Demand-side Management over a Finite Time Horizon An Approximate Dynamic Programming Approach

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Outline

1. Problem Description
   - Co-optimization of Electricity and Reserve over a Single Time Period
   - Planning Demand over a Horizon
2. Approximate Policies
   - Linking Constraint
   - Decomposition
   - Utility - Consumption Curves
   - Heuristics
3. Stochastic Versions
   - How does it work?
   - Wait and See
   - Here and Now
   - Hybrid
4. Case Study

- The primary formulation is a bi-level optimization problem.

Maximize [Consumer's Profit]

Subject to:

[Consumer's Consumption and ILR Constraints]

vSPD:

Maximize [Social Welfare]

Subject to: [Node Balance Constraints]

[Reserve Constraints]

[Network Constraints]

[Generation/Demand Constraints]

Bi-level Optimization

Strategic Consumer

- The single node equivalent is:

$$
\begin{aligned}
\text{Maximize} \quad & u(q^c) - q^c.\pi^e + q^{ILR}.\pi^r \\
\text{s.t:} \quad & 0 \leq q^c \leq C^c \\
& 0 \leq q^{ILR} \leq C_r \\
& q^c - q^{ILR} \geq V \\
& \text{Max.} \quad \sum_{i \in I} p_i^c x_i^c - \sum_{j \in J} p_j^g x_j^g - \sum_{k \in K} p_k^r x_k^r \\
& \qquad \text{s.t.} \quad \sum_{i \in I} x_i^c = \sum_{j \in J} x_j^g - q^c \qquad [\pi^e] \\
& \qquad\qquad R - \sum_{k \in K} x_k^r = q^{ILR} \qquad [\pi^r] \\
& \qquad\qquad \text{[Tranche and Bathtub Constraints]}
\end{aligned}
$$

Mixed-Integer Problem

Strategic Consumer

- We use KKT conditions of the dispatch problem, as well as big-M parameters to reformulate our bi-level problem to a MIP.
- We define our MIP in a generic form at time period t as below:

$$[\text{MIP}]_{t,v} \quad \max \Pi_t(\mathbf{x}_t) = vx_t - \mathcal{C}_t(\mathbf{x}_t)$$
$$\text{s.t. } \mathbf{x}_t \in \mathcal{S}_t$$

- Here $\mathbf{x}_t$ is a vector of decision variables at time t (in our model it consists of consumption and ILR), and x_t is the consumption level at time t.
- v is the value of consuming one unit of electricity, and $\mathcal{C}_t$ is the cost function at time t.

Planning Demand over a Horizon

Strategic Consumer

- We optimize the consumption and ILR levels for the strategic consumer, given a total consumption level G over a time horizon $\mathcal{T}$.

$$[\text{MIP}] \quad \max \Pi(\mathbf{x}) = -\sum_{t\in\mathcal{T}} \mathcal{C}_t(\mathbf{x}_t) + \sum_{t\in\mathcal{T}} vx_t$$

$$\text{s.t. } \mathbf{x}_t \in \mathcal{S}_t \qquad \forall t \in \mathcal{T}$$

$$\sum_{t\in\mathcal{T}} x_t = G$$

- Note that $\sum_{t\in\mathcal{T}} vx_t^* = vG$, where $\mathbf{x}^*$ is the optimal solution to [MIP].

$$[\text{CM-MIP}] \quad \min \sum_{t\in\mathcal{T}} \mathcal{C}_t(\mathbf{x}_t)$$

$$\text{s.t. } \mathbf{x}_t \in \mathcal{S}_t \qquad \forall t \in \mathcal{T}$$

$$\sum_{t\in\mathcal{T}} x_t = G$$

- Given vG is constant, the optimal solution to [MIP] solves [CM-MIP].

Linking Constraint

- The constraint that links all the time periods together is the total consumption requirement.
- Although our problem is not convex, we can use the idea of Lagrangian relaxation to tackle this problem.

Linking Constraint

Price-taker consumer

- Imagine we had a price-taker consumer that minimizes its cost.
- In this setting our model is an LP, therefore we have convexity and duals on constraints.

$$
\begin{aligned}
\text{[LP]} \quad \min \quad & \sum_{t \in \mathcal{T}} \mathcal{C}_t(\mathbf{x}_t) & \\
\text{s.t.} \quad & S_t(\mathbf{x}_t) \leq 0 & \forall t \in \mathcal{T} \\
& \sum_{t \in \mathcal{T}} x_t = G & [u]
\end{aligned}
$$

Linking Constraint

Price-taker consumer

- We can write [LP] in the form below.

$$[\text{LP}] \quad \min \sum_{t\in\mathcal{T}} \mathcal{C}_t(\mathbf{x}_t) - u(\sum_{t\in\mathcal{T}} x_t - G)$$
$$\text{s.t. } S_t(\mathbf{x}_t) \leq 0 \qquad \forall t \in \mathcal{T}$$

- Given G is a constant, if we find the right $u = \hat{u}$ that solves [LP], we can separate [LP] by time periods. Hence for each t we have:

$$[\text{LP}]_t \quad \min \mathcal{C}_t(\mathbf{x}_t) - \hat{u}x_t$$
$$\text{s.t. } S_t(\mathbf{x}_t) \leq 0$$

Decomposing our MIP

- Can we use u to decompose our [CM-MIP]?
- Can we find the right u for our [CM-MIP]?
- What does this u mean in a non-convex model?

Decomposing our MIP

- Let's visit our generic model again.

$$[\text{CM-MIP}] \quad \min \sum_{t \in \mathcal{T}} \mathcal{C}_t(\mathbf{x}_t)$$
$$\text{s.t.} \quad \mathbf{x}_t \in \mathcal{S}_t \qquad \forall t \in \mathcal{T}$$
$$\sum_{t \in \mathcal{T}} x_t = G$$

- If we could price the total consumption constraint, the problem could be decomposed to each time period.

Decomposing our MIP

- We define $[\text{U-MIP}]_u$ as the problem with the total consumption constraint removed, and instead valued in the objective.

$$[\text{U-MIP}]_u \quad \max\ U(\mathbf{x}) = -\sum_{t\in\mathcal{T}} \mathcal{C}_t(\mathbf{x}_t) + \sum_{t\in\mathcal{T}} u x_t$$
$$\text{s.t. } \mathbf{x}_t \in \mathcal{S}_t \qquad \forall t \in \mathcal{T}$$

- Given the condition $G = \hat{G}$ and $\hat{G} \in \mathcal{G}$, where $G := \sum_{t\in\mathcal{T}} x_t^*$, we prove that the optimal solution of $[\text{U-MIP}]_u$ solves [CM-MIP].

Marginal utility, or Multiplier?

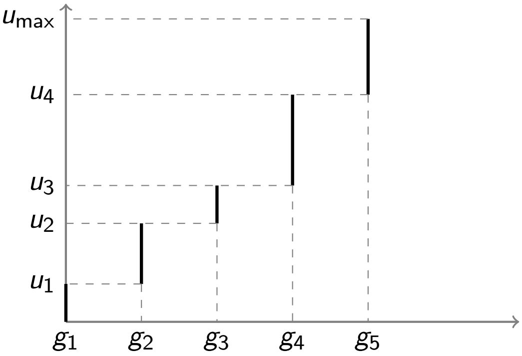
Price Maker Consumer

- If we find the right u for our model, we can solve each time period separately.
- How does an approximate dual look like in our non-convex model?

Utility-Consumption Curves

Price Maker Consumer

- If we solve the $[\text{U-MIP}]_{t,u}$, $\forall u \in [0, u_{\max}]$. We have the figure below:



- Scott and Read (1996) used demand curves for release in a hydro-scheduling setting.

Lemma

Let $\mathbf{x}^*$ *be the optimal solution to* $[MIP]_{t,u}$*, for some* t *and* u*.* $x^*(u)$ *is monotone* $\forall u \geq 0$*.*

Proof.

Let $\Pi(\mathbf{x}^*)$ be the optimal objective value of $[\text{MIP}]_{t,u}$. We define $[\text{MIP}]_{t,u'}$, for some $u' \geq u$, where $\Pi'(\mathbf{x}^{*\prime})$ is the optimal objective value of $[\text{MIP}]_{t,u'}$.

$$[\text{MIP}]_{t,u'} \max - \mathcal{C}(\mathbf{x}) + ux + (u' - u)x$$
$$\text{s.t. } \mathbf{x} \in \mathcal{S}$$

$\Pi(\mathbf{x}^*) = \Pi'(\mathbf{x}^*) - (u' - u)x^*$ and $\Pi(\mathbf{x}^{*\prime}) = \Pi'(\mathbf{x}^{*\prime}) - (u' - u)x^{*\prime}$
$\Pi(\mathbf{x}^*) \geq \Pi(\mathbf{x}^{*\prime}) \implies \Pi'(\mathbf{x}^*) \geq \Pi'(\mathbf{x}^{*\prime}) - (u' - u)(x^{*\prime} - x^*)$
If $x'^* \leq x^* \implies \Pi'(\mathbf{x}^*) \geq \Pi'(\mathbf{x}^{*\prime}) \implies \perp$ □

Utility-Consumption Curves

Price Maker Consumer

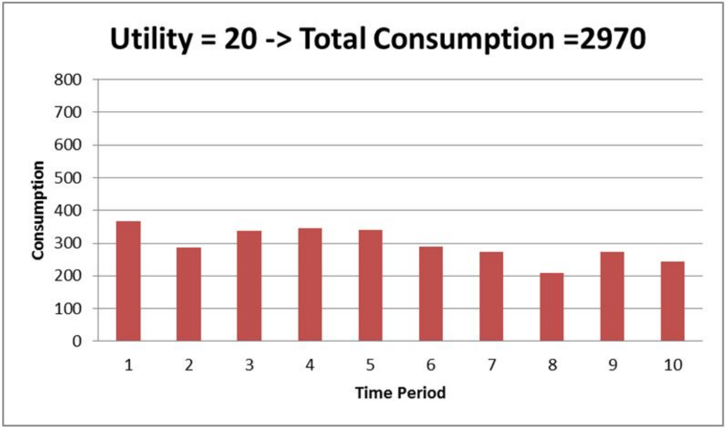


Figure: Utility vs Consumption

Utility-Consumption Curves
Price Maker Consumer

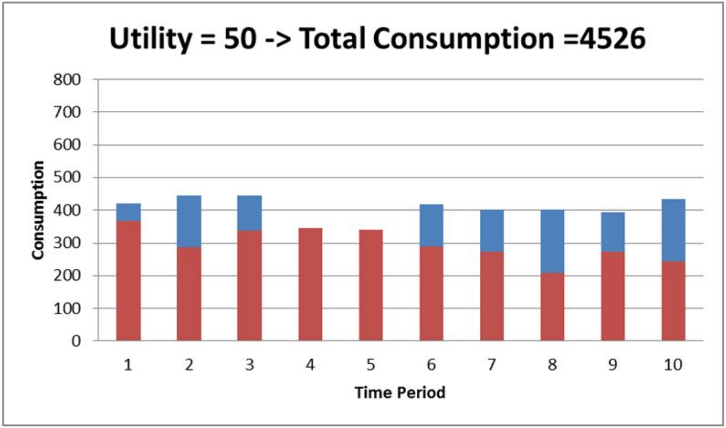


Figure: Utility vs Consumption

Utility-Consumption Curves

Price Maker Consumer

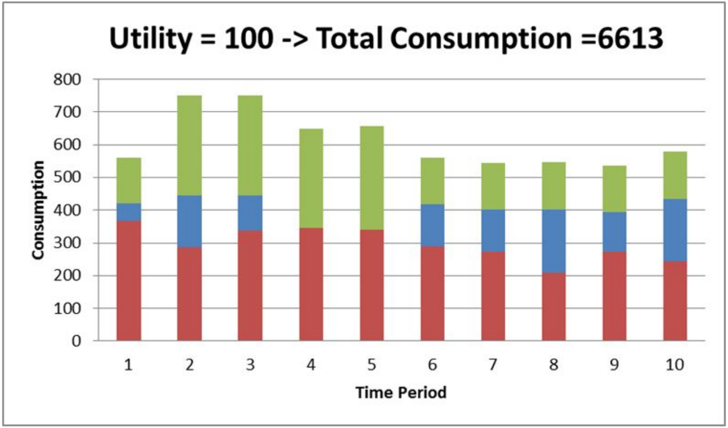


Figure: Utility vs Consumption

Utility-Consumption Curves

Price Maker Consumer

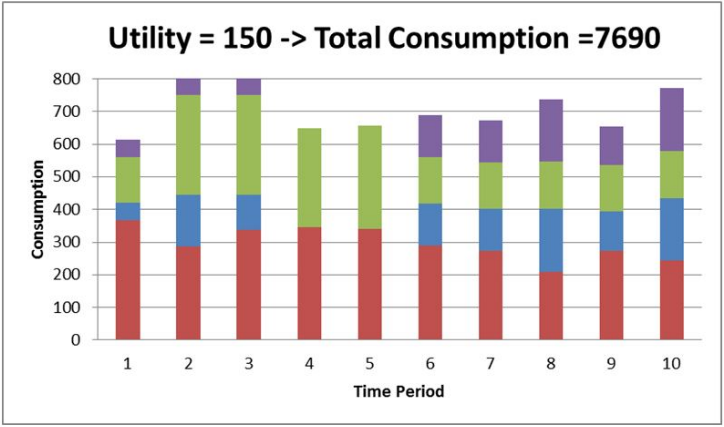
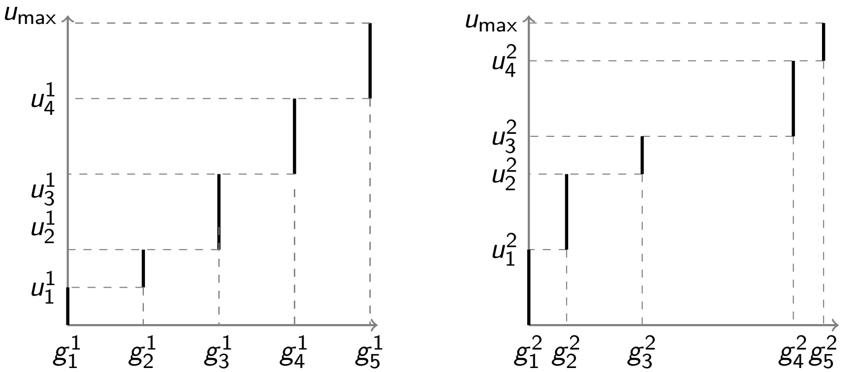


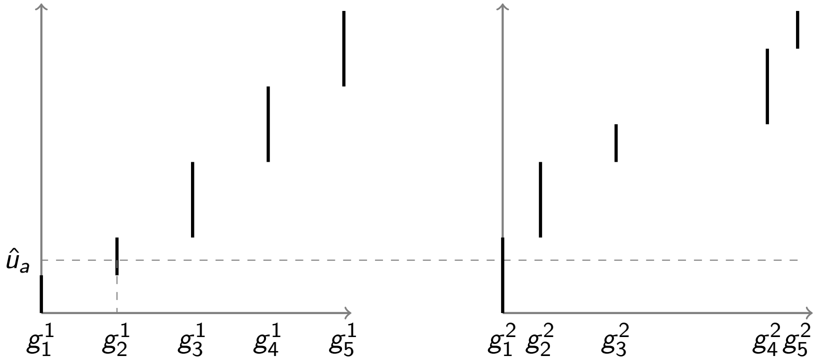
Figure: Utility vs Consumption

- First we build the U-C curve for each $t \in \mathcal{T}$. Assume we have $\mathcal{T} = \{1, 2\}$.



Aggregate Utility-Consumption Curves

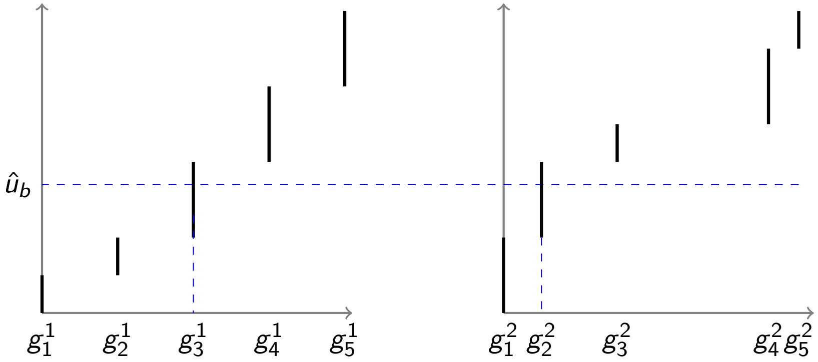
Price Maker Consumer



- $G(\hat{u}_a) = g_2^1 + g_1^2$

Aggregate Utility-Consumption Curves

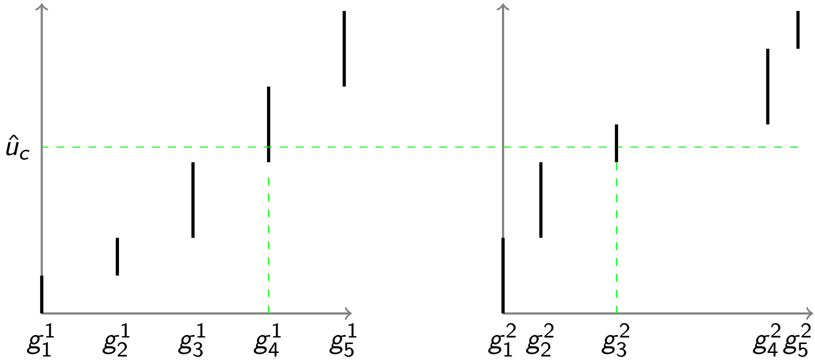
Price Maker Consumer



- $G(\hat{u}_b) = g_3^1 + g_2^2$

Aggregate Utility-Consumption Curves

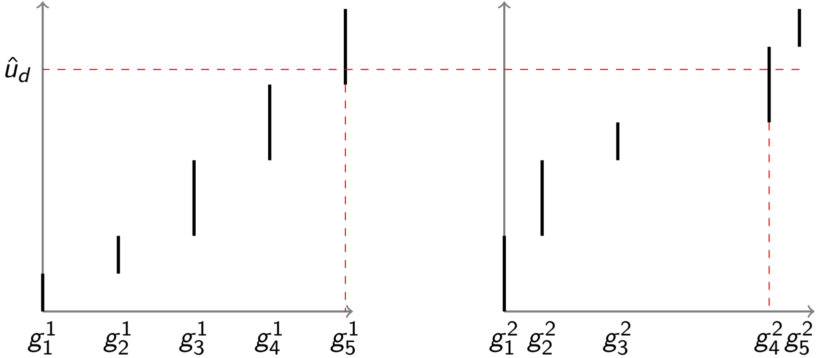
Price Maker Consumer



- $G(\hat{u}_c) = g_4^1 + g_3^2$

Utility-Consumption Curves

Price Maker Consumer

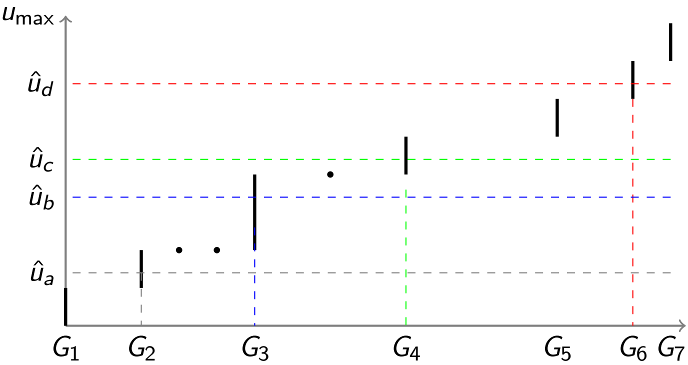


- $G(\hat{u}_d) = g_5^1 + g_4^2$

Aggregate Utility-Consumption Curves

Price Maker Consumer

- The aggregated curve is as below:



Utility-Consumption curves

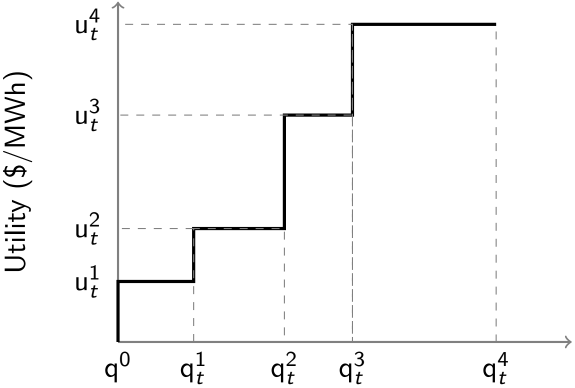
Price Maker Consumer

- In order to make policies, we construct the U-C curves $\forall t \in \mathcal{T}$.
- By aggregating the curves we will have a U-C curve, with more pieces, where the points on the G axis, determine the set of total consumption values $\mathcal{G}$ that we can choose from.
- After choosing the right $G \in \mathcal{G}$, we can look up our U-C curve and find the right $\hat{u}$ for our model.
- Therefore we can solve each $[\text{U-MIP}]_t$ separately, given $\hat{u}$.
- But what if our designated total consumption $\hat{G}$ is not in $\mathcal{G}$?

Heuristics

Price Maker Consumer

- We define the U-C curve for $t \in \mathcal{T}$ as below:



- We connect the vertical lines with virtual horizontal lines, to make step functions.

Heuristics

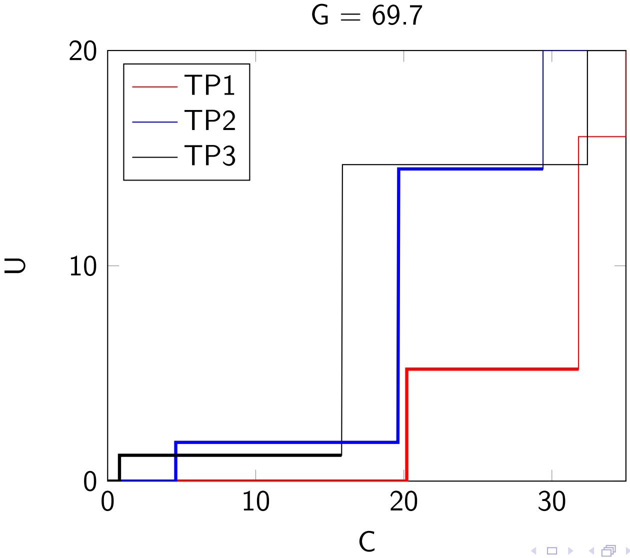
Price Maker Consumer

- This step-wise function has the same attributes as a supply function for a generator in the dispatch model.
- We view each time period as a node, and the total consumption value is the demand.
- We introduce [HEU], where we minimize utility $\times$ consumption.

$$
\begin{aligned}
\text{[HEU]} \quad \min \quad & \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{I}_t} u_t^i x_t^i \\
\text{s.t.} \quad & \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{I}_t} x_t^i = G && [\hat{u}] \\
& 0 \leq x_t^i \leq q_t^i - q_{t-1}^i && \forall i \in \mathcal{I}_t, \forall t \in \mathcal{T}
\end{aligned}
$$

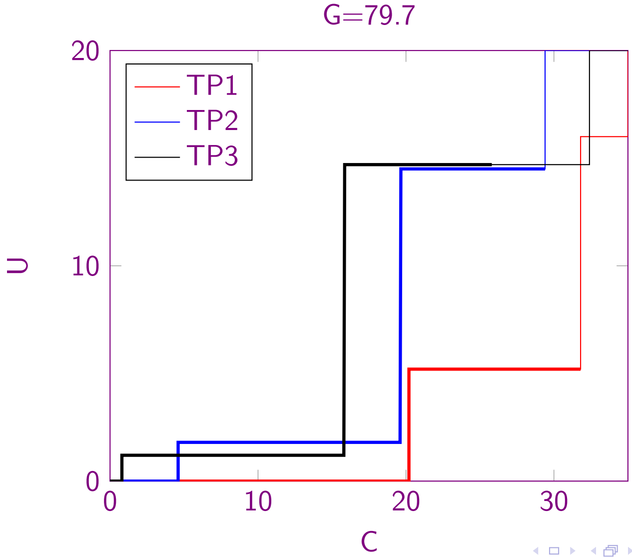
Heuristic Policy Example

- At this G, the Heuristic policy and optimal policy coincide.



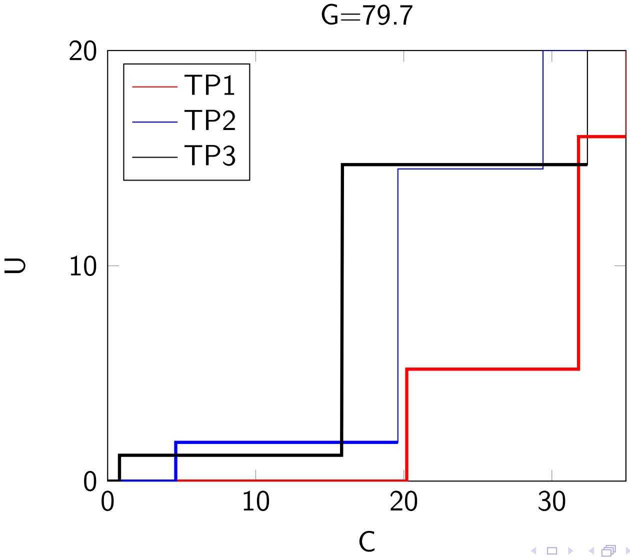
Heuristic Policy Example

- At this G, Heuristic policy add consumption to TP3.



Heuristic Policy Example

- Here, Optimal policy add consumption to TP3 and reduce from TP2.



Modelling Uncertainty

- We model stochasticity by defining scenarios $\omega \in \Omega_t, \forall t \in \mathcal{T}$.
- We will maximize expected utility $\mathbb{E}_{\omega \in \Omega_t, t \in \mathcal{T}} U(\mathbf{x})$.
- By utilizing intuitions from the price-taker model, we prove that we can use the same decomposition method for the stochastic model with some adjustments*.

Stochastic Dynamic Programming

Price taker consumer

- Imagine we have a price taker consumer that solves a stochastic dynamic program, which minimizes cost, with total consumption to go (g_t) as the state variable.
- For TP = T and $\forall \omega \in \Omega_T$ we have:

$$C_T^\omega(g_T) = \min \ \mathcal{C}(\mathbf{x}_T^\omega)$$
$$\text{s.t.} \ \ x_T^\omega = g_T \qquad [u_T^\omega]$$

- For TP $= t \in \{1, 2, .., T-1\}$ and $\forall \omega \in \Omega_t$ we have:

$$[\text{D-LP}]_t \ \ C_t^\omega(g_t) = \min \ \mathcal{C}(\mathbf{x}_t^\omega) + \mathbf{E}_{\nu \in \Omega^{t+1}}[C_{t+1}^\nu(g_{t+1}^\omega)]$$
$$\text{s.t.} \ \ x_t^\omega + g_{t+1}^\omega = g_t \qquad [u_t^\omega]$$

Stochastic Dynamic Programming

Price taker consumer

- By using the KKTs of the [D-LP]$_t$ we have:

$$\frac{\partial \mathbf{E}_{\nu \in \Omega^{t+1}}[C^{\nu}_{t+1}(g^{\omega}_{t+1})]}{\partial g^{\omega}_{t+1}} = u^{\omega}_t$$

$$\frac{\partial \mathcal{C}'(\mathbf{x}^{\omega}_t)}{\partial x^{\omega}_t} = u^{\omega}_t$$

- Also, as we have an LP, the global optimum of the objective function gives us the optimal solution values.

$$\mathcal{C}'(\mathbf{x}^{\omega}_t) = \mathbf{E}'_{\nu \in \Omega^{t+1}}[C^{\nu}_{t+1}(g^{\omega}_{t+1})]$$

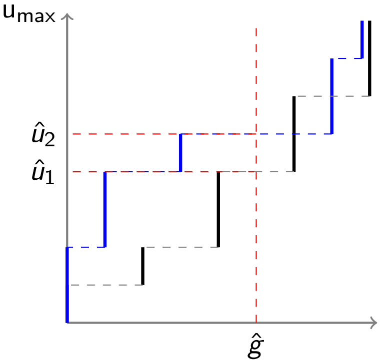
- We will utilize the equality above, to define the marginal utility vs marginal cost relationship in the price maker model.

Wait and See Model

Price maker consumer

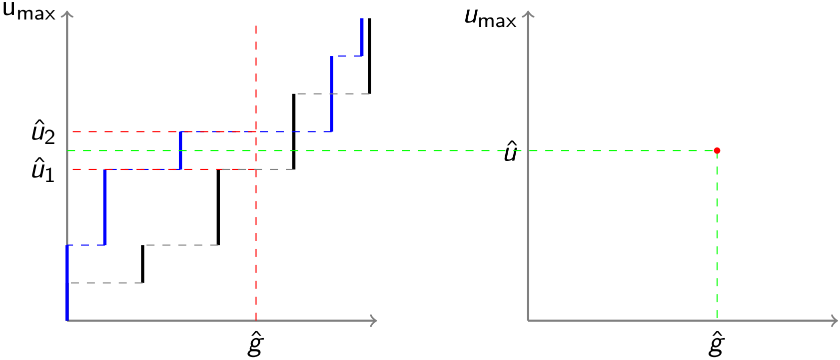
- We start with the wait and see setting, where we at each t we know the realized ω_t before taking an action, but future is uncertain.
- We make the expected U-C curve. For each consumption level q using the following algorithm.
- For $t = T$: $\boldsymbol{u}_T(q) = \mathbf{E}_{\omega \in \Omega_T} u_T^{\omega}(q)$.
- In order to aggregate the curve in backward induction, we use the intuition from the price taker model $(\mathbf{E}'_{\nu \in \Omega^{t+1}}[C_{t+1}^{\nu}(g_{t+1}^{\omega})] = u_t^{\omega})$.
- For $t \neq T$: if $u_t^{\omega}(\hat{q}_t) = \mathbf{E}_{\omega \in \Omega_{t+1}} u_{t+1}^{\omega}(\hat{q}_{t+1})$, we aggregate the consumption levels $q = \hat{q}_t + \hat{q}_{t+1}$ to build the U-C curve for time t and scenario ω $(u_t^{\omega}(q))$.
- For $t \neq T$: $\boldsymbol{u}_t(q) = \mathbf{E}_{\omega \in \Omega_t} u_t^{\omega}(q)$.

Constructing Wait and See Curves



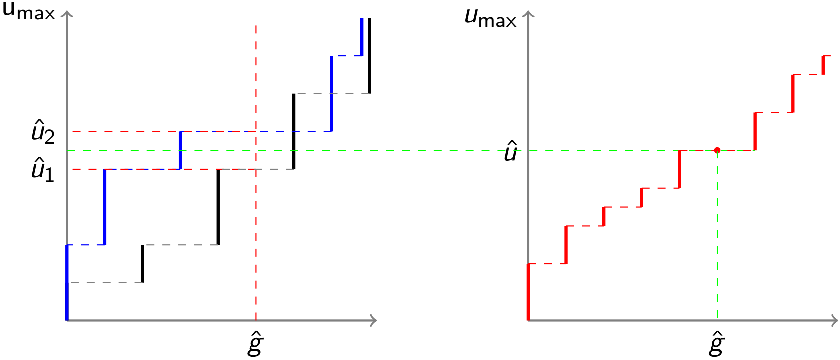
- $\hat{u}(\hat{g}) = \hat{\rho}_1 \hat{u}_1(\hat{g}) + \hat{\rho}_2 \hat{u}_2(\hat{g})$

Constructing Wait and See Curves



- $\hat{u}(\hat{g}) = \hat{\rho}_1 \hat{u}_1(\hat{g}) + \hat{\rho}_2 \hat{u}_2(\hat{g})$

Constructing Wait and See Curves



- $\hat{u}(\hat{g}) = \hat{\rho}_1 \hat{u}_1(\hat{g}) + \hat{\rho}_2 \hat{u}_2(\hat{g})$

Here and Now Model

- We may solve our model with a here and now decision process.
- In this setting, we know which scenario ω is realized only after we make a decision.
- In order to capture such uncertainty, we solve a stochastic $[\text{S-MIP}]_t$, $\forall t \in \mathcal{T}$, which maximizes expected profit over $\omega \in \Omega_t$.
- We solve $[\text{S-MIP}]_t$ $\forall u \in [0, u^{max}]$, and Construct the U-C curves (the same way as the deterministic model).

Hybrid

- We can combine the here and now and wait and see methods to enhance our scenario space.
- Here we use the here and now attributes of uncertainty (e.g. Sudden shutdown of a generator) to make scenarios for [S-MIP].
- And the wait and see properties (e.g. energy offers), to build Expected U-C curves.

Stage-wise Dependent Model

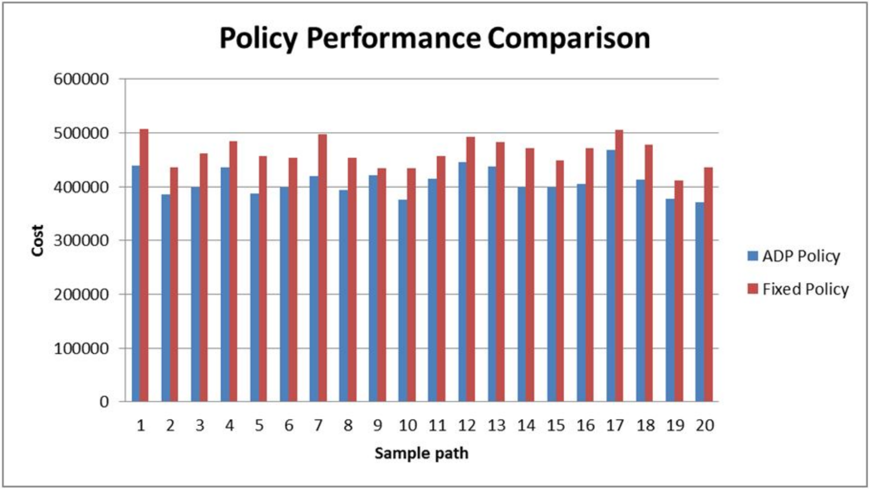
- We can use the Markovian nature of price processes in electricity markets to make the transition matrix.
- In order to model stage-wise dependency, we make U-C curves with conditional expectation, while the rest of the algorithm stays the same.

Case Study

- We implemented this ADP for a large consumer of electricity in South Island.
- We simulated the optimal policies, using historic data of weekdays in winter 2016.
- At each time period the sample space consists of 15 scenarios.

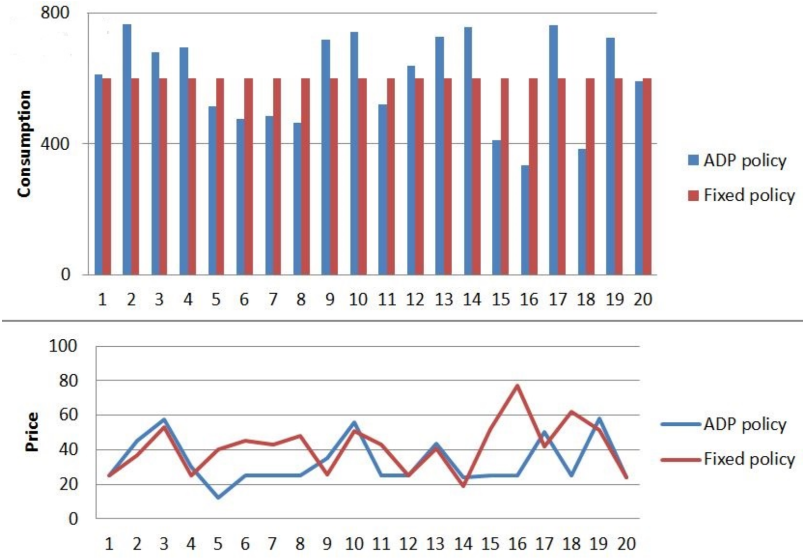
	Average cost per TP
ADP Policy	20456
Fixed Policy	23177

- The ADP policy reduce average cost by 11 percent.


Policy Performance Comparison
Cost
600000
500000
400000
300000
200000
100000
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
Sample path
ADP Policy
Fixed Policy

Case Study

Consumption - Price Comparison



Questions?

- Thank you.

Decomposing our MIP

- We define $[\text{U-MIP}]_u$ as the problem with the total consumption constraint removed, and instead valued in the objective.

$$
\begin{aligned}
[\text{U-MIP}]_u \quad & \max\ U(\mathbf{x}) = -\sum_{t\in\mathcal{T}} \mathcal{C}_t(\mathbf{x}_t) + \sum_{t\in\mathcal{T}} u x_t \\
& \text{s.t. } \mathbf{x}_t \in \mathcal{S}_t \qquad \forall t \in \mathcal{T}
\end{aligned}
$$

- Given one condition, we prove that the optimal solution of $[\text{U-MIP}]_u$ solves [CM-MIP].

Decomposing our MIP

Lemma

Let $\mathbf{x}^*$ *be the optimal solution to [U-MIP]*$_u$, *for some* $u \geq 0$, *and define* $G := \sum_{t \in \mathcal{T}} x_t^*$, *then* $\mathbf{x}^*$ *solves [CM-MIP].*

Proof.

$$\Pi(\mathbf{x}^*) = -\sum_{t \in \mathcal{T}} \mathcal{C}_t(\mathbf{x}_t^*) + \sum_{t \in \mathcal{T}} u x_t^* \geq -\sum_{t \in \mathcal{T}} \mathcal{C}_t(\hat{\mathbf{x}}_t) + \sum_{t \in \mathcal{T}} u \hat{x}_t \quad \forall \hat{x}_t \in \mathcal{X}$$

$$\Rightarrow \sum_{t \in \mathcal{T}} \mathcal{C}_t(\hat{\mathbf{x}}_t) \geq \sum_{t \in \mathcal{T}} u(\hat{x}_t - x_t^*) + \sum_{t \in \mathcal{T}} \mathcal{C}_t(\mathbf{x}_t^*)$$

$$\Rightarrow \sum_{t \in \mathcal{T}} \mathcal{C}_t(\hat{\mathbf{x}}_t) \geq \sum_{t \in \mathcal{T}} \mathcal{C}_t(\mathbf{x}_t^*)$$

□

SDP decomposition - Notation

- In order to describe our decomposition algorithm we use the following notation.
- We index our decision variables by nodes $n \in \mathcal{N}$ in the scenario tree.
- We define $\mathcal{N}_t$ as the set of nodes in time period t and ρ_n as the probability of ending up in node n.
- P is the matrix of paths, where row m ($\mathbf{P}_m$) corresponds to the path leading to node m (at $t = T$).
- u_n is the marginal utility of consumption at node n.

SDP decomposition - W&S Proof

- We aim to solve the stochastic cost minimization problem [S-CM-MIP], defined as below:

$$
\begin{aligned}
\text{[S-CM-MIP]} \quad \min \sum_{t \in \mathcal{T}} \sum_{n \in \mathcal{N}_t} \rho_n \mathcal{C}_n(\mathbf{x}_n) & \\
\text{s.t. } \mathbf{x}_t \in \mathcal{S}_t & \qquad \forall n \in \mathcal{N} \\
\sum_{x_n \in \mathbf{P}_m} x_n \geq G & \qquad \forall m \in \mathcal{N}_T
\end{aligned}
$$

- We solve the stochastic utility maximization MIP [S-U-MIP] with the marginal utility u for each node $n \in \mathcal{N}$ separately for all possible u values.

$$
\begin{aligned}
\text{[S-U-MIP]}_n \quad \max u_n x_n - \mathcal{C}_n(\mathbf{x}_n) & \qquad \forall n \in \mathcal{N} \\
\text{s.t. } \mathbf{x}_t \in \mathcal{S}_t & \qquad \forall n \in \mathcal{N}
\end{aligned}
$$

SDP decomposition - W&S Proof

- We define $\mathcal{X}_n$ as the set of all optimal consumption values of $[\text{S-U-MIP}]_n$, $\forall u_n \in \mathcal{U}$.
- Now we solve [XS-CM-MIP], while we pick our x_n values from $\mathcal{X}_n$:

$$
\begin{aligned}
\text{[XS-CM-MIP]} \;\; \min \sum_{t\in\mathcal{T}} \sum_{n\in\mathcal{N}_t} \rho_n \mathcal{C}_n(\mathbf{x}_n) & \\
\text{s.t.} \;\; \mathbf{x}_t \in \mathcal{S}_t & \qquad \forall n \in \mathcal{N} \\
\sum_{x_n \in \mathbf{P}_m} x_n \geq G & \qquad \forall m \in \mathcal{N}_T \\
x_n \in \mathcal{X}_n & \qquad \forall n \in \mathcal{N}.
\end{aligned}
$$

- Here we have $\hat{G}_m = \sum_{x_n^* \in \mathbf{P}_m} x_n^*$. We also know that $\forall m \in \mathcal{N}_T$ we have $\hat{G}_m \geq G$.
- We define u_n^* as the marginal utility corresponding to the chosen optimal consumption value of node n (x_n^*). We call $\mathbf{y}^*$ the vector of optimal consumption values of [XS-CM-MIP].

SDP decomposition - W&S Proof

- Now we solve the [GS-CM-MIP], with an additional constraint on the total consumption for each path.

$$[\text{GS-CM-MIP}] \quad \mathcal{C}(\mathbf{x}) = \min \sum_{t\in\mathcal{T}} \sum_{n\in\mathcal{N}_t} \rho_n \mathcal{C}_n(\mathbf{x}_n)$$

$$\text{s.t. } \mathbf{x}_t \in \mathcal{S}_t \qquad \forall n \in \mathcal{N}$$

$$\sum_{x_n\in\mathbf{P}_m} x_n = \hat{G}_m \qquad \forall m \in \mathcal{N}_T$$

- Using a similar proof to that of previous lemma we prove that the optimal consumption value of [[XS-CM-MIP] solves [GS-CM-MIP].

SDP decomposition - W&S Proof

- We also show that the expected utilities in U-C curves are calculated correctly:

$$u_n^* = \frac{\sum_{m \in \mathcal{N}_T | x_n^* \in \mathbf{P}_m} \rho_m \mu_m}{\sum_{m \in \mathcal{N}_T | x_n^* \in \mathbf{P}_m} \rho_m}$$

- μ_m is the optimal marginal utility associated with the leaf node m.

Utility-Consumption Curves

Monotonicity Proof

Lemma

Let $\mathbf{x}^*$ *be the optimal solution to [MIP]*$_{t,u}$*, for some* t *and* u*, and* $\Pi(\mathbf{x}^*)$ *be the optimal objective value of [MIP]*$_{t,u}$*.* $x^*(u)$ *is monotone* $\forall u \geq 0$*.*

Proof.

We define $[\text{MIP}]_{t,u'}$, for some $u' \geq u$, where $\Pi'(\mathbf{x}^{*\prime})$ is the optimal objective value of $[\text{MIP}]_{t,u'}$.

$$[\text{MIP}]_{t,u'} \max -\mathcal{C}(\mathbf{x}) + ux + (u' - u)x$$
$$\text{s.t. } \mathbf{x} \in \mathcal{S}$$

$\Pi(\mathbf{x}^*) = \Pi'(\mathbf{x}^*) - (u' - u)x^*$ and $\Pi(\mathbf{x}^{*\prime}) = \Pi'(\mathbf{x}^{*\prime}) - (u' - u)x^{*\prime}$
$\Pi(\mathbf{x}^*) \geq \Pi(\mathbf{x}^{*\prime}) \implies \Pi'(\mathbf{x}^*) \geq \Pi'(\mathbf{x}^{*\prime}) - (u' - u)(x^{*\prime} - x^*)$
If $x'^* \leq x^* \implies \Pi'(\mathbf{x}^*) \geq \Pi'(\mathbf{x}^{*\prime}) \implies \perp$

□