right] = \mathbb{E}_{\mathbb{Q}}\left[j_A(u_A, \cdot) \mid \widetilde{\mathfrak{I}}_A \vee \widetilde{\mathfrak{I}}_P \vee \widetilde{\mathfrak{D}}_P \right] \,, \quad \forall u_A \in \mathbb{U}_A$$

Under assumption SICE, we have that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_a}\left[j_a \circ S_{(\lambda_a, \lambda_b)} \right] &= \mathbb{E}_{\mathbb{P}_a}\left[j_a \circ (\widetilde{\lambda}_A, \mathrm{Id}_{\mathbb{U}_P \times \Omega}) \circ S_{\widetilde{\lambda}_P} \right] \\
&= \mathbb{E}_{\mathbb{P}_A \circ S_{\widetilde{\lambda}_P}^{-1}}\left[j_a \circ (\widetilde{\lambda}_A, \mathrm{Id}_{\mathbb{U}_P \times \Omega}) \right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[T_{\widetilde{\lambda}_P} j_a \circ (\widetilde{\lambda}_A, \mathrm{Id}_{\mathbb{U}_P \times \Omega}) \right]
\end{aligned}$$

Bayesian Nash equilibrium under assumption SICE

Bayesian Nash equilibrium

Under assumption SICE,
the couple $(\widetilde{\lambda}_A^\star, \widetilde{\lambda}_P^\star) \in \widetilde{\Lambda}_A^{ad} \times \widetilde{\Lambda}_P^{ad}$ of reduced admissible strategies
is a Bayesian Nash equilibrium if (in case of payoffs)

$$\mathbb{E}_{\mathbb{Q}}\Big[j_A \circ (\widetilde{\lambda}_A^\star, \mathrm{Id}_{\mathbb{U}_P \times \Omega})\Big] \geq \mathbb{E}_{\mathbb{Q}}\Big[j_A \circ (\widetilde{\lambda}_A, \mathrm{Id}_{\mathbb{U}_P \times \Omega})\Big] \quad \forall \widetilde{\lambda}_A \in \widetilde{\Lambda}_A^{ad}$$

$$\mathbb{E}_{\mathbb{P}_P}\Big[j_P \circ S_{(\widetilde{\lambda}_A^\star, \widetilde{\lambda}_P^\star)}\Big] \geq \mathbb{E}_{\mathbb{P}_P}\Big[j_P \circ S_{(\widetilde{\lambda}_A^\star, \widetilde{\lambda}_P)}\Big] \quad \forall \widetilde{\lambda}_P \in \widetilde{\Lambda}_P^{ad}$$

There exist an optimal strategy of the follower A that does not depend on the leader P strategy

$$\min_{\widetilde{\lambda}_A \in \widetilde{\Lambda}_A^{ad}} \mathbb{E}_{\mathbb{Q}}\Big[j_A \circ (\widetilde{\lambda}_A, \mathrm{Id}_{\mathbb{U}_P \times \Omega})\Big] = \min_{\widetilde{\lambda}_A \,,\ \widetilde{\lambda}_A^{-1}(\mathcal{U}_A) \subset \widetilde{\mathfrak{I}}_A} \mathbb{E}_{\mathbb{Q}}\Big[j_A \circ (\widetilde{\lambda}_A, \mathrm{Id}_{\mathbb{U}_P \times \Omega})\Big]$$

$$= \mathbb{E}_{\mathbb{Q}}\Big[\min_{u_A \in \mathbb{U}_A} \mathbb{E}_{\mathbb{Q}}\big[j_A(u_A, \cdot) \mid \widetilde{\mathfrak{I}}_A\big]\Big]$$

Bayesian Nash equilibria can be obtained by bi-level optimization under assumption SICE

Suppose assumption SICE holds true

- The (upper level) optimization problem for the follower A

$$\min_{u_A \in \mathbb{U}_A} \mathbb{E}_{\mathbb{Q}}\left[j_A(u_A, \cdot) \mid \widetilde{\mathfrak{I}}_A\right]$$

  provides (under technical assumptions, by a measurable selection theorem) an $\widetilde{\mathfrak{I}}_A$-measurable solution

$$\widetilde{\lambda}_A^\star : \mathbb{U}_P \times \Omega \to \mathbb{U}_A \ , \ \ \sigma(\widetilde{\lambda}_A^\star) \subset \widetilde{\mathfrak{I}}_A$$

- Then, the (lower level) optimization problem for the leader P is

$$\min_{\widetilde{\lambda}_P \in \widetilde{\Lambda}_P^{ad}} \mathbb{E}_{\mathbb{P}_P}\left[j_P \circ S_{(\widetilde{\lambda}_A^\star, \widetilde{\lambda}_P)}\right]$$

Outline of the presentation

Witsenhausen intrinsic model and game theory with information

Nash equilibrium with information

Witsenhausen intrinsic model and principal-agent models

Games solvable by dynamic programming

Open questions

Research questions

- How should we talk about games using WIM?
  - How can we define players using the notion of agent?
  - How does the notion of subgame perfect Nash equilibrium translate within this framework?
  - Can we extend the Bayesian Nash Equilibrium concept to general risk measures?
- WIM: game theoretical results
  - What would Nash theorem be in the WIM setting?
  - When do we have a generalized "backward induction" mechanism?
- Applications of WIM
  - Can we re-organize the games bestiary using WIM?
  - Can we use the WIM framework for mechanism design?

How we can talk about games using WIM

- A player p is defined as a subset $\mathbb{A}_p$ of $\mathbb{A}$
- To each player p, we attach a criterion j_p and a belief $\mathbb{P}_p$
- If P is the set of all players, then $(\mathbb{A}_p)_{p \in P}$ is a partition of $\mathbb{A}$

We obtain a Nash theorem in the WIM setting

Theorem

Any finite, solvable, Witsenhausen game has a mixed NE

Proof

- The set of policies is finite, as policies map the finite history set towards finite decision sets
- To each policy profile, we associate a payoff vector
- We thus obtain a matrix game and we can apply Nash theorem

Generalized existence result

Step one, discretization

- We introduce $g_a^{(n)}$ the injection from $\mathbb{U}_a^{(n)}$ into $\mathbb{U}_a$

$$g_a^{(n)} : \mathbb{U}_a^{(n)} \hookrightarrow \mathbb{U}_a$$

- We introduce $h^{(n)}$ that maps $\mathbb{H}$ into $\mathbb{H}^{(n)}$ with $h^{(n)}_{\mathbb{H}^{(n)}} = Id_{\mathbb{H}^{(n)}}$
- $\left(\Lambda_a^{ad}\right)^{(n)} = \{\lambda_a \in \Lambda_a^{(n)}, \sigma(g_a^{(n)} \circ \lambda_a \circ h^{(n)}) \subseteq \mathcal{I}_a\}$

Current difficulties:

- Definition of the discretization, in particular $h^{(n)}$, to obtain a limit
- Continuity of the solution map

Applications

- The WIM is of particular interest for non sequential games
- In particular we envision applications for networks, auctions and decentralized energy systems

Thank you :-)