Submitted to *Operations Research*
manuscript (Please, provide the manuscript number!)

Dynamic risked equilibrium

Michael Ferris
Computer Sciences Department and Wisconsin Institute for Discovery, University of Wisconsin, Madison, WI 53706, ferris@cs.wisc.edu

Andy Philpott
Electric Power Optimization Centre, University of Auckland, New Zealand, a.philpott@auckland.ac.nz

We study a competitive partial equilibrium in markets where risk-averse agents solve multistage stochastic optimization problems formulated in scenario trees. The agents trade a commodity that is produced from an uncertain supply of resources. Both resources and the commodity can be stored for later consumption. Several examples of a multistage risked equilibrium are outlined, including aspects of battery and hydroelectric storage in electricity markets, distributed ownership of competing technologies relying on shared resources, and aspects of water control and pricing. The agents are assumed to have nested coherent risk measures based on one-step risk measures with polyhedral risk sets that have a non-empty intersection over agents. Agents can trade risk in a complete market of Arrow-Debreu securities. In this setting we define a risk-trading competitive market equilibrium and establish two welfare theorems: competitive equilibrium will yield a social optimum (with a suitably defined social risk measure) when agents have strictly monotone one-step risk measures. Conversely, a social optimum with an appropriately chosen risk measure will yield a risk-trading competitive market equilibrium when all agents have strictly monotone risk measures. The paper also demonstrates versions of these theorems when risk measures are not strictly monotone.

Key words: coherent risk measure, partial equilibrium, perfect competition, welfare theorem

History: This paper was first submitted on March 20, 2018.

1. Introduction

In many competitive situations, manufacturers of a product that is sold over several periods use storage to improve their profits. Storage of the finished product enables the manufacturers to store the product in periods when prices are low for later sale in periods when they are high. In practice, prices are uncertain and so the optimal storage policy becomes the solution to a stochastic control problem in which manufacturers seek to maximize expected profits if risk neutral, or some risk-adjusted profit if they are risk averse.

In some cases it is also possible to store the raw materials used in production. An example arises in renewable electricity production in which intermittent generation (wind or photovoltaic energy) can be stored in a battery for later sale. Similarly hydroelectric reservoirs can store energy for later conversion to electricity, or farmers can store pasture (or silage, its harvested form) for later conversion into milk by dairy cows. The process by which the storage is replenished has a random element (e.g. wind, sunlight, catchment inflows, and beneficial weather, in the respective examples we cite). Storage of raw materials enables the producer to maximize their capacity utilization when sale prices are high, while possibly holding back production during low-priced periods.

Our current interest focuses on a situation in which prices of the finished product are determined by an equilibrium of several competing producers, where the total sales of product from the manufacturers equals the demand from consumers in each period. Demand is defined in terms of price by a known decreasing demand function.

The simplest case occurs when the future is known with certainty and producers have convex costs. Then an equilibrium time-varying price can be derived from a Lagrangian decomposition of a social planning model that seeks to maximize the consumer and producer surplus summed over all periods. The Second Welfare Theorem (see e.g. Feldman and Serrano (2006)) in this setting is a straightforward consequence of Lagrangian duality theory, and states that the optimal social plan can be interpreted as a perfectly competitive equilibrium at the prices that solve the dual problem. The First Welfare Theorem, stating that any perfectly competitive equilibrium maximizes the consumer and producer surplus in the social plan is also immediate from this duality.

When the parameters of the model are uncertain, but governed by a known stochastic process, the social planning problem becomes a multistage stochastic programming problem. Multistage stochastic optimization models have been well studied (see e.g. Birge and Louveaux (2011), Shapiro et al. (2014)). If all agents act as price takers, and seek to maximize expected operating profits, then the first and second welfare theorems translate naturally into the stochastic setting. When the stochastic supply process (or its approximation) is represented by a scenario tree (see Birge and Louveaux (2011)) the Lagrangian theory can be applied to the extensive form of the deterministic equivalent social planning problem and its dual to yield versions of these theorems.

Multistage stochastic optimization becomes more complicated when agents are risk-averse. As in the risk-neutral case, decisions are measurable with respect to the filtration defined by the random parameters. For a scenario tree this means that decisions are made in each node given the history of information accrued in previously visited nodes. When combined with realizations of the random parameters, these decisions lead to a stochastic process of payoffs defined at the nodes of the scenario tree. A risk-averse optimizer then requires a preference relation over these random payoff processes to be able to compare different policies. In a two-stage setting, risk preferences can be approached through a wide variety of models including but not limited to utility theory (Von Neumann and Morgenstern (2007)), mean-variance optimization (Markowitz (1952)), value-at-risk (Jorion (2000)), stochastic dominance (Levy (1992)), prospect theory (Tversky and Kahneman (1992)), dual utility theory (Yaari (1987)), and coherent risk measures (Artzner et al. (1999)). For a summary and comparison of these and other approaches to optimzation under risk see e.g. Anderson (2013) or Shapiro et al. (2014).

A theory of extending one-step risk preferences to a multistage setting using conditional risk mappings is described in Shapiro et al. (2014). Conditional risk mappings add current costs to risk-adjusted uncertain future costs expressed as a certainty-equivalent value defined in terms of a single-step coherent risk measure as defined by Artzner et al. (1999). The translation equivariance and monotonicity axioms of coherent risk measures then enable the evaluation of the risk-adjusted cost of a random cost sequence using a recursive formula. When information is revealed over time, and agents make optimal decisions given their current history of observations of random outcomes, this enables the solution of a risk averse dynamic optimization problem using dynamic programming. Mean-variance measures of risk do not extend like this, and following the same approach using utility theory is only possible for a limited set of utility functions (e.g. linear or exponential (Howard and Matheson (1972))) that have translation-equivariant certainty-equivalent forms. For example the one-step exponential utility function leads to the class of entropic risk measures (Föllmer and Knispel (2011)) which are translation-equivariant and monotone.

When one-step risk measures are translation-equivariant, monotone, convex and positively homogeneous (i.e. coherent), the certainty-equivalent value of future costs at a node of the scenario tree can be expressed using duality theory as the conditional expectation of future costs with respect to a probability measure that is chosen to be the worst in a convex set of conditional probability distributions (see e.g. Shapiro et al. (2014)). There is assumed to be such a set of distributions (which we denote the *risk set*) defined for every node of the scenario tree.

Asset pricing models with risk-averse agents have been widely studied in a two-stage equilibrium setting. The classical economics literature has many examples of this including the capital asset pricing model (CAPM) (Sharpe (1964)) and asset-pricing models of Arrow and Debreu (Arrow (1973)) in complete markets. The CAPM model has been successfully applied to forward contracting in electricity markets (a key application area) by Bessembinder and Lemmon (2002). However, since it is based on a single-step mean-variance risk measure which is not translation equivariant, it is hard to see how to extend the CAPM model to a multistage setting.

Following Heath and Ku (2004) and Ralph and Smeers (2015) our work is more closely related to the asset-pricing models of Arrow (1973). In this setting, the welfare theorems rely heavily on the concept of *market completeness* achieved though a set of *Arrow-Debreu securities* that span all possible random future outcomes. In a classical two-stage setting, a complete market of Arrow-Debreu securities will ensure by a no-arbitrage argument that every collection of contingent payoffs in stage 2 can be priced at stage 1 using the state prices of the Arrow-Debreu instruments (see e.g. Varian (1987) for an elementary explanation of this principle). The results of Heath and Ku

(2004) (assuming finite probability distributions) and Ralph and Smeers (2015) (for continuous distributions) provide asset prices for a complete market of Arrow-Debreu securities in exchange economies of agents with risk-averse coherent risk measures. They show that when the relative interiors of the risk sets of agents intersect, agents will trade Arrow-Debreu securities at equilibrium prices that are a probability measure lying in this intersection. These prices have an interpretation as risk-adjusted probabilities (state prices) that all agents agree on when evaluating their payoffs, and so provide risk-adjusted probabilities for a social planner to evaluate total system welfare. Using common probability distributions yields risk-adjusted welfare in equilibrium that has the same risk-adjusted value in the social plan.

Our goal in this paper is to extend the welfare theorems for partial equilibrium to a multistage setting with risk-averse agents. We assume perfect competition throughout the paper, so agents are assumed to be price takers. The recent paper Philpott et al. (2016) (building on the models of Heath and Ku (2004) and Ralph and Smeers (2015) studies a special case of this problem for multistage electricity markets when some producers operate hydroelectric reservoirs with uncertain inflows. Under an assumption that agents can trade risk using a complete set of Arrow-Debreu securities, Philpott et al. (2016) show that a risk-averse social planning solution with an appropriately chosen risk measure can be interpreted as a competitive equilibrium in which the agents trade risk. This result corresponds to the Second Welfare Theorem.

The result in Philpott et al. (2016) is specific to electricity systems with hydroelectric generators. In this paper we extend this theorem to systems that operate with storage in a more general setting. Like the hydroelectricity case, agents can store raw materials (water) for later electricity production, but we also admit the possibility of storing the commodity (corresponding to e.g. battery storage in the electricity setting). Agents might own and operate a single production or storage facility, or a collection of both production and storage facilities in different locations. The storage facilites could be a linked system of raw material storage sites (such as a river chain of hydro reservoirs) or a system of final product storage sites (e.g. warehouses linked by roads, or batteries linked by electricity transmission lines).

We also add to the theoretical results in Philpott et al. (2016), by giving new proofs of both first and second welfare theorems. Our second welfare theorem (Theorem 4 and Corollary 2) is an extension of Theorem 11 in Philpott et al. (2016) to the more general case. The proof in this general case is arguably simpler. It also illuminates the role of strict monotonicity of risk measures in stochastic risked equilibrium. Theorem 3 and Corollary 1 (which are both new) give our version of the First Welfare Theorem (which is not discussed in Philpott et al. (2016)).

Our motivation in studying welfare theorems comes from a desire to understand imperfectly competitive markets. The analogue of the Second Welfare Theorem shows that a social planner could argue that their actions in solving a risk-averse social planning problem replicates what one might expect to see in a perfectly competitive market with a complete market for trading risk. A number of wholesale electricity markets (e.g. Brazil and Chile) operate on this principle, whereby regulated energy prices are computed using an agreed social planning model rather than emerging from a market trading process.

The (newly established) analogue of the First Welfare Theorem shows that if markets are perfectly competitive and endowed with a complete market for trading risk, then one might expect agents in them to arrive at an equilibrium using policies that maximize risk-adjusted social welfare. In other words, Theorem 3 and Corollary 1 provide a perfectly competitive benchmark against which real markets might be measured. In the real world, where markets are imperfect, the optimal value of a social planning model provides an upper bound on what might be achieved in welfare terms by reducing market imperfections.

It is worth remarking that the welfare results we establish suffer from some restrictive assumptions. Markets are not perfectly competitive, and nearly always incomplete. The assumption of a complete set of priced Arrow-Debreu securities to cover every possible random event is clearly impossible. A number of authors (see e.g. de Maere d'Aertrycke and Smeers (2013), Abada et al. (2017), Kok et al. (2018)) have explored the effect of replacing this assumption in two-stage models with a limited set of traded instruments. In some experiments this restriction can significantly reduce welfare, while in others it has only a minimal effect on welfare losses compared with outcomes from a risk-averse social plan. The results in this paper demonstrate that such welfare losses are an artifact of incompleteness in the risk market rather than some other imperfection, and so help in identifying potential market interventions that might reduce them.

In summary, the contributions of the paper are as follows:

1. We extend the definition of multistage risked equilibrium given in Philpott et al. (2016) to a more general model.
2. We provide a simpler proof of our second welfare theorem as applied to multistage risked equilibrium with risk trading.
3. We state and give a proof of a first welfare theorem (which is new) as applied to multistage risked equilibrium with risk trading.
4. We illuminate the role that strict monotonicity of risk measures plays in multistage risked equilibrium.

The paper is laid out as follows. In the next section we describe the underlying model and its constituent stochastic, dynamic and optimizing agent components, and provide several motivating examples that can be cast into the framework. Section 3 provides a viewpoint of dynamic risk measures, with specific examples, introduces the notion of dynamic consistency, determines optimality conditions for a system optimization problem that incorporates a dynamic risk measure, and links this to a multistage risked equilibrium problem. Section 4 adds the notion of risk trading to these equilibria, and provides the main results, providing counterparts of the first and second welfare theorems in the multistage risked setting. We conclude the paper with a summary of the results and some suggestions for future research. The proofs of the main results of the paper are given in the appendices. We have split these into appendices A, B and C containing results related to coherent risk measures, some technical results linking conditional tree multipliers to unconditional multipliers, and the proofs of the main results, respectively.

2. Models

In our model, random events are defined by a discrete-time stochastic process, with a finite set of events in each stage. Such a process can be modeled using a *scenario tree* with nodes $n \in \mathcal{N}$ and leaves in $\mathcal{L}$. The probability of the event represented by node n is denoted $\phi(n)$. By convention we number the root node $n = 0$. The unique predecessor of node $n \neq 0$ is denoted by n_-. We denote the set of children of node $n \in \mathcal{N} \setminus \mathcal{L}$ by n_+, and denote its cardinality by $|n_+|$. The set of predecessors of node n on the path from n to node 0 is denoted $\mathcal{P}(n)$ (so $\mathcal{P}(n) = \{n, n_-, n_{--}, \ldots, 0\}$), where we use the natural definitions for n_{--}. The set of successors of node n is $\mathcal{S}(n) = \{n\} \cup n_+ \cup n_{++} \cup \ldots.\}$ where n_{++} is defined in the obvious way. The depth $\delta(n)$ of node n is the number of nodes on the path to node 0, so $\delta(0) = 1$ and we assume that every leaf node has the same depth, say $\delta_{\mathcal{L}}$. The depth of a node can be interpreted as a time index $t = 1, 2, \ldots, T = \delta_{\mathcal{L}}$. A pictorial representation of a scenario tree with four time stages is given in Figure 1.

We assume that there are a number of agents in the model, indexed by $a \in \mathcal{A}$. At each node n in the scenario tree, the agents observe a realization of random parameters, and seek optimal actions $u_a(n)$ to minimize their current and risk-adjusted future disbenefit. The current disbenefit of agent a in node n consists of a cost $C_{an}(u_a(n))$, and expenses and rewards from trading with other agents. Ignoring these wealth transfers, the current system disbenefit in node n is the total cost $\sum_{a \in \mathcal{A}} C_{an}(u_a(n))$. Here for producer a, C_{an} measures production cost, and for consumer a, C_{an} measures consumption disbenefit that increases as (negative) u_a increases towards 0. We assume that each C_{an} is convex.

Each producing agent a consumes resources at node n that come from a vector $x_a(n_-)$ of storages that are released at rates defined by the vector $u_a(n)$ yielding total production $g_{an}(u_a(n))$. The storage is replenished by agent actions (such as charging a battery with purchased electricity) or by (possibly) random supplies (such as inflows or photovoltaic input). Denoting the latter by $\omega_a(n)$ gives a stochastic process defined by

$$x_a(n) \leq x_a(n_-) + \sum_{b \in \mathcal{A}} T_{ab} u_b(n) + \omega_a(n).$$

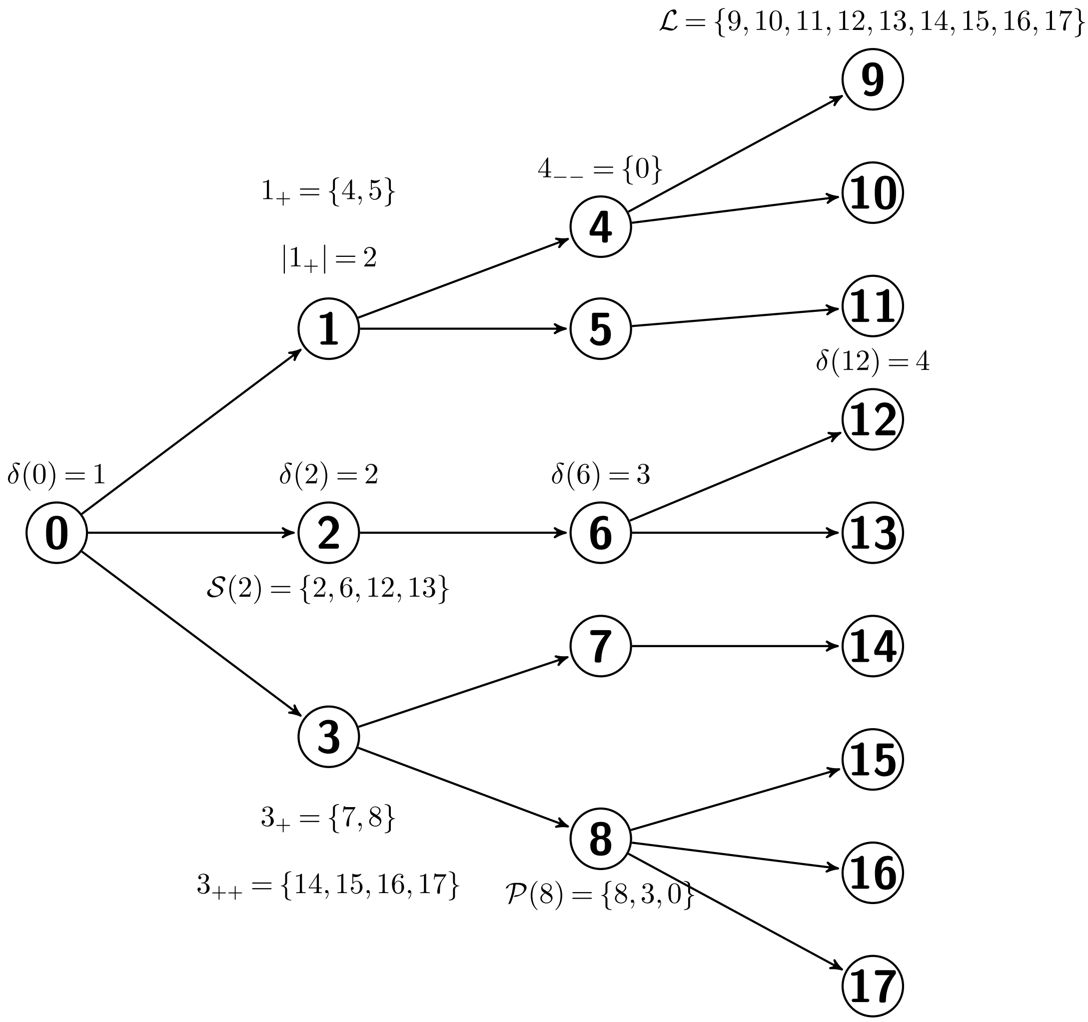

Figure 1 A scenario tree with nodes $\mathcal{N}=\{1,2,\ldots,17\}$, and $T=4$

Note that the matrix T_{ab} in the dynamics allows for a network of connections between storage devices controlled by different agents, and the inequality allows for free disposal (or spilling) at the storage device. The dynamics could be expressed a little more generally using a diagonal matrix S_a for gains or losses and making S and T dependent on node as

$$x_a(n) \le S_a(n)x_a(n_-) + \sum_{b\in\mathcal{A}} T_{ab}(n)u_b(n) + \omega_a(n),$$

but since this does not change the subsequent analysis in any substantive way, we assume $S_a(n) \equiv I$ and $T_{ab}(n) \equiv T_{ab}$ in what follows. The actions u_a and storages x_a are constrained to lie in respective sets $\mathcal{U}_a$ and $\mathcal{X}_a$. Finally for each leaf node $n \in \mathcal{L}$, we define $V_{an}(x_a(n))$ to represent the value of residual storage $x_a(n)$ held by agent a at node n.

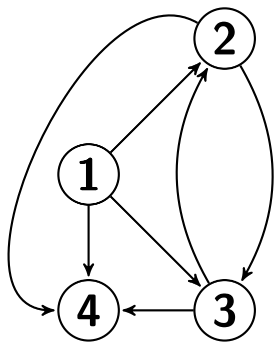

Figure 2 Small network example with production at node 1, consumption at node 4 and storage at nodes 2, 3

Given a scenario tree we can now formulate a risk-neutral model that seeks to minimize total expected social disbenefit.

$$\begin{aligned}
\text{SO: } \min_{u,x} \; & \sum_{n\in\mathcal{N}} \phi(n) \sum_{a\in\mathcal{A}} C_{an}(u_a(n)) - \sum_{n\in\mathcal{L}} \phi(n) \sum_{a\in\mathcal{A}} V_{an}(x_a(n)) \\
\text{s.t. } & x_a(n) \le x_a(n_-) + \sum_{b\in\mathcal{A}} T_{ab} u_b(n) + \omega_a(n), \quad n\in\mathcal{N}, \quad a\in\mathcal{A}, && (1)\\
& \sum_{a\in\mathcal{A}} g_{an}(u_a(n)) \ge 0 \quad n\in\mathcal{N}, && (2)\\
& u_a(n)\in\mathcal{U}_a, \quad x_a(n)\in\mathcal{X}_a, \quad n\in\mathcal{N}, \quad a\in\mathcal{A}.
\end{aligned}$$

We now outline several examples that fit into this framework, and demonstrate the interplay between production units and storage devices, and the agents that own and operate them.

Example 1. The first example involves a collection of consumers, producers and storage entities. These are connected via a network, an example of which is given in Figure 2. This network is encoded into a matrix T whose rows correspond to locations, and columns correspond to edges in the network:

$$\begin{bmatrix} a=1 \\ a=2 \\ a=3 \\ a=4 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & -1 & & & & & \\ & 1 & & & -1 & -1 & 1 & & \\ & & 1 & & 1 & & -1 & -1 & \\ & & & 1 & & 1 & & 1 & -1 \end{bmatrix}$$

This example assumes ownership of a single entity (at the given location) by an agent $a \in \mathcal{A}$ and operates only on a single good that can be produced, stored or consumed at a location, and transported from one location to another. Thus agents $a \in \mathcal{A}$ index the rows of the matrix (corresponding to the network nodes), with $x(n) \in \mathbb{R}^{|\mathcal{A}|}$. Column 1 of T corresponds to production occurring at location 1, while column 9 corresponds to a consumer at location 4, and we will call these edges production or consumption edges respectively. The other columns correspond to arcs in Figure 2. Each agent controls the flows along the arcs emanating from the node that she controls. For example agent $a = 2$ controls flows in arcs (2,3) and (2,4), so $u_a(n)$ has two components corresponding to columns 5 and 6 of T. The net flow into node a is $\sum_{b \in \mathcal{A}} T_{ab} u_b(n)$, so when $a = 2$ we have $T_{a1} = [0 \quad 1 \quad 0 \quad 0]$, $T_{a2} = [-1 \quad -1]$, $T_{a3} = [1 \quad 0]$, and $T_{a4} = [0]$. In total, $u(n) \in \mathbb{R}^{|\mathcal{E}|}$ where $\mathcal{E}$ is the collection of arcs in the network. Constraints (such as capacities or operational considerations) on flows and storage are captured by the sets $\mathcal{U}_a$ and $\mathcal{X}_a$. Treating the variables x and u in vectorized form, this gives a stochastic process defined by

$$x(n) \leq x(n_-) + Tu(n) + \omega(n),$$

where the inequality allows for free disposal of the good. It is possible to generalize T to account for transport gains/losses, and to add a diagonal matrix S to account for storage inefficiencies as mentioned above.

In this example, the functions $g_{an}(u_k(n))$ are $u_k(n)$ if arc k emanates from location a and $-u_k(n)$ otherwise and thus they are separable over $u_a(n)$, $a \in \mathcal{A}$. The cost functions are production cost, consumption disbenefit, and 0 for storage devices, while $V_{an}(x_a(n))$ captures the value of residual storage $x_a(n)$ at any leaf node of the scenario tree. Situations that are covered by this type of formulation include a production/distribution network where storage devices are warehouses and arcs represent transportation links, and also the situation of a distributed system of batteries that could be used to store energy generated by fossil fuel or renewable energy production systems.

Example 2. The second example generalizes the previous situation to allow agents to be firms controlling a collection of production, storage and demand facilities in different locations. Thus $a \in \mathcal{A}$ now indexes firms, and $x_a(n) \in \mathbb{R}^{m(a)}$ is a vector of storage amounts at the locations controlled by firm a. The vector $u_a(n)$ again represents the flows that occur from nodes that are controlled by a. Data representing costs, capacities, production and terminal values are suitably extended from the previous setting. This example allows a modeler to look at the affects of plant ownership within a competitive equilibrium setting.

Example 3. The above examples do not involve raw materials. The third example extends the framework to differentiate between raw materials and a finished good. Assume for simplicity at this time that we do not have a distribution/storage network for the finished good but simply have given demand for that good at a collection of locations, and ability to produce that good from raw materials at those locations. (Alternatively, one could model a single demand for the finished good and assume generation at any location can be used to help satisfy that demand.) We can think of raw materials as being water flowing along a river network, or fuel stored in stockpile locations. In the first setting, the river network is modeled by a collection of trees. Water (raw material) flows through the tree (to a root node) and can be used by a hydro generator situated at a node to produce electricity (finished good), and that water continues to flow through the river network to the next reservoir where it could be used for future generation. This fits naturally into the formulation above where T is now the classical node-arc incidence matrix representing the river network (without production and consumer arcs), $u_a(n)$ are the water releases on a given arc and $x_a(n)$ are the reservoir storages. The function $g_{an}(u_k(n))$ encodes the production of electricity at the turbines to satisfy demand at that location. Water is not destroyed in this production process and continues to flow through the river network. Spillage is naturally handled by the inequality in the dynamics.

However, if instead we think of the raw resource as being fuel in a given stockpile, then the network models a transportation network for that fuel. Also, the production process consumes fuel to generate electricity at the node. Thus, T is no longer just the node-arc incidence matrix of the fuel transportation network, but has consumption arcs added at each demand location that consume the fuel to produce electricity. On these arcs the electricity production function becomes $g_{an}(u_k(n))$, and the cost function represents the cost of using that material and production.

Example 4. The final example extends the above example to capture both a transportation network for the raw materials, and a distribution network for the final good. We assume for simplicity of exposition that the distribution network is acyclic. Consider the fuel and electricity example, and construct a network that is the union of the fuel transportation network, and the electricity distribution network, combined with arcs that join a fuel node at a given location to an electricity production node at that location. Thus $u_k(n)$ represents flow of fuel on arc k of the transportation network, or flow of electricity along the distribution/storage network or the production of electricity from fuel on the arcs that join these two networks together. Flow along these arcs convert fuel into electricity (in a linear fashion or using the slight generalization of $g_{an}(u_k(n))$) that can be incorporated as a loss or gain multiplier along that arc in the formulation of T. Thus, T contains the node arc incidence matrix of the both networks, augmented with generalized arcs to represent the conversion of raw quantities into finished goods. The vector $x_a(n)$ has components that correspond to the amount of raw material stored at a node (operated by a) in the fuel transportation network, or the amount of electricity stored at a node in the electricity distribution network. The flow around the electricity distribution network satisfies load-flow constraints that represent Kichhoff's Laws, and the consumption arcs in the fuel network and the production arcs in the electricity distribution network are replaced by conversion arcs linking the two networks. The remainder of the cost and generation functions are unchanged.

The situation for hydroelectric generation is a little more involved since water is not consumed as it generates electricity. To model this, we consider the union of the river network and the electricity distribution network, augmented by arcs that join a generation location on the water network to a bus on the distribution network. The river network is effectively modeled as a forest so there is only one water flow emanating from each node. However, flow out of a hydro production node generates water flow into the downstream node and an amount of electricity at the bus (determined by the production function). This can be captured by a generalized column in the matrix T with 3 entries instead of 2. Apart from this change, the model follows the fuel example.

Pumped storage is an extension of this model that incorporates additional arcs from the electricity distribution network back into the river (or raw material) network. The essential idea is that energy can be converted back into a raw resource (at a given location) with pumping efficiency modeled via a multiplier factor on the additional arc.

3. Dynamic risk measures

The agents in our models are risk averse when contemplating a sequence of decisions that have random future consequences. To model this behavior we consider a single-stage model with finite sample space indexed by $m \in \mathcal{M}$. Each decision maker faced with a random disbenefit $Z(m)$, $m \in \mathcal{M}$, measures its risk using a *coherent risk measure* ρ as defined axiomatically by Artzner et al. (1999). Thus $\rho(Z)$ is a real number representing the risk-adjusted disbenefit of Z.

It is well-known that any coherent risk measure $\rho(Z)$ has a dual representation expressing it as

$$\rho(Z) = \sup_{\nu \in \mathcal{D}} \mathbb{E}_\nu[Z],$$

where $\mathcal{D}$ is a convex subset of probability measures on $\mathcal{M}$ (see e.g. Artzner et al. (1999), Heath and Ku (2004)). $\mathcal{D}$ is called the *risk set* of the coherent risk measure. We use the notation $[p]_\mathcal{M}$ to denote any vector $\{p(m), m \in \mathcal{M}\}$. So any probability measure $\nu \in \mathcal{D}$ can be written $[\nu]_\mathcal{M}$, where $\nu(m)$ defines the probability of event m. The dual representation using a risk set plays an important role in the analysis we carry out in this paper. We refer to the case where the risk set is a singleton as *risk neutral*.

A number of examples of coherent risk measures are discussed in Shapiro et al. (2014) including worst-case and average value at risk (also known as conditional value at risk). Given a random disbenefit Z the *average value at risk* of Z at level $1-\alpha$ is defined as

$$\mathrm{AVaR}_{1-\alpha}(Z) = \inf_t \{t + \frac{1}{\alpha}\mathbb{E}[(Z-t)_+]\}.$$

Given a finite sample space indexed by $m \in \mathcal{M}$, with $\phi(m)$ the probability of m, $\mathrm{AVaR}_{1-\alpha}(Z)$ has a polyhedral risk set

$$\mathcal{D} = \{\nu : \sum_{m\in\mathcal{M}} \nu(m) = 1, \quad 0 \le \alpha\nu(m) \le \phi(m), \quad m \in \mathcal{M}\}.$$

The *good-deal* risk measure originated in the work of Cochrane and Saa-Requejo (2000) and has been widely applied in capacity planning equilibrium models (see e.g. Abada et al. (2017)). The risk sets of the good-deal risk measure are not polyhedral. They consist of probability distributions that are consistent with bounds on Sharpe ratios. Although the risk measure is evaluated using the worst distribution in $\mathcal{D}$, the distributions in $\mathcal{D}$ are constrained by requiring expected payoffs with these distributions to be no more than a fixed multiple of their standard deviation, thus precluding outcomes that are too good to be true. Details are provided in Abada et al. (2017).

In the rest of this paper we assume that risk sets are polyhedrons with known extreme points $\{[p^k]_{\mathcal{M}}, k \in \mathcal{K}\}$, where $\mathcal{K}$ is a finite index set. This condition is not essential to the theory we derive, but it simplifies the analysis without losing much generality. Assuming a polyhedral risk set we write

$$\sup_{\nu\in\mathcal{D}} \mathbb{E}_\nu[Z] = \sup_{\nu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \nu(m)Z(m) = \max_{k\in\mathcal{K}} \sum_{m\in\mathcal{M}} p^k(m)Z(m),$$

since the maximum of a linear function over $\mathcal{D}$ is attained at an extreme point. By a standard dualization, this gives

$$\sup_{\nu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \nu(m)Z(m) = \begin{cases} \min\ \theta \\ \text{s.t. } \theta \ge \sum_{m\in\mathcal{M}} p^k(m)Z(m),\ k \in \mathcal{K}. \end{cases}$$

LEMMA 1. *Suppose $\mathcal{D}$ is a polyhedral risk set with extreme points $\{[p^k]_{\mathcal{M}},\, k \in \mathcal{K}\}$ and $Z(m)$, $m \in \mathcal{M}$ is given. Then*

$$\theta = \sup_{\nu \in \mathcal{D}} \sum_{m \in \mathcal{M}} \nu(m) Z(m)$$

if and only if there is some γ^k, $k \in \mathcal{K}$, with

$$\begin{aligned} &\sum_{k \in \mathcal{K}} \gamma^k = 1 \\ &0 \le \gamma^k \perp \theta - \sum_{m \in \mathcal{M}} p^k(m) Z(m) \ge 0, \quad k \in \mathcal{K}. \end{aligned}$$

Furthermore, $\bar{\nu}$, defined by $\bar{\nu}(m) = \sum_{k \in \mathcal{K}} \gamma^k p^k(m)$, is in $\mathcal{D}$ and attains the supremum.

By definition, a coherent risk measure is *monotone*. This means that

$$Z_a \ge Z_b \Rightarrow \rho(Z_a) \ge \rho(Z_b).$$

A stronger condition is *strict monotonicity*. This requires that

$$Z_a \ge Z_b \text{ and } Z_a \ne Z_b \Rightarrow \rho(Z_a) > \rho(Z_b).$$

If strictly monotone coherent risk measures have polyhedral risk sets then these lie strictly inside the positive orthant.

LEMMA 2. *Suppose ρ is a coherent risk measure with a polyhedral risk set $\mathcal{D}$. Then $\mathcal{D} \subset int(\mathbb{R}_+^{|\mathcal{M}|})$ if and only if ρ is strictly monotone.*

We incorporate the risk measures discussed above into a multistage setting in which agents make production and consumption decisions over several time stages to minimize risk-adjusted expected disbenefit.

For a multistage decision problem, we require a dynamic version of risk. The concept of coherent dynamic risk measures was introduced in Riedel (2004) and is described for general Markov decision problems in Ruszczyński (2010). Formally one defines a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\emptyset, \Omega\} = \mathcal{F}_1 \subset \mathcal{F}_2 \ldots \subset \mathcal{F}_T \subset \mathcal{F}$ of σ-fields where all data in node 0 is assumed to be deterministic and decisions at time t are $\mathcal{F}_t$-measurable random variables (see Ruszczyński (2010)). Working with finite probability spaces defined by a scenario tree simplifies this description.

Given a tree defined by $\mathcal{N}$, suppose the random sequence of actions $\{u(n), n \in \mathcal{N}\}$ results in a random sequence of disbenefits $\{Z(n), n \in \mathcal{N}\}$. We seek to measure the risk of this disbenefit sequence when viewed by a decision maker at node 0. At node n the decision maker is endowed with a one-step risk set $\mathcal{D}(n)$ that measures the risk of random risk-adjusted costs accounted for in $m \in n_+$. Thus elements of $\mathcal{D}(n)$ are finite probability distributions of the form $[p]_{n_+}$.

The risk-adjusted disbenefit $\theta(n)$ of all random future outcomes at node $n \in \mathcal{N} \setminus \mathcal{L}$ can be defined recursively. We denote the future risk-adjusted disbenefit in each leaf node $n \in \mathcal{L}$ by $\bar{\theta}(n)$. Then $\theta(n)$ is defined recursively to be

$$\theta(n) = \begin{cases} \bar{\theta}(n), & n \in \mathcal{L}, \\ \sup\limits_{\nu \in \mathcal{D}(n)} \sum\limits_{m \in n_+} \nu(m)(Z(m) + \theta(m)), & n \in \mathcal{N} \setminus \mathcal{L}. \end{cases} \tag{3}$$

When viewed in node n, $\theta(n)$ can be interpreted to be the fair one-time charge we would be willing to incur instead of the sequence of random future costs $Z(m)$ incurred in all successor nodes of n. In other words the measure $\theta(n)$ is a *certainty equivalent cost* or *risk-adjusted expected cost* of all the future costs in the subtree rooted at node n.

Since we assume for $n \in \mathcal{N} \setminus \mathcal{L}$ that $\mathcal{D}(n)$ is a polyhedron with extreme points $\{[p^k], k \in \mathcal{K}(n)\}$, the recursive structure defined by (3) can then be simplified to

$$\begin{aligned} &\sup_{\nu \in \mathcal{D}(n)} \sum_{m \in n_+} \nu(m)(Z(m) + \theta(m)) \\ &= \begin{cases} \min\ \theta \\ \text{s.t. } \theta \geq \sum\limits_{m \in n_+} p^k(m)\left(Z(m) + \theta(m)\right), \ k \in \mathcal{K}(n). \end{cases} \end{aligned} \tag{4}$$

We now recall the system optimization problem SO, and modify this by adding variables θ so that it minimizes *risk-adjusted* system disbenefit using a dynamic risk measure. The risk-averse system optimization problem is then formulated as follows.

$$\begin{aligned}
\text{SO}(\mathcal{D})\text{:}\ \min_{u,x,\theta} & \sum_{a\in\mathcal{A}} C_{an}(u_a(0)) + \theta(0) \\
\text{s.t. } \theta(n) \geq & \sum_{m\in n_+} p^k(m)\left(\sum_{a\in\mathcal{A}} C_{an}(u_a(m)) + \theta(m)\right), \quad [\gamma^k(n)] \\
& k \in \mathcal{K}(n),\ n \in \mathcal{N}\setminus\mathcal{L}, \qquad (5) \\
& x_a(n) \leq x_a(n_-) + \sum_{b\in\mathcal{A}} T_{ab}u_b(n) + \omega_a(n), \quad a\in\mathcal{A}, n\in\mathcal{N}, \quad [\alpha_a(n)] \qquad (6) \\
& \sum_{a\in\mathcal{A}} g_{an}(u_a(n)) \geq 0 \quad n\in\mathcal{N}, \quad [\pi(n)] \qquad (7) \\
& \theta(n) = -\sum_{a\in\mathcal{A}} V_{an}(x_a(n)), \quad n\in\mathcal{L}, \\
& u_a(n)\in\mathcal{U}_a, \quad x_a(n)\in\mathcal{X}_a, \quad n\in\mathcal{N}, \quad a\in\mathcal{A}.
\end{aligned}$$

3.1. Dynamic consistency

The solution of SO($\mathcal{D}$) gives a policy of decisions $\{\bar{u}_a(n), n\in\mathcal{N}\}$ and resulting stocks $\{\bar{x}_a(n), n\in\mathcal{N}\}$. We digress briefly here to discuss the notion of dynamic consistency as applied to such a solution. Recall the set of successors $\mathcal{S}(n)$ of node n is the maximal subtree in $\mathcal{N}$ with root node n. Following Carpentier et al. (2012) we make the following definition.

Definition 1. An optimal solution $\{\bar{u}_a(n), \bar{x}_a(n), n\in\mathcal{N}\}$ to SO($\mathcal{D}$) is called *dynamically consistent* if for every $\bar{n}\in\mathcal{N}$, $\{\bar{u}_a(n), \bar{x}_a(n), n\in\mathcal{S}(\bar{n})\}$ is an optimal solution to SO($\mathcal{D}$) formulated in $\mathcal{S}(\bar{n})$ where node 0 is replaced by node $\bar{n}$ and we choose initial endowments $x_a(\bar{n}_-) = \bar{x}_a(\bar{n}_-)$.

Dynamic consistency of solutions to SO($\mathcal{D}$) is guaranteed under the following assumption.

Assumption 1. *For every $n\in\mathcal{N}\setminus\mathcal{L}$, $\mathcal{D}(n)\subset$ int$(\mathbb{R}_+^{|n_+|})$.*

Under this assumption, Lemma 2 ensures that one-step risk measures are strictly monotone. As shown in Shapiro (2017), this implies that optimal solutions to the tree problem with risk sets $\mathcal{D}(n)$, $n\in\mathcal{N}\setminus\mathcal{L}$ correspond to dynamic programming policies that compute optimal solutions by backwards recursion. In other words the optimal policy for SO($\mathcal{D}$) will be dynamically consistent.

To see that Assumption 1 is necessary, observe that if it does not hold then it is possible for

$$\bar{\nu} \in \arg \max_{\nu \in \mathcal{D}(n)} \sum_{m \in n_+} \nu(m)(Z(m) + \theta(m)) \tag{8}$$

to have $\bar{\nu}(\bar{m}) = 0$ for some $\bar{m}$. If so, then evaluating the risk at node 0 will ignore all disbenefits in the subtree of nodes in $\mathcal{N}$ rooted at $\bar{m}$. Decisions in these nodes will not affect the overall risk-adjusted disbenefit in node 0 unless they change nodal disbenefits enough to change $\bar{\nu}$ in (8). If these decisions are suboptimal given that the decision maker is in the state of the world defined by $\bar{m}$, then the policy defined by all the decisions is not dynamically consistent.

Of course it is true that one can construct a dynamically consistent policy (by dynamic programming) even though the decision maker assigns zero probability to events in some nodes. We will show that such policies correspond to optimality conditions defined over the whole scenario tree. These will be sufficient but may not be necessary conditions for an optimal solution to an instance of SO($\mathcal{D}$) that violates Assumption 1.

3.2. Optimality conditions

We now define optimality consitions for the problem SO($\mathcal{D}$). Recall for any set $\mathcal{X}$ we define the *normal cone* at $\bar{x}$ to be

$$N_{\mathcal{X}}(\bar{x}) = \{d : d^\top (x - \bar{x}) \leq 0 \text{ for all } x \in \mathcal{X}\},$$

and recall that $\bar{x}$ minimizes a convex function $f(x)$ over convex set $\mathcal{X}$ if and only if

$$0 \in \nabla_x f(\bar{x}) + N_{\mathcal{X}}(\bar{x}).$$

When the set $\mathcal{X}$ has a particular representation in terms of nonlinear functions, these optimality conditions have a specific form (often termed the KKT conditions) provided that a constraint qualification holds. To facilitate use of these conditions within our proofs, we will assume that the following condition is satisfied throughout this paper.

ASSUMPTION 2. *The nonlinear constraints in SO($\mathcal{D}$) satisfy a constraint qualification that ensures that SO($\mathcal{D}$) is equivalent to its KKT conditions.*

The weakest condition Gould and Tolle (1971) that ensures this equivalence is referred to as the Guinard constraint qualification, and the stronger Slater constraint qualification is often used since it is easier to verify.

Since SO($\mathcal{D}$) is a convex optimization problem and the constraint qualification Assumption 2 holds, Assumption 1 implies that the following set of conditions SE($\mathcal{D}$) are necessary and sufficient for optimality in SO($\mathcal{D}$).

SE($\mathcal{D}$):

$$0 = 1 - \sum_{k \in \mathcal{K}(n)} \gamma^k(n), \quad n \in \mathcal{N} \setminus \mathcal{L}$$

$$0 \le \gamma^k(n) \perp \theta(n) - \sum_{m \in n_+} p^k(m) \left(\sum_{a \in \mathcal{A}} C_{an}(u_a(m)) + \theta(m) \right) \ge 0, \quad k \in \mathcal{K}(n), n \in \mathcal{N} \setminus \mathcal{L}$$

$$\theta(n) = -\sum_{a \in \mathcal{A}} V_{an}(x_a(n)), \quad n \in \mathcal{L}$$

$$0 \in \nabla_{u_a(n)} \left[C_{an}(u_a(n)) - \pi(n) g_{an}(u_a(n)) - \sum_{b \in \mathcal{A}} \alpha_b(n) T_{ba} u_a(n) \right] + N_{\mathcal{U}_a}(u_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{N}$$

$$0 \in \alpha_a(n) - \sum_{m \in n_+} \sum_{k \in \mathcal{K}(n)} \gamma^k(n) p^k(m) \alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{N} \setminus \mathcal{L}$$

$$0 \in \alpha_a(n) - \nabla_{x_a(n)} V_{an}(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{L}$$

$$0 \le \alpha_a(n) \perp -x_a(n) + x_a(n_-) + \sum_{b \in \mathcal{A}} T_{ab} u_b(n) + \omega_a(n) \ge 0, \quad a \in \mathcal{A}, n \in \mathcal{N}$$

$$0 \le \pi(n) \perp \sum_{a \in \mathcal{A}} g_{an}(u_a(n)) \ge 0, \quad n \in \mathcal{N}.$$

THEOREM 1. *(i) Any solution to SE($\mathcal{D}$) provides (u, x, θ) that solves SO($\mathcal{D}$) and satisfies*

$$\begin{aligned} \theta(n) &= \max_{\nu \in \mathcal{D}(n)} \sum_{m \in n_+} \nu(m) \left(\sum_{a \in \mathcal{A}} C_{an}(u_a(m)) + \theta(m) \right) \\ &= \sum_{m \in n_+} \bar{\nu}(m) \left(\sum_{a \in \mathcal{A}} C_{an}(u_a(m)) + \theta(m) \right), \end{aligned}$$

where $\bar{\nu}(m) = \sum_{k \in \mathcal{K}(n)} \gamma^k(n) p^k(m)$.

(ii) Under Assumption 1 any solution to SO($\mathcal{D}$) satisfies SE($\mathcal{D}$) for some π, α, γ.

3.3. Equilibrium

Given a set of agents $a \in \mathcal{A}$ we can define a risk-averse competitive equilibrium as follows. We first define an agent optimization problem that minimizes their risk-adjusted disbenefit at given prices.

$$
\begin{aligned}
\mathrm{P}_a(\pi, \alpha, \mathcal{D}_a)\colon \min_{u_a, x_a, \theta_a} \; & Z_a(0; u, x) + \theta_a(0) \\
\text{s.t. } & \theta_a(n) \geq \sum_{m \in n_+} p_a^k(m)(Z_a(m; u, x) + \theta_a(m)), \\
& k \in K_a(n),\, n \in \mathcal{N} \setminus \mathcal{L}, \\
& \theta_a(n) = -V_{an}(x_a(n)), \quad n \in \mathcal{L}, \\
& u_a(n) \in \mathcal{U}_a,\, x_a(n) \in \mathcal{X}_a, \quad n \in \mathcal{N},
\end{aligned}
$$

where we use the shorthand notation

$$
\begin{aligned}
Z_a(n; u, x) = {} & C_{an}(u_a(n)) - \pi(n) g_{an}(u_a(n)) + \alpha_a(n)\,(x_a(n) - x_a(n_-) - \omega_a(n)) \\
& - \sum_{b \in \mathcal{A}} \alpha_b(n) T_{ba} u_a(n), \quad n \in \mathcal{N}.
\end{aligned} \tag{9}
$$

Here $\pi(n)$ is the commodity price at node n and $\alpha_a(n)$ is the resource price at node n at a's location. Recall that agents are assumed throughout this paper to behave as price takers, so prices will be determined in equilibrium by market clearing rather than anticipated by agents behaving as Cournot players. This means that $\alpha_a(n)$ is the price paid by *every* agent for resource at a's location, rather than individual agent prices that could emerge from a generalized Nash equilibrium in the Cournot setting.

For a producer, the first two terms in (9) are the production cost minus sales revenue. The third term is the cost incurred in node n in retaining extra resources for later use, and the final term $\sum_{b \in \mathcal{A}} \alpha_b(n) T_{ba} u_a(n)$ is the payment received from downstream beneficiaries for releases of resources. Observe that they pay at price $\alpha_b(n)$ that will typically be less than $\alpha_a(n)$ as agent a has extracted value from the resource en route to b. In the hydroelectric setting $\alpha_b(n)$ is a payment received for released water from downstream reservoirs. In the case where a single agent owns both reservoirs (i.e. a and b identify the same agent) the payment can be viewed as the loss in risk-adjusted expected water value incurred by the release.

Definition 2. A *multistage risked equilibrium* $\mathrm{RE}(\mathcal{D}_{\mathcal{A}})$ is a stochastic process of prices $\{\pi(n), n \in \mathcal{N}\}$, $\{\alpha_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, and a corresponding collection of actions $\{u_a(n), n \in \mathcal{N}\}$ with the property that (u_a, x_a, θ_a) solves the problem $\mathrm{P}_a(\pi, \alpha, \mathcal{D}_a)$ and

$$0 \leq \pi(n) \quad \perp \quad \sum_{a \in \mathcal{A}} g_{an}(u_a(n)) \geq 0, \quad n \in \mathcal{N},$$

$$0 \leq \alpha_a(n) \quad \perp \quad -x_a(n) + x_a(n_-) + \sum_{b \in \mathcal{A}} T_{ab} u_b(n) + \omega_a(n) \geq 0, \quad a \in \mathcal{A}, n \in \mathcal{N}.$$

In a multistage risked equilibrium, the system clearing agent announces a set of prices $\{\pi(n), n \in \mathcal{N}\}$, $\{\alpha_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, and each agent chooses a sequence of actions adapted to the filtration defined by the scenario tree that minimizes their risk-adjusted disbenefit with these prices as viewed in node 0 of the tree. Since agents are price takers they do not anticipate possible responses of rival agents in later periods when making decicions now, although these responses have an implicit effect through future market clearing prices.

The existence of multistage risked equilibrium depends on the formulation of each problem $\mathrm{P}_a(\pi, \alpha, \mathcal{D}_a)$. Existence proofs for particular formulations typically invoke general results (see e.g. Rosen (1965) or Arrow and Debreu (1954)) based on fixed-point theorems that require bounds on the set of actions and convex disbenefit functions. Existence results for risked equilibrium models for capacity expansion can be found in de Maere d'Aertrycke and Smeers (2013), Abada et al. (2017), Kok et al. (2018) and Ralph and Smeers (2015).

Uniqueness of risked equilibrium is more problematic. Consider a model with three agents and a random supply $\xi(\omega)$ that takes values 1 and 3 with equal probability. Let each agent be endowed with the worst case risk measure (denoted $\mathbb{F}$). This is not strictly monotone. A risked equilibrium is a set of prices $\pi(\omega)$ and actions x^*, $y^*(\omega)$, $z^*(\omega)$ satisfying:

$$x^* \in \arg\max_{x \geq 0} \mathbb{F}[\pi(\omega)x - \frac{1}{2}x^2]$$

$$y^*(\omega) \in \arg\max_{y(\omega) \geq 0} \mathbb{F}[\pi(\omega)y(\omega) - y(\omega)^2],$$

$$z^*(\omega) \in \arg\max_{z(\omega)\geq 0} \mathbb{F}[16z(\omega) - z(\omega)^2 - \pi(\omega)z(\omega)],$$

$$0 \leq x^* + y^*(\omega) + \xi(\omega) - z^*(\omega) \perp \pi(\omega) \geq 0, \omega \in \Omega.$$

There are infinitely many equilibria. For example we might choose $\pi(\omega_1) = \frac{9}{2}$, $\pi(\omega_2) = \frac{5}{2}$, $x = \frac{5}{2}$, $y_1 = \frac{9}{4}$, $y_2 = \frac{5}{4}$, $z_1 = \frac{23}{4}$, $z_2 = \frac{27}{4}$. Here $x = \frac{5}{2}$ maximizes $\min_\omega\{\pi(\omega)x - \frac{1}{2}x^2\}$, y_i maximizes $\pi(\omega_i)y - y^2$, and z_i maximizes $16z - z^2 - \pi(\omega_i)z$. For each agent a, the values of x, y, and z in this solution solve $\mathrm{P}_a(\pi, \alpha, \mathcal{D}_a)$. Note that $y_1 = \frac{9}{4}$ and $z_2 = \frac{27}{4}$ are both optimal solutions for agents 2 and 3 in scenarios ω_1 and ω_2, even though the (optimal) outcomes of these actions have no effect on the risk-adjusted disbenefit of these agents.

A second equilibrium is $\pi(\omega_1) = 5$, $\pi(\omega_2) = 3$, $x = 3$, $y_1 = \frac{3}{2}$, $y_2 = \frac{3}{2}$, $z_1 = \frac{11}{2}$, $z_2 = \frac{15}{2}$. Observe that $x = 3$ maximizes $\min_\omega\{\pi(\omega)x - \frac{1}{2}x^2\}$, $y_2 = \frac{3}{2}$ maximizes $\min_\omega\{\pi(\omega)y - y^2\}$, and $z_1 = \frac{23}{4}$ maximizes $\min_\omega\{16z - z^2 - \pi(\omega)z\}$, but y_1 and z_2 are not optimal for agents 2 and 3 in scenarios ω_1 and ω_2, but chosen to make markets clear in each supply outcome.

It is tempting to suppose that strict monotonicity of each agent's risk measure would be sufficient for uniqueness of equilibrium. This is not true as demonstrated by the counterexample in Gérard et al. (2018).

4. Risk trading

We now turn our attention to the situation where agents with polyhedral risk sets can trade financial contracts to reduce their risk. We will show that the system optimal solution to a social planning problem corresponds to a perfectly competitive equilibrium with risk trading.

We use the notation $Z_a(n)$, $n \in \mathcal{N}$ to denote the disbenefit of agent a, and $\mathcal{D}_a(n)$ to denote the risk set of agent a, which is a polyhedral set with extreme points $\{[p_a^k]_{n_+}, k \in \mathcal{K}_a(n)\}$. In order to get some alignment between the objectives of agents and a social planner, we establish a connection between their risk sets using the following assumption and definitions.

Assumption 3. *For $n \in \mathcal{N} \setminus \mathcal{L}$*

$$\bigcap_{a\in\mathcal{A}} \mathcal{D}_a(n) \neq \emptyset.$$

Definition 3. For $n \in \mathcal{N} \setminus \mathcal{L}$ the *social planning risk set* is

$$\mathcal{D}_s(n) = \bigcap_{a \in \mathcal{A}} \mathcal{D}_a(n).$$

The financial instruments that are traded are assumed to take a specific form.

Definition 4. Given any node $n \in \mathcal{N} \setminus \mathcal{L}$, an *Arrow-Debreu security* for node $m \in n_+$ is a contract that charges a price $\mu(m)$ in node n to receive a payment of 1 in node $m \in n_+$, and zero in other nodes $m' \neq m$, $m' \in n_+$.

We shall assume throughout this section that the market for risk is *complete*. Formally this means that the set of Arrow-Debreu securities traded at each node n spans the set of possible outcomes in n_+. It is important to emphasize that the trade in these instruments yields a common market price $\mu(m)$ that is paid by all agents in node n for each of the securites indexed by $m \in n_+$.

Assumption 4. *At every node $n \in \mathcal{N} \setminus \mathcal{L}$, there is an Arrow-Debreu security for each child node $m \in n_+$ that is traded in node n at an equilibrium price $\mu(m)$.*

To reduce its risk, suppose that each agent a in node n purchases $W_a(m)$ Arrow-Debreu securities for node $m \in n_+$. Each agent a's optimization problem with risk trading is then formulated as

$$\begin{aligned}
&\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a): \\
&\qquad \min_{u_a, x_a, W_a, \theta_a} \; Z_a(0; u, x, W) + \theta_a(0) \\
&\qquad\qquad \text{s.t. } \theta_a(n) \geq \sum_{m \in n_+} p_a^k(m)(Z_a(m; u, x, W) - W_a(m) + \theta_a(m)), \\
&\qquad\qquad\qquad k \in K_a(n),\; n \in \mathcal{N} \setminus \mathcal{L}, \\
&\qquad\qquad \theta_a(n) = -V_{an}(x_a(n)), \quad n \in \mathcal{L}, \\
&\qquad\qquad u_a(n) \in \mathcal{U}_a,\; x_a(n) \in \mathcal{X}_a, \quad n \in \mathcal{N},
\end{aligned}$$

where we use the shorthand notation

$$\begin{aligned}
Z_a(n; u, x, W) = {} & C_{an}(u_a(n)) - \pi(n) g_{an}(u_a(n)) + \alpha_a(n)\,(x_a(n) - x_a(n_-) - \omega_a(n)) \\
& - \sum_{b \in \mathcal{A}} \alpha_b(n) T_{ba} u_a(n) + \sum_{m \in n_+} \mu(m) W_a(m), \quad n \in \mathcal{N}. \qquad (10)
\end{aligned}$$

Here the agent minimizes immediate cost plus the (insurance) cost of the security along with future costs, in the understanding that the security will pay back in the next period according to the situation realized. The interpretation of the notation in (10) is the same as in (9), with the exception of variables $W_a(m)$ that denote the number of Arrow-Debreu securities of type m bought by agent a at node n. The agent pays a market price $\mu(m)$ for each of these. The payoff for security m only occurs in scenario m as reflected in the first inequality of $\mathrm{AO}_a(\pi,\alpha,\mu,\mathcal{D}_a)$. Observe that $W_a(m)$ can be negative (if the security is sold) and is unbounded in this formulation. In equilibrium $W_a(m)$ will be traded at an equilibrium price $\mu(m)$. We show below that Assumption 3 above (which is a form of no-arbitrage condition) will ensure that the trade in Arrow-Debreu securities is bounded at these prices.

We can define a complementarity form of $\mathrm{AO}_a(\pi,\alpha,\mu,\mathcal{D}_a)$as follows.

$$\mathrm{AE}_a(\pi,\alpha,\mu,\mathcal{D}_a):$$

$$0 = 1 - \sum_{k\in\mathcal{K}_a(n)} \gamma^k(n), \quad n \in \mathcal{N}\setminus\mathcal{L} \tag{11a}$$

$$0 \le \gamma^k(n) \perp \theta_a(n) - \sum_{m\in n_+} p_a^k(m)\Big(Z_a(m;u,x,W) - W_a(m) + \theta_a(m)\Big) \ge 0,$$

$$k \in \mathcal{K}_a(n), n \in \mathcal{N}\setminus\mathcal{L} \tag{11b}$$

$$\theta_a(n) = -V_{an}(x_a(n)), \quad n \in \mathcal{L} \tag{11c}$$

$$0 \in \nabla_{u_a(n)} Z_a(n;u,x,W) + N_{\mathcal{U}_a}(u_a(n)), \quad n \in \mathcal{N} \tag{11d}$$

$$0 \in \alpha_a(n) - \sum_{m\in n_+} \mu(m)\alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{N}\setminus\mathcal{L} \tag{11e}$$

$$0 \in \alpha_a(n) - \nabla_{x_a(n)} V_{an}(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{L} \tag{11f}$$

$$0 = \mu(m) - \sum_{k\in\mathcal{K}_a(n)} \gamma^k(n) p_a^k(m), \quad m \in n_+, n \in \mathcal{N}\setminus\mathcal{L}, \tag{11g}$$

where $Z_a(n;u,x,W)$ is defined by (10).

THEOREM 2. *(i) Any solution to $AE_a(\pi,\alpha,\mu,\mathcal{D}_a)$ provides a solution (u_a,x_a,W_a,θ_a) to the optimization problem $AO_a(\pi,\alpha,\mu,\mathcal{D}_a)$, and satisfies*

$$\theta_a(n) = \max_{\nu\in\mathcal{D}_a(n)} \sum_{m\in n_+} \nu(m)\left(Z_a(m;u,x,W) - W_a(m) + \theta_a(m)\right)$$

$$= \sum_{m \in n_+} \mu(m) \left(Z_a(m; u, x, W) - W_a(m) + \theta_a(m) \right).$$

(ii) If Assumption 1 holds, then any solution of $AO_a(\pi, \alpha, \mu, \mathcal{D}_a)$ provides a solution to $AE_a(\pi, \alpha, \mu, \mathcal{D}_a)$ for some γ.

Theorem 2 provides a link between the solution to the optimization problem AO faced by an agent at node 0 and the optimality conditions AE that a dynamically consistent optimal solution would satisfy at each node. Any solution to AE will solve AO. The converse is true when Assumption 1 holds.

The remainder of the paper seeks to connect competitive equilibrium in a market where agents trade risk to the solution of a social optimization problem. We do this by linking system optimization (SO) to a complementarity problem (SE) that is equivalent to a system of variational inequalities (RTVI). This system is in turn linked to the competitive equilibrium with risk trading (RTE). A broad outline of our proof strategy is given in Figure 3.

Suppose each agent solves the optimization problem $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ taking prices π, α, and μ as given. If these prices clear the markets for respective quantities, then we have a competitive equilibrium with risk trading.

Definition 5. A *multistage risk-trading equilibrium* $\mathrm{RTE}(\mathcal{D}_\mathcal{A})$ is a stochastic process of prices $\{\pi(n), n \in \mathcal{N}\}$, $\{\alpha_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, $\{\mu(n), n \in \mathcal{N} \setminus \{0\}\}$, and a corresponding collection of actions for each $a \in \mathcal{A}$, $\{(u_a(n), x_a(n), \theta_a(n)), n \in \mathcal{N}\}, \{W_a(n), n \in \mathcal{N} \setminus \{0\}\}$ with the property that $(u_a, x_a, W_a, \theta_a)$ solves the problem $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ and

$$0 \le \pi(n) \quad \perp \quad \sum_{a \in \mathcal{A}} g_{an}(u_a(n)) \ge 0, \quad n \in \mathcal{N}, \tag{12}$$

$$\begin{aligned} 0 \le \alpha_a(n) \quad \perp \quad & -x_a(n) + x_a(n_-) + \sum_{b \in \mathcal{A}} T_{ab} u_b(n) + \omega_a(n) \ge 0, \\ & \qquad a \in \mathcal{A}, n \in \mathcal{N}, \end{aligned} \tag{13}$$

$$0 \le \mu(n) \quad \perp \quad -\sum_{a \in \mathcal{A}} W_a(n) \ge 0, \quad n \in \mathcal{N} \setminus \{0\}. \tag{14}$$

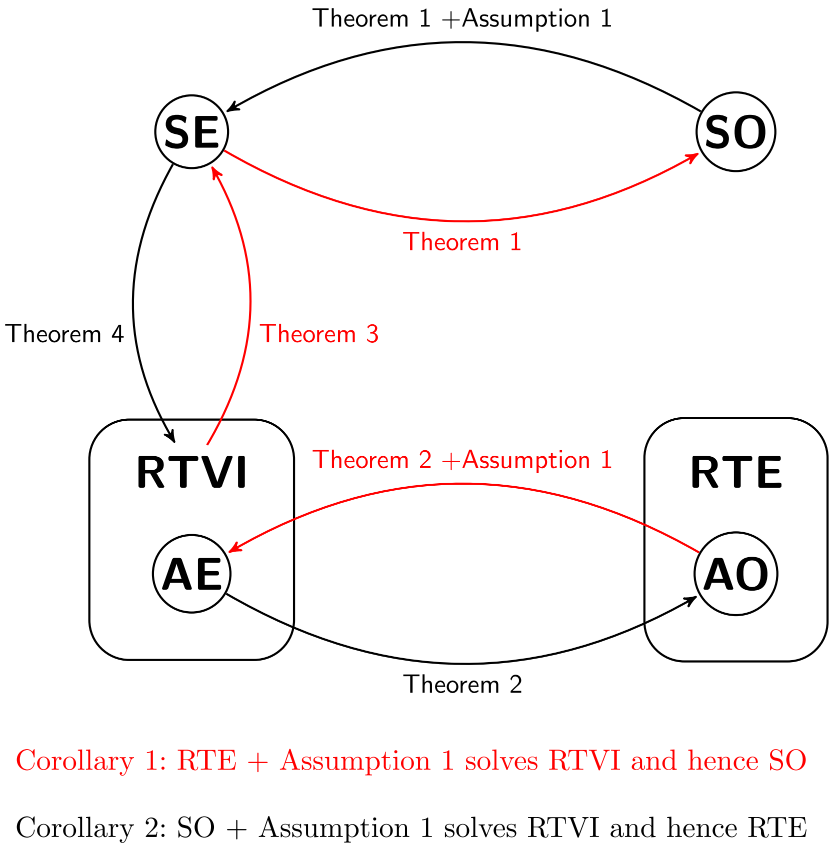

Figure 3 An outline of the interplay of the main results.

In the absence of Assumption 1, the solution set of $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ might strictly contain that of $\mathrm{AE}_a(\pi, \alpha, \mu, \mathcal{D}_a)$. We can then define a constrained form of $\mathrm{RTE}(\mathcal{D}_\mathcal{A})$ as follows.

Definition 6. A *multistage risk-trading variational inequality* $\mathrm{RTVI}(\mathcal{D}_\mathcal{A})$ is a stochastic process of prices $\{\pi(n), n \in \mathcal{N}\}$, $\{\alpha_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, $\{\mu(n), n \in \mathcal{N} \setminus \{0\}\}$, and a corresponding collection of actions for each $a \in \mathcal{A}$, $\{(u_a(n), x_a(n), \theta_a(n)), n \in \mathcal{N}\}, \{W_a(n), n \in \mathcal{N} \setminus \{0\}\}$ with the property that for some γ, $(u_a, x_a, W_a, \theta_a, \gamma)$ solves the problem $\mathrm{AE}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ and satisfies (12), (13), and (14).

Note that RTVI is a more restrictive form of RTE, that is equivalent when Assumption 1 holds. This difference in models means that we write $-\sum_{a\in\mathcal{A}} W_a(n) \geq 0$ rather than an equation. Since buying $W_a(m)$ to payoff in scenario m decreases cost in this scenario we require no more securities $W_a(m)$ to be bought than sold, so at the least $\sum_a W_a(m) \leq 0$. Under Assumption 1, the markets for contracts clear with strictly postive prices μ. This ensures that $\sum_a W_a(m) = 0$. If Assumption 1 does not hold then it is possible to have $\mu = 0$ in equilibrium. This could occur e.g. for a worst-case risk measure as in the two scenario example problem at the end of section 3. It can be shown for this example that an agent purchasing an Arrow-Debreu security $W_a(1) > 0$ that pays off in worst case outcome 1 in order to make 2 the worst outcome, could be matched by a seller who sells more than $W_a(1)$ at price 0, as long as outcome 1 does not become the seller's worst-case outcome after the sale. Thus we might have $\sum_a W_a(1) < 0$ in equilibrium, so more securities are sold than are bought.

First and second welfare theorems can be derived for RTVI($\mathcal{D}_\mathcal{A}$)

THEOREM 3. *Consider a set of agents $a \in \mathcal{A}$, each endowed with polyhedral node-dependent risk sets $\mathcal{D}_a(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$ satisfying Assumption 3. Suppose $\{\bar{\pi}(n), n \in \mathcal{N}\}$, $\{\bar{\alpha}_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, and $\{\bar{\mu}(n), n \in \mathcal{N} \setminus \{0\}\}$ form a multistage risk-trading variational inequality RTVI($\mathcal{D}_\mathcal{A}$) in which agent a solves $AE_a(\bar{\pi}, \bar{\alpha}, \bar{\mu}, \mathcal{D}_a)$ with a policy defined by $(\bar{u}_a(\cdot), \bar{x}_a(\cdot), \bar{\theta}_a(\cdot))$ together with a policy of trading Arrow-Debreu securities defined by $\{\bar{W}_a(n), n \in \mathcal{N} \setminus \{0\}\}$. For every $n \in \mathcal{N}$ define $\bar{\theta}(n) = \sum_{a \in \mathcal{A}} \bar{\theta}_a(n)$. Then*

(i) $\bar{\mu} \in \mathcal{D}_a$ for all $a \in \mathcal{A}$, and hence $\bar{\mu} \in \mathcal{D}_s$,

(ii)

$$\bar{\theta}(n) = \sum_{m \in n_+} \bar{\mu}(m) \left(\sum_{a \in \mathcal{A}} C_{an}(\bar{u}_a(m)) + \bar{\theta}(m) \right), \quad n \in \mathcal{N} \setminus \mathcal{L}. \tag{15}$$

(iii) there exist multipliers γ such that $(\bar{u}, \bar{x}, \bar{\theta}, \gamma, \bar{\pi}, \bar{\alpha})$ is a solution to SE($\mathcal{D}_0$) with $\mathcal{D}_0 = \{\bar{\mu}\}$,

(iv) there exist multipliers γ such that $(\bar{u}, \bar{x}, \bar{\theta}, \gamma, \bar{\pi}, \bar{\alpha})$ is a solution to SE($\mathcal{D}_s$)

where $\bar{\mu}(n) = \sum_{k \in K_s(n)} \gamma^k(n) p_a^k(m)$.

We are also able to establish a converse result to Theorem 3, the proof of which is in Appendix C.

THEOREM 4. *Consider a set of agents $a \in \mathcal{A}$, each endowed with a polyhedral node-dependent risk set $\mathcal{D}_a(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$ satisfying Assumption 3. Now let $(u, x, \theta^s, \gamma, \pi, \alpha)$ be a solution to SE($\mathcal{D}_s$) with risk sets $D_s(n) = \bigcap_{a \in \mathcal{A}} \mathcal{D}_a(n)$. Let μ be defined by*

$$\mu(m) = \sum_{k \in \mathcal{K}(n)} \gamma^k(n) p^k(m), \quad m \in n_+, \quad n \in \mathcal{N} \setminus \mathcal{L}.$$

Then there exist $\{\theta_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$ *such that the prices* $\{\pi(n), n \in \mathcal{N}\}$, $\{\alpha_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, $\{\mu(n), n \in \mathcal{N} \setminus \{0\}\}$ *and actions of each agent* $a \in \mathcal{A}$ $\{(u_a(n), x_a(n), \theta_a(n)), n \in \mathcal{N}\}$, $\{W_a(n), n \in \mathcal{N} \setminus \{0\}\}$ *form a multistage risk-trading variational inequality RTVI*($\mathcal{D}_{\mathcal{A}}$).

Under Assumption 1 we can establish versions of the welfare theorems in which each agent solves a multistage optimization problem $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ to yield a risk-trading equilibrium. These are the following corollaries, the proofs of which are immediate from Theorem 3 and Theorem 4 and the fact that Assumption 1 gives the equivalence of $\mathrm{AE}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ and $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$, and $\mathrm{SE}(\mathcal{D}_s)$ and $\mathrm{SO}(\mathcal{D}_s)$.

COROLLARY 1. *Suppose Assumption 1 holds. Consider a set of agents* $a \in \mathcal{A}$*, each endowed with a polyhedral node-dependent risk set* $\mathcal{D}_a(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$ *satisfying Assumption 3. Suppose* $\{\bar{\pi}(n), n \in \mathcal{N}\}$, $\{\bar{\alpha}_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, *and* $\{\bar{\mu}(n), n \in \mathcal{N} \setminus \{0\}\}$ *form a multistage risk-trading equilibrium RTE*($\mathcal{D}_{\mathcal{A}}$) *in which agent a solves* $AO_a(\pi, \alpha, \mu, \mathcal{D}_a)$ *with a policy defined by* $(\bar{u}_a(\cdot), \bar{x}_a(\cdot), \bar{\theta}_a(\cdot))$ *together with a policy of trading Arrow-Debreu securities defined by* $\{\bar{W}_a(n), n \in \mathcal{N} \setminus \{0\}\}$*. Then* $(\bar{u}, \bar{x}, \bar{\theta})$ *is a solution to SO*($\mathcal{D}_s$) *where* $D_s(n) = \bigcap_{a \in \mathcal{A}} \mathcal{D}_a(n)$ *and* $\bar{\theta}(n) = \sum_{a \in \mathcal{A}} \bar{\theta}_a(n)$.

COROLLARY 2. *Suppose Assumption 1 holds. Consider a set of agents* $a \in \mathcal{A}$*, each endowed with a polyhedral node-dependent risk set* $\mathcal{D}_a(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$ *satisfying Assumption 3. Now let* (u, x, θ^s) *be a solution to SO*($\mathcal{D}_s$) *with risk sets* $D_s(n) = \bigcap_{a \in \mathcal{A}} \mathcal{D}_a(n)$*. Suppose this gives rise to Lagrange multipliers* $\{\pi(n), n \in \mathcal{N}\}$, $\{\alpha_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$ *for constraints (7) and (6) respectively. Then for some* γ

1. $(u, x, \theta^s, \gamma, \pi, \alpha)$ *satisfies SE*($\mathcal{D}_s$),
2. *If* $\mu(m) = \sum_{k \in \mathcal{K}(n)} \gamma^k(n) p^k(m), \quad m \in n_+,\ n \in \mathcal{N} \setminus \mathcal{L}$ *then there exist* $\{\theta_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$ *such that the prices* $\{\pi(n), n \in \mathcal{N}\}$, $\{\alpha_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$ *and actions for each* $a \in \mathcal{A}$ $\{(u_a(n), x_a(n), \theta_a(n)), n \in \mathcal{N}\}$, $\{W_a(n), n \in \mathcal{N} \setminus \{0\}\}$ *form a multistage risk-trading equilibrium RTE*($\mathcal{D}_{\mathcal{A}}$).

5. Conclusions

This paper provides a theory for multistage risked equilibria. Its main contributions are threefold.

Firstly, we extend the definition of multistage risked equilibrium given in Philpott et al. (2016) to a more general model that allows storage and pricing of transfers of shared resources, along with a number of examples that demonstrate the richness of the equilibrium framework that we propose.

Secondly, we have established versions of the first and second welfare theorems in a setting where agents can trade risk. We give a proof of the first welfare theorem (which is new) and a simpler proof of the second welfare theorem as applied to multistage risked equilibrium with risk trading. The First Welfare Theorem provides a perfectly competitive benchmark against which real markets might be measured. In the real world, where markets are imperfect, the optimal value of a social planning model provides an upper bound on what might be achieved in welfare terms by reducing market imperfections. The multistage risked equilibrium can be used to determine a competitive plan in the incomplete case, but we point out the difficulties in this approach related to both existence and non-uniqueness of solutions. Observe that the welfare results rely on Assumption 3. The risk sets of the agents must intersect to enable trade to be bounded. In a non-polyhedral setting we would require the stronger condition that the intersection of the relative interiors of the risk sets is nonempty (see e.g. Ralph and Smeers (2015)). If one agent believes that the risk-adjusted price of a given Arrow-Debreu contract strictly exceeds that asked by a prospective seller, then an infinite trade will result.

Thirdly, we illuminate the role that strict monotonicity of risk measures plays in multistage risked equilibrium. Our optimization versions of the welfare theorems (Corollaries 1 and 2) rely on Assumption 1. This is equivalent to the assertion that the one-step risk measure is strictly monotone, thus guaranteeing a nested risk measure that yields a time-consistent optimal solution. Competitive equilibrium specifies an optimal action for each agent in every state of the world, even if this is discounted in equilibrium to have zero risk-adjusted disbenefit. It is therefore necessary for a social plan to specify a set of actions for the agents in such states. This can be done either by constraining it to be time consistent using the formulation SE($\mathcal{D}_s$) in the absence of Assumption 1, or by imposing strict monotonicity on each agent's one-step risk measure.

Acknowledgments

This project has been supported in part by the Air Force Office of Scientific Research, the Department of Energy, the National Science Foundation, and the USDA National Institute of Food and Agriculture. The second author wishes to acknowledge the support of the New Zealand Marsden Fund grant UOA1520.

References

Abada I, Ehrenmann A, Smeers Y (2017) Modeling gas markets with endogenous long-term contracts. *Operations Research* 65(4):856–877.

Anderson EJ (2013) *Business risk management: models and analysis* (John Wiley & Sons).

Arrow K (1973) The role of securities in the optimal allocation of risk-bearing. *Readings in Welfare Economics*, 258–263 (Springer).

Arrow K, Debreu G (1954) Existence of an equilibrium for a competitive economy. *Econometrica: Journal of the Econometric Society* 265–290.

Artzner P, Delbaen F, Eber JM, Heath D (1999) Coherent measures of risk. *Mathematical Finance* 9:203–228.

Bessembinder H, Lemmon M (2002) Equilibrium pricing and optimal hedging in electricity forward markets. *The Journal of Finance* 57(3):1347–1382.

Birge JR, Louveaux F (2011) *Introduction to Stochastic Programming* (New York: Springer), 2nd edition.

Carpentier P, Chancelier JP, Cohen G, De Lara M, Girardeau P (2012) Dynamic consistency for stochastic optimal control problems. *Annals of Operations Research* 200(1):247–263.

Cochrane J, Saa-Requejo J (2000) Beyond arbitrage: Good-deal asset price bounds in incomplete markets. *Journal of political economy* 108(1):79–119.

de Maere d'Aertrycke G, Smeers Y (2013) Liquidity risks on power exchanges: a generalized nash equilibrium model. *Mathematical Programming* 140(2):381–414.

Feldman AM, Serrano R (2006) *Welfare Economics and Social Choice Theory* (Kluwer Academic Publishers), 2nd edition.

Föllmer H, Knispel T (2011) Entropic risk measures: Coherence vs. convexity, model ambiguity and robust large deviations. *Stochastics and Dynamics* 11:333–351.

Gérard H, Leclère V, Philpott A (2018) On risk averse competitive equilibrium. *Operations Research Letters* 46(1):19–26.

Gould FJ, Tolle JW (1971) A Necessary and Sufficient Qualification for Constrained Optimization. *SIAM Journal on Applied Mathematics* 20:164–172.

Heath D, Ku H (2004) Pareto equilibria with coherent measures of risk. *Mathematical Finance* 14(2):163–172.

Howard R, Matheson J (1972) Risk-sensitive Markov decision processes. *Management Science* 18(7):356–369.

Jorion P (2000) *Value at risk* (McGraw-Hill Professional Publishing).

Kok C, Philpott A, Zakeri G (2018) Value of transmission capacity in electricity markets with risk averse agents. Technical report, Electric Power Optimization Centre, University of Auckland, URL `www.epoc.org.nz`.

Levy H (1992) Stochastic dominance and expected utility: survey and analysis. *Management Science* 38(4):555–593.

Markowitz H (1952) Portfolio selection. *The Journal of Finance* 7(1):77–91.

Philpott AB, Ferris MC, Wets RJB (2016) Equilibrium, uncertainty and risk in hydro-thermal electricity systems. *Mathematical Programming B* 157(2):483–513.

Ralph D, Smeers Y (2015) Risk trading and endogenous probabilities in investment equilibria. *SIAM Journal on Optimization* 25(4):2589–2611.

Riedel F (2004) Dynamic coherent risk measures. *Stochastic Processes and their Applications* 112(2):185–200.

Rosen JB (1965) Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica: Journal of the Econometric Society* 520–534.

Ruszczyński A (2010) Risk-averse dynamic programming for Markov decision processes. *Mathematical Programming* 125:235–261.

Shapiro A (2017) Interchangeability principle and dynamic equations in risk-averse stochastic programming. *Operations Research Letters* 45:377–381.

Shapiro A, Dentcheva D, Ruszczyński A (2014) *Lectures on Stochastic Programming: Modeling and Theory* (SIAM), second edition.

Sharpe WF (1964) Capital asset prices: A theory of market equilibrium under conditions of risk. *The Journal of Finance* 19(3):425–442.

Tversky A, Kahneman D (1992) Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty* 5(4):297–323.

Varian H (1987) The arbitrage principle in financial economics. *Journal of Economic Perspectives* 1(2):55–72.

Von Neumann J, Morgenstern O (2007) *Theory of games and economic behavior (commemorative edition)* (Princeton University Press).

Yaari ME (1987) The dual theory of choice under risk. *Econometrica: Journal of the Econometric Society* 95–115.

Appendices

Appendix A: Coherent risk measures: proofs of lemmas

Proof of Lemma 1. For the forward implication, just choose $\gamma^k = 1$ for the term involving extreme point k that achieves the supremum. For the reverse implication, since $\theta \geq \sum_{m\in\mathcal{M}} p^k(m)Z(m)$ for each extreme point, it follows that $\theta \geq \sum_{m\in\mathcal{M}} \nu(m)Z(m)$ for each $\nu \in \mathcal{D}$ and hence $\theta \geq \sup_{\nu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \nu(m)Z(m)$. But complementary slackness shows that

$$\theta = \sum_{m\in\mathcal{M}} \bar{\nu}(m)Z(m),$$

where $\bar{\nu}$ is defined in the statement of the theorem and is clearly in $\mathcal{D}$ so $\theta \leq \sup_{\nu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \nu(m)Z(m)$ and thus equality holds. $\square$

Proof of Lemma 2. Suppose $\mathcal{D}$ lies in $\mathrm{int}(\mathbb{R}_+^{|\mathcal{M}|})$. To show strict monotonicity, we suppose $Z_a \geq Z_b$ and $Z_a(\bar{m}) > Z_b(\bar{m})$ for some $\bar{m} \in \mathcal{M}$. Let $\rho(Z_a) = \sum_{m\in\mathcal{M}} \nu_a^*(m)Z_a(m)$, and $\rho(Z_b) = \sum_{m\in\mathcal{M}} \nu_b^*(m)Z_b(m)$. Then strict monotonicity follows from $\nu_b^*(\bar{m}) > 0$ since

$$\begin{aligned}\rho(Z_a) &= \sum_{m\in\mathcal{M}} \nu_a^*(m)Z_a(m)\\ &\geq \sum_{m\in\mathcal{M}} \nu_b^*(m)Z_a(m)\end{aligned}$$

$$
\begin{aligned}
&> \sum_{m\in\mathcal{M}} \nu_b^*(m) Z_b(m) \\
&= \rho(Z_b).
\end{aligned}
$$

Conversely, suppose $\mathcal{D}$ does not lie in $\mathrm{int}(\mathbb{R}_+^{|\mathcal{M}|})$, thus containing some point $\bar{\nu}$ with a zero component, say $\bar{\nu}(m_1) = 0$. Choose $Z(m) = 0$, $m = m_2, m_3, \ldots, m_{|\mathcal{M}|}$, and $Z(m_1) < 0$. Then $\bar{\nu} \in \arg\max_{\nu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \nu(m)Z(m)$, since $\sum_{m\in\mathcal{M}} \nu(m)Z(m) \le 0$ for every $\nu \in \mathcal{D}$. Let

$$
Z'(m) = \begin{cases} Z(m_1) - 1, & m = m_1 \\ Z(m), & \text{otherwise} \end{cases}
$$

so $Z' \le Z$ with $Z' \ne Z$. But $\bar{\nu} \in \arg\max_{\nu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \nu(m)Z'(m)$, so

$$
\rho(Z') = \sum_{m\in\mathcal{M}} \bar{\nu}(m)Z'(m) = \sum_{m\in\mathcal{M}} \bar{\nu}(m)Z(m) = \rho(Z),
$$

violating the strict monotonicity of ρ. $\square$

Appendix B: Tree multipliers

Consider a scenario tree with polyhedral risk sets $\mathcal{D}(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$, each having a finite set of extreme points $\{[p^k]_{n_+}, k \in \mathcal{K}(n)\}$. Any set of nonnegative numbers of the form $\{\gamma^k(n), k \in \mathcal{K}(n), n \in \mathcal{N} \setminus \mathcal{L}\}$ is called a set of *tree multipliers*. A set of tree multipliers is *conditional* if for every $n \in \mathcal{N} \setminus \mathcal{L}$, $\sum_{k\in\mathcal{K}(n)} \gamma^k(n) = 1$. A set of tree multipliers $\{\lambda^k(n), k \in \mathcal{K}(n), n \in \mathcal{N} \setminus \mathcal{L}\}$ is *unconditional* if

$$
0 = 1 - \sum_{k\in\mathcal{K}(0)} \lambda^k(0), \tag{16}
$$

$$
0 = -\sum_{k\in\mathcal{K}(n)} \lambda^k(n) + \sum_{j\in\mathcal{K}(n_-)} \lambda^j(n_-)p^j(n), \quad n \in \mathcal{N} \setminus \mathcal{L}, n \ne 0. \tag{17}
$$

Observe that any set of conditional tree multipliers γ corresponds to a unique set of unconditional tree multipliers λ defined recursively by setting $\lambda(0) = \gamma(0)$, and defining

$$
\lambda(n) = \gamma(n) \sum_{j\in\mathcal{K}(n_-)} \lambda^j(n_-)p^j(n), \quad n \in \mathcal{N} \setminus \{0\}. \tag{18}
$$

Since $\lambda(0) \ge 0$, repeated application of (18) implies $\lambda(n) \ge 0$ for every $n \in \mathcal{N} \setminus \{0\}$, so $\lambda^k(n)$ are well-defined tree multipliers. These are easily verified to be unconditional since for $n \in \mathcal{N} \setminus \mathcal{L}$

$$
\sum_{k\in\mathcal{K}(n)} \lambda^k(n) = \sum_{j\in\mathcal{K}(n_-)} \lambda^j(n_-)p^j(n),
$$

giving (17) and

$$
\sum_{k\in\mathcal{K}(0)} \lambda^k(0) = \sum_{k\in\mathcal{K}(0)} \gamma^k(0) = 1,
$$

giving (16). Conversely any unconditional set of tree multipliers corresponds to a unique set of conditional tree multipliers as long as Assumption 1 holds. To see this define $\gamma(0) = \lambda(0)$, and

$$\gamma^k(n) = \lambda^k(n)/(\sum_{\ell \in \mathcal{K}(n_-)} \lambda^\ell(n_-)p^\ell(n)) \quad k \in \mathcal{K}(n), \quad n \in \mathcal{N} \setminus \{0\}. \tag{19}$$

By Assumption 1 every component of $p^k(m)$, $m \in 0_+$ is strictly positive, and the vector $(\lambda(0))$ is nonnegative and nonzero by (16), so $\gamma^k(m)$ is well defined by (19) for $m \in 0_+$. However, (17) implies that the vector $(\lambda^k(m))$ is nonnegative and nonzero, and hence recursively that

$$\sum_{j \in \mathcal{K}(n_-)} \lambda^j(n_-)p^j(n) > 0, \quad n \in \mathcal{N} \setminus \{0\}. \tag{20}$$

Finally (17) and (19) imply $\sum_{k \in \mathcal{K}(n)} \gamma^k(n) = 1$, showing that $\{\gamma^k(n),\ k \in \mathcal{K}(n),\ n \in \mathcal{N} \setminus \mathcal{L}\}$ is conditional.

To ease notation in what follows, given any unconditional tree multipliers λ we define tree multipliers σ by

$$\sigma(n) = \begin{cases} 1, & n = 0, \\ \sum\limits_{j \in \mathcal{K}(n_-)} \lambda^j(n_-)p^j(n), & n \in \mathcal{N} \setminus \{0\}. \end{cases} \tag{21}$$

Observe by (18) that (21) implies

$$\lambda^k(n) = \gamma^k(n)\sigma(n), \quad k \in \mathcal{K}(n), n \in \mathcal{N}, \tag{22}$$

whence multiplying by $p^k(m)$ and summing gives

$$\sum_{k \in \mathcal{K}(n)} \lambda^k(n)p^k(m) = \sigma(m) = (\sum_{k \in \mathcal{K}(n)} \gamma^k(n)p^k(m))\sigma(n), \quad m \in n_+, \quad n \in \mathcal{N} \setminus \mathcal{L}. \tag{23}$$

Conditional and unconditional multipliers satisfy the following lemma.

LEMMA 3. *If $\theta(n), n \in \mathcal{N}$ and a conditional set of tree multipliers $\{\gamma^k(n),\ k \in \mathcal{K}(n),\ n \in \mathcal{N} \setminus \mathcal{L}\}$ satisfies*

$$0 \le \gamma^k(n) \perp \theta(n) - \sum_{m \in n_+} p^k(m)\,(C(m) + \theta(m)) \ge 0, \quad k \in \mathcal{K}(n), n \in \mathcal{N} \setminus \mathcal{L}, \tag{24}$$

then there exist unconditional multipliers λ satisfying

$$0 \le \lambda^k(n) \perp \theta(n) - \sum_{m \in n_+} p^k(m)\,(C(m) + \theta(m)) \ge 0, \quad k \in \mathcal{K}(n), n \in \mathcal{N} \setminus \mathcal{L}. \tag{25}$$

Conversely, if (λ, θ) satisfies (16),(17),(25), and Assumption 1 holds, then $\sigma(n)$ defined by (21) is strictly positive for every $n \in \mathcal{N}$, and there exists conditional tree multipliers $\gamma^k(n) = \frac{\lambda^k(n)}{\sigma(n)}$, $\quad n \in \mathcal{N} \setminus \mathcal{L}$ satisfying (24).

Proof. Given a set of conditional tree multipliers γ construct unconditional multipliers λ from (18) and $\lambda(0) = 1$. Given these values, $\sigma \geq 0$ is defined by (21), so

$$0 \leq \sigma(n)\gamma^k(n) \perp \theta(n) - \sum_{m \in n_+} p^k(m)\left(C(m) + \theta(m)\right) \geq 0,$$

yielding (25) via (22). Conversely, Assumption 1 implies (20), so we have $\sigma(n) > 0$. The relationship (24) then follows from (25) by dividing through by $\sigma(n) > 0$. $\square$

Appendix C: Proofs of main results

Proof of Theorem 1. The following Karush-Kuhn-Tucker conditions for SO($\mathcal{D}$) are necessary and sufficient for optimality in SO($\mathcal{D}$).

KKT($\mathcal{D}$):

$$\begin{aligned}
0 &= 1 - \sum_{k \in \mathcal{K}(0)} \lambda^k(0), \\
0 &= -\sum_{k \in \mathcal{K}(n)} \lambda^k(n) + \sum_{j \in \mathcal{K}(n_-)} \lambda^j(n_-) p^j(n), \quad n \in \mathcal{N} \setminus \mathcal{L}, n \neq 0 \\
0 &\leq \lambda^k(n) \perp \theta(n) - \sum_{m \in n_+} p^k(m) \left(\sum_{a \in \mathcal{A}} C_{an}(u_a(m)) + \theta(m) \right) \geq 0, \quad k \in \mathcal{K}(n), n \in \mathcal{N} \setminus \mathcal{L} \\
\theta(n) &= -\sum_{a \in \mathcal{A}} V_{an}(x_a(n)), \quad n \in \mathcal{L} \\
0 &\in \nabla_{u_a(0)} \left[C_{an}(u_a(0)) - \tilde{\pi}(0) g_{an}(u_a(0)) - \sum_{b \in \mathcal{A}} \tilde{\alpha}_b(0) T_{ba} u_a(0) \right] + N_{\mathcal{U}_a}(u_a(0)), \quad a \in \mathcal{A} \\
0 &\in \nabla_{u_a(n)} \left[\sum_{k \in \mathcal{K}(n_-)} \lambda^k(n_-) p^k(n) C_{an}(u_a(n)) - \tilde{\pi}(n) g_{an}(u_a(n)) - \sum_{b \in \mathcal{A}} \tilde{\alpha}_b(n) T_{ba} u_a(n) \right] \\
&\quad + N_{\mathcal{U}_a}(u_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{N} \setminus \{0\} \\
0 &\in \tilde{\alpha}_a(n) - \sum_{m \in n_+} \tilde{\alpha}_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{N} \setminus \mathcal{L} \\
0 &\in \tilde{\alpha}_a(n) - \sum_{k \in \mathcal{K}(n_-)} \lambda^k(n_-) p^k(n) \nabla_{x_a(n)} V_{an}(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{L} \\
0 &\leq \tilde{\alpha}_a(n) \perp -x_a(n) + x_a(n_-) + \sum_{b \in \mathcal{A}} T_{ab} u_b(n) + \omega_a(n) \geq 0, \quad a \in \mathcal{A}, n \in \mathcal{N} \\
0 &\leq \tilde{\pi}(n) \perp \sum_{a \in \mathcal{A}} g_{an}(u_a(n)) \geq 0, \quad n \in \mathcal{N}.
\end{aligned}$$

The proof proceeds to show the equivalence of these conditions to a solution of SE($\mathcal{D}$) under the hypotheses of the theorem.

(i) Suppose $(u, \theta, x, \gamma, \alpha, \pi)$ is a solution of SE($\mathcal{D}$). Since γ are conditional multipliers, and $\theta(n) = -\sum_{a \in \mathcal{A}} V_{an}(x_a(n))$, $n \in \mathcal{L}$, and (θ, γ) satisfies (24), Lemma 3 provides unconditional multipliers

λ (and therefore σ from (21)) such that (25) holds for $C(m) = \sum_{a\in\mathcal{A}} C_{an}(u_a(m))$. Using these observations, the problem SE($\mathcal{D}$) leads to the conditions:

$$0 = 1 - \sum_{k\in\mathcal{K}(0)} \lambda^k(0),$$
$$0 = -\sum_{k\in\mathcal{K}(n)} \lambda^k(n) + \sum_{j\in\mathcal{K}(n_-)} \lambda^j(n_-)p^j(n), \quad n\in\mathcal{N}\setminus\mathcal{L}, n\neq 0$$
$$0 \le \lambda^k(n) \perp \theta(n) - \sum_{m\in n_+} p^k(m) \sum_{a\in\mathcal{A}} (C_{an}(u_a(m)) + \theta(m)) \ge 0, \quad k\in\mathcal{K}(n), n\in\mathcal{N}\setminus\mathcal{L}$$
$$\theta(n) = -\sum_{a\in\mathcal{A}} V_{an}(x_a(n)), \quad n\in\mathcal{L}$$
$$0 \in \nabla_{u_a(n)} \left[\sigma(n)C_{an}(u_a(n)) - \sigma(n)\pi(n)g_{an}(u_a(n)) - \sigma(n)\sum_{b\in\mathcal{A}} \alpha_b(n)T_{ba}u_a(n)\right] + N_{\mathcal{U}_a}(u_a(n)), \quad a\in\mathcal{A}, n\in\mathcal{N}$$
$$0 \in \sigma(n)\alpha_a(n) - \sum_{m\in n_+}\sum_{k\in\mathcal{K}(n)} \lambda^k(n)p^k(m)\alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad a\in\mathcal{A}, n\in\mathcal{N}\setminus\mathcal{L}$$
$$0 \in \sigma(n)\alpha_a(n) - \sigma(n)\nabla_{x_a(n)}V_{an}(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad a\in\mathcal{A}, n\in\mathcal{L}$$
$$0 \le \sigma(n)\alpha_a(n) \perp -x_a(n) + x_a(n_-) + \sum_{b\in\mathcal{A}} T_{ab}u_b(n) + \omega_a(n) \ge 0, \quad a\in\mathcal{A}, n\in\mathcal{N}$$
$$0 \le \sigma(n)\pi(n) \perp \sum_{a\in\mathcal{A}} g_{an}(u_a(n)) \ge 0, \quad n\in\mathcal{N}.$$

The relationships involving normal cones follow from multiplication by $\sigma(n)$ and (22), while the complementarity conditions follow from Lemma 3 and multiplication by $\sigma(n)$. If we let $\tilde{\alpha}_a(n) = \sigma(n)\alpha_a(n)$ and $\tilde{\pi}(n) = \sigma(n)\pi(n)$ then recalling (21) these conditions yield KKT($\mathcal{D}$), the KKT conditions for SO($\mathcal{D}$). Since any solution of SE($\mathcal{D}$) satisfies (24) in Lemma 3, (22) and Lemma 1 imply that

$$\theta(n) = \sup_{\nu\in\mathcal{D}(n)} \sum_{m\in n_+} \nu(m) \left(\sum_{a\in\mathcal{A}} C_{an}(u_a(m)) + \theta(m)\right)$$

is attained by $\bar{\nu}(m) = \sum_{k\in\mathcal{K}(n)} \gamma^k(n)p^k(m)$, which gives the last statement of (i).

(ii) For the converse result, suppose that we have a solution $(u, x, \theta, \lambda, \tilde{\pi}, \tilde{\alpha})$ of the KKT conditions of SO($\mathcal{D}$) as shown above. Then Assumption 1 and Lemma 3 provide $\sigma(n) > 0$ and a conditional set of multipliers $\gamma^k(n) = \lambda^k(n)/\sigma(n)$ satisfying (24) for $C(m) = \sum_{a\in\mathcal{A}} C_{an}(u_a(m))$. Substituting $\alpha_a(n) = \tilde{\alpha}_a(n)/\sigma(n)$ and $\pi(n) = \tilde{\pi}(n)/\sigma(n)$ into the KKT conditions of SO($\mathcal{D}$) and using (24) and (23) leads to

$$1 = \sum_{k\in\mathcal{K}(n)} \gamma^k(n), \quad n\in\mathcal{N}\setminus\mathcal{L}$$
$$0 \le \gamma^k(n) \perp \theta(n) - \sum_{m\in n_+} p^k(m) \sum_{a\in\mathcal{A}} (C_{an}(u_a(m)) + \theta(m)) \ge 0, \quad k\in\mathcal{K}(n), n\in\mathcal{N}\setminus\mathcal{L}$$
$$\theta(n) = -\sum_{a\in\mathcal{A}} V_{an}(x_a(n)), \quad n\in\mathcal{L}$$

$$0 \in \nabla_{u_a(n)} \left[\sigma(n) C_{an}(u_a(n)) - \sigma(n)\pi(n) g_{an}(u_a(n)) - \sigma(n) \sum_{b\in\mathcal{A}} \alpha_b(n) T_{ba} u_a(n) \right] + N_{\mathcal{U}_a}(u_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{N}$$

$$0 \in \sigma(n)\alpha_a(n) - \sum_{m\in n_+} \sigma(n) \sum_{k\in\mathcal{K}(n)} \gamma^k(n) p^k(m) \alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{N} \setminus \mathcal{L}$$

$$0 \in \sigma(n)\alpha_a(n) - \sigma(n)\nabla_{x_a(n)} V_{an}(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{L}$$

$$0 \le \sigma(n)\alpha_a(n) \perp -x_a(n) + x_a(n_-) + \sum_{b\in\mathcal{A}} T_{ab} u_b(n) + \omega_a(n) \ge 0, \quad a \in \mathcal{A}, n \in \mathcal{N}$$

$$0 \le \sigma(n)\pi(n) \perp \sum_{a\in\mathcal{A}} g_{an}(u_a(n)) \ge 0, \quad n \in \mathcal{N}.$$

Dividing through by $\sigma(n)$ appropriately leads to a solution of SE($\mathcal{D}$) as required. □

Proof of Theorem 2. As we outlined above for the system optimization problem, $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ is equivalent to its KKT conditions, which are derived by applying nonnegative Lagrange multipliers $\lambda^k(n)$ to the inequality constraints. Since θ is unconstrained, λ satisfies (16) and (17), so they are unconditional tree multipliers. This enables us to substitute $\sigma(n)$ for $\sum_{j\in\mathcal{K}_{\dashv}(n_-)} \lambda^j(n_-) p_a^j(n)$ to give the following KKT conditions for $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$.

KKT_a:

$$0 = 1 - \sum_{k\in\mathcal{K}_a(0)} \lambda^k(0), \tag{26a}$$

$$0 = - \sum_{k\in\mathcal{K}_a(n)} \lambda^k(n) + \sigma(n), \quad n \in \mathcal{N} \setminus \mathcal{L}, n \neq 0 \tag{26b}$$

$$0 \le \lambda^k(n) \perp \theta_a(n) - \sum_{m\in n_+} p_a^k(m) \left(Z_a(m; u, x, W) - W_a(m) + \theta_a(m) \right) \ge 0,$$
$$k \in \mathcal{K}_a(n), n \in \mathcal{N} \setminus \mathcal{L} \tag{26c}$$

$$\theta_a(n) = -V_{an}(x_a(n)), \quad n \in \mathcal{L}. \tag{26d}$$

$$0 \in \nabla_{u_a(0)} Z_a(0; u, x, W) + N_{\mathcal{U}_a}(u_a(0)), \tag{26e}$$

$$0 \in \sigma(n) \nabla_{u_a(n)} Z_a(n; u, x, W) + N_{\mathcal{U}_a}(u_a(n)), \quad n \in \mathcal{N} \setminus \{0\} \tag{26f}$$

$$0 \in \sigma(n)\alpha_a(n) - \sum_{m\in n_+} \sigma(m)\alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{N} \setminus \mathcal{L} \tag{26g}$$

$$0 \in \sigma(n)\alpha_a(n) - \sigma(n)\nabla_{x_a(n)} V_{an}(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{L} \tag{26h}$$

$$0 = \mu(m) - \sum_{k\in\mathcal{K}_a(0)} \lambda^k(0) p_a^k(m), \quad m \in 0_+ \tag{26i}$$

$$0 = \sigma(q_-)\mu(q) - \sigma(q), \quad q \in n_{++} \cap \mathcal{N}, \quad n \in \mathcal{N}, \tag{26j}$$

with $Z_a(n; u, x, W)$ defined by (10).

(i) First suppose that $(u_a, x_a, W_a, \theta_a, \gamma)$ is a solution of $\mathrm{AE}_a(\pi, \alpha, \mu, \mathcal{D}_a)$. Observe that (11b) implies that

$$\theta_a(n) \geq \sum_{m \in n_+} \nu(m)\left(Z_a(m; u, x, W) - W_a(m) + \theta_a(m)\right)$$

for every $\nu \in \mathcal{D}_a(n)$. Summing the complementarity condition (11b) over k and combining with (11a) gives

$$\begin{aligned}
\theta_a(n) &= \sum_{k \in \mathcal{K}_a(n)} \gamma^k(n)\theta_a(n) \\
&= \sum_{m \in n_+} \sum_{k \in \mathcal{K}_a(n)} \gamma^k(n) p_a^k(m)\left(Z_a(m; u, x, W) - W_a(m) + \theta_a(m)\right) \\
&= \sum_{m \in n_+} \mu(m)\left(Z_a(m; u, x, W) - W_a(m) + \theta_a(m)\right)
\end{aligned}$$

after substituting using (11g). It also follows from Lemma 3 that there exists λ which satisfies (25) with $C(m) = Z_a(m; u, x, W) - W_a(m)$ and $\theta_a(m)$ replacing $\theta(m)$. Given λ we can define σ using (21).

Putting these relationships together and substituting into the $\mathrm{AE}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ conditions (observing that $\sigma(n) \geq 0$) gives

$$\begin{aligned}
&0 = 1 - \sum_{k \in \mathcal{K}_a(0)} \lambda^k(0), \\
&0 = -\sum_{k \in \mathcal{K}_a(n)} \lambda^k(n) + \sum_{j \in \mathcal{K}_{\dashv}(n_-)} \lambda^j(n_-) p_a^j(n), \quad n \in \mathcal{N} \setminus \mathcal{L}, n \neq 0 \\
&0 \leq \lambda^k(n) \perp \theta_a(n) - \sum_{m \in n_+} p_a^k(m)\Big(Z_a(m; u, x, W) - W_a(m) + \theta_a(m)\Big) \geq 0, \quad k \in \mathcal{K}_a(n), n \in \mathcal{N} \setminus \mathcal{L} \\
&\quad \theta_a(n) = -V_{an}(x_a(n)), \quad n \in \mathcal{L} \\
&0 \in \sigma(n)\nabla_{u_a(n)} Z_a(n; u, x, W) + N_{\mathcal{U}_a}(u_a(n)), \quad n \in \mathcal{N} \\
&0 \in \sigma(n)\alpha_a(n) - \sum_{m \in n_+} \sigma(m)\alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{N} \setminus \mathcal{L} \\
&0 \in \sigma(n)\alpha_a(n) - \sigma(n)\nabla_{x_a(n)} V_{an}(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{L} \\
&0 = \sigma(n)\mu(m) - \sum_{k \in \mathcal{K}_a(n)} \lambda^k(n) p_a^k(m), \quad m \in n_+, n \in \mathcal{N} \setminus \mathcal{L},
\end{aligned}$$

with $Z_a(n, u, x, W)$ defined by (10).

Clearly we recover (26a)–(26h). It simply remains to show that λ satisfies (26i) and (26j). Since $\lambda^k(0) = \gamma^k(0)$, (26i) is immediate from (11g). Since $\sigma(n) = \sum_{j \in \mathcal{K}_{\dashv}(n_-)} \lambda^j(n_-) p_a^j(n)$, (11g) is equivalent to $\sigma(n)\mu(m) = \sigma(m)$ for $m \in n_+$, $n \in \mathcal{N} \setminus \mathcal{L}$, which gives (26j) if we identify q with m.

(ii) For the converse, suppose that we have a solution of (26), then Lemma 3 coupled with Assumption 1 provides $\sigma(n) > 0$ and conditional multipliers $\gamma^k(n) = \lambda^k(n)/\sigma(n)$ that satisfy (24) for

$C(m) = Z_a(m; u, x, W) - W_a(m)$ and $\theta(n) = \theta_a(n)$. Thus (11a), (11b) and (11c) are satisfied in the definition of the $\mathrm{AE}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ problem. Now (11g) follows by dividing (26j) by $\sigma(q_-)$ and using (21) and (22). Noting (23) and then dividing (26g) and (26h) by $\sigma(n)$ then gives (11e) and (11f) respectively. The relationship (11d) follows from the definition of σ and (26e) and (26f). $\square$

Proof of Theorem 3. (i) If we have a solution of $\mathrm{AE}_a(\bar{\pi}, \bar{\alpha}, \bar{\mu}, \mathcal{D}_a)$ for each $a \in \mathcal{A}$, it follows from (11g) that $[\bar{\mu}]_{n_+} \in \mathcal{D}_a(n)$ for each n and thus $\bar{\mu} \in \mathcal{D}_a$ for all a, and hence $\bar{\mu} \in \mathcal{D}_s$ by Definition 3.
(ii) For each $a \in \mathcal{A}$ it follows from Theorem 2 that for $n \in \mathcal{N} \setminus \mathcal{L}$,

$$\bar{\theta}_a(n) = \sum_{m \in n_+} \bar{\mu}(m) \left(\bar{Z}_a(m; \bar{u}, \bar{x}, \bar{W}) - \bar{W}_a(m) + \bar{\theta}_a(m) \right). \tag{27}$$

Summing over $a \in \mathcal{A}$ and invoking (14) gives

$$\bar{\theta}(n) = \sum_{a \in \mathcal{A}} \bar{\theta}_a(n) = \sum_{m \in n_+} \bar{\mu}(m) \left(\sum_{a \in \mathcal{A}} \bar{Z}_a(m; \bar{u}, \bar{x}, \bar{W}) + \bar{\theta}(m) \right).$$

Recalling the definition of $\bar{Z}_a(m; \bar{u}, \bar{x}, \bar{W})$ from (10), summing over $a \in \mathcal{A}$, and invoking (12), (13) and (14) gives (15).
(iii) Suppose $\mathcal{D}_0 = \{\bar{\mu}\}$. It follows that $\mathcal{K}_0(n) = \{1\}$ for $n \in \mathcal{N} \setminus \mathcal{L}$ where $p_0^1(m) = \bar{\mu}(m)$, for $m \in n_+$. Define $\gamma^1(n) = 1$, $n \in \mathcal{N} \setminus \mathcal{L}$. It then follows that the first, second and fifth conditions of $SE(\mathcal{D}_0)$ simplify to

$$\begin{aligned}
\gamma^1(n) &= 1, \quad n \in \mathcal{N} \setminus \mathcal{L}, \\
\theta(n) &= \sum_{m \in n_+} \bar{\mu}(m) \left(\sum_{a \in \mathcal{A}} C_{an}(u_a(m)) + \theta(m) \right), \quad n \in \mathcal{N} \setminus \mathcal{L}, \\
0 &\in \alpha_a(n) - \sum_{m \in n_+} \bar{\mu}(m) \alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{N} \setminus \mathcal{L}.
\end{aligned}$$

Combining these with the other conditions in (11), (12) and (13) shows that $(\bar{u}, \bar{x}, \bar{\theta}, \gamma, \pi, \alpha)$ solves $SE(\mathcal{D}_0)$.
(iv) Suppose $(\bar{u}_a, \bar{x}_a, \bar{W}_a, \bar{\theta}_a, \gamma_a)$ solves $\mathrm{AE}_a(\bar{\pi}, \bar{\alpha}, \bar{\mu}, \mathcal{D}_a)$. Let $\bar{Z}_a(n; \bar{u}, \bar{x}, \bar{W})$ be defined using (10) so that it follows from (12), (13) and (14) that

$$\begin{aligned}
&\sum_{a \in \mathcal{A}} \sum_{m \in n_+} \bar{\mu}(m) \left(C_{an}(\bar{u}_a(m)) + \bar{\theta}_a(m) \right) \\
&\qquad = \sum_{a \in \mathcal{A}} \sum_{m \in n_+} \bar{\mu}(m) \left(\bar{Z}_a(m; \bar{u}, \bar{x}, \bar{W}) - \bar{W}_a(m) + \bar{\theta}_a(m) \right),
\end{aligned}$$

which by (15)

$$= \sum_{a \in \mathcal{A}} \bar{\theta}_a(n)$$

and by (11a) and (11b) and Lemma 1

$$=\sum_{a\in\mathcal{A}}\sup_{\nu\in\mathcal{D}_a(n)}\sum_{m\in n_+}\nu(m)\left(\bar{Z}_a(m;\bar{u},\bar{x},\bar{W})-\bar{W}_a(m)+\bar{\theta}_a(m)\right)$$

so that Assumption 3 and Definition 3 imply

$$\geq\sum_{a\in\mathcal{A}}\sup_{\nu\in\mathcal{D}_s(n)}\sum_{m\in n_+}\nu(m)\left(\bar{Z}_a(m;\bar{u},\bar{x},\bar{W})-\bar{W}_a(m)+\bar{\theta}_a(m)\right)$$

and interchanging supremum and summation

$$\geq\sup_{\nu\in\mathcal{D}_s(n)}\sum_{m\in n_+}\nu(m)\sum_{a\in\mathcal{A}}\left(\bar{Z}_a(m;\bar{u},\bar{x},\bar{W})-\bar{W}_a(m)+\bar{\theta}_a(m)\right)$$

since feasibility implies $-\sum_{a\in\mathcal{A}}\bar{W}_a(m)\geq 0$

$$\geq\sup_{\nu\in\mathcal{D}_s(n)}\sum_{m\in n_+}\nu(m)\sum_{a\in\mathcal{A}}\left(\bar{Z}_a(m;\bar{u},\bar{x},\bar{W})+\bar{\theta}_a(m)\right)$$

by (12), (13) and (14)

$$=\sup_{\nu\in\mathcal{D}_s(n)}\sum_{m\in n_+}\nu(m)\sum_{a\in\mathcal{A}}\left(C_{an}(\bar{u}_a(m))+\bar{\theta}_a(m)\right)$$

by (i)

$$\geq\sum_{m\in n_+}\bar{\mu}(m)\sum_{a\in\mathcal{A}}\left(C_{an}(\bar{u}_a(m))+\bar{\theta}_a(m)\right).$$

Hence equality holds throughout and thus $[\bar{\mu}]_{n_+}$ solves

$$\sup_{\nu\in\mathcal{D}_s(n)}\sum_{m\in n_+}\nu(m)\left(\sum_{a\in\mathcal{A}}C_{an}(\bar{u}_a(m))+\bar{\theta}(m)\right).$$

Lemma 1 then shows that these conditions are equivalent to the first two conditions of SE($\mathcal{D}_s$), which combined with the other conditions in $\mathrm{AE}_a(\bar{\pi},\bar{\alpha},\bar{\mu},\mathcal{D}_a)$ gives the remaining conditions of SE($\mathcal{D}_s$). □

To prove Theorem 4, we will require a preliminary lemma that uses the following formulations. For each $n\in\mathcal{N}\setminus\mathcal{L}$, suppose $Z_a^s(m)$, $\theta^s(m)$ and $\theta_a^s(m)$ are given for each $m\in n_+$ and satisfy $\theta^s(m)=\sum_{a\in\mathcal{A}}\theta_a^s(m)$. Consider the problems:

$$\mathrm{R}(n,\mathcal{D}_s)\colon\ \max_{\nu\in\mathcal{D}_s(n)}\sum_{m\in n_+}\nu(m)\left(\sum_{a\in\mathcal{A}}Z_a^s(m)+\theta^s(m)\right)$$

$$\begin{aligned}
&\mathrm{T}(n,\mathcal{D}_{\mathcal{A}}):\\
&\min_{[[W_a]_{n_+}]_{a\in\mathcal{A}},\theta_a(n)} \sum_{a\in\mathcal{A}} \theta_a(n)\\
&\qquad\text{s.t. } \theta_a(n) \geq \sum_{m\in n_+} p_a^k(m)\left(Z_a^s(m) - W_a(m) + \theta_a^s(m)\right), \quad k\in\mathcal{K}_a(n), a\in\mathcal{A}\\
&\qquad\quad -\sum_{a\in\mathcal{A}} W_a(m) \geq 0, \quad m\in n_+
\end{aligned}$$

$$\begin{aligned}
&\mathrm{TD}(n,\mathcal{D}_{\mathcal{A}}):\\
&\max_{\mu,\phi} \sum_{m\in n_+}\sum_{a\in\mathcal{A}} \left(\sum_{k\in\mathcal{K}_a(n)} p_a^k(m)\phi_a^k(n)\right)\left(Z_a^s(m) + \theta_a^s(m)\right)\\
&\quad \sum_{k\in\mathcal{K}_a(n)} \phi_a^k(n) = 1, \quad a\in\mathcal{A},\\
&\quad \mu(m) = \sum_{k\in\mathcal{K}_a(n)} p_a^k(m)\phi_a^k(n), \quad m\in n_+, a\in\mathcal{A}\\
&\quad \mu(m) \geq 0, \quad m\in n_+, \quad \phi_a^k(n)\geq 0, \quad k\in\mathcal{K}_a(n), a\in\mathcal{A}
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{TOC}(n,\mathcal{D}_{\mathcal{A}}):\\
&0 = 1 - \sum_{k\in\mathcal{K}_a(n)} \phi_a^k(n), \quad a\in\mathcal{A}\\
&0 = \mu(m) - \sum_{k\in\mathcal{K}_a(n)} \phi_a^k(n) p_a^k(m), \quad m\in n_+, a\in\mathcal{A}\\
&0 \leq \phi_a^k(n) \perp \theta_a(n) - \sum_{m\in n_+} p_a^k(m)\left(Z_a^s(m) - W_a(m) + \theta_a^s(m)\right), \quad k\in\mathcal{K}_a(n), a\in\mathcal{A}\\
&0 \leq \mu(m) \perp -\sum_{a\in\mathcal{A}} W_a(m) \geq 0, \quad m\in n_+
\end{aligned}$$

The formulation R evaluates the one-stage risk of the random disbenefit $\sum_{a\in\mathcal{A}} Z_a$ using the coherent risk measure with risk set $D_s(n) = \bigcap_{a\in\mathcal{A}} \mathcal{D}_a(n)$. The problem T on the other hand accumulates the risk measure of each agent a in a setting where they can exchange welfare W (constrained so that it cannot be created out of nothing). If the model has a variable $W_a(m)$ defined for each outcome $m \in n_+$, then the following analysis demonstrates that an exchange exists in node n that will yield the risk-adjusted value of the total social disbenefit faced by all agents if evaluated with risk set $\mathcal{D}_s(n)$.

LEMMA 4. *Let $n \in \mathcal{N}$ and suppose $\mathcal{D}_{\mathcal{A}}$ satisfies Assumption 3. The problems T, TD, TOC and R all have optimal solutions with the same optimal value. Any solution to one of these problems yields a solution to all of the others.*

Proof. Observe that T and TD are dual linear programs, and TOC gives the optimality conditions for T. The constraints of TD entail that $\mu(m), m \in n_+$ is a finite probability distribution that is constrained to lie in each $\mathcal{D}_a(n)$. Definition 3 means that TD is equivalent to R. So any optimal solution of one of these four formulations yields solutions to all the others. Observe that the feasible region of TD is compact and and nonempty by Assumption 3, so T, TD, TOC and R all have optimal solutions with the same optimal value. □

Proof of Theorem 4. Suppose $(u, x, \theta^s, \gamma, \pi, \alpha)$ is a solution of $\mathrm{SE}(\mathcal{D}_s)$. It follows from Theorem 1 that defining $\mu(m) = \sum_{k\in\mathcal{K}_s(n)} \gamma^k(n)p_s^k(m) \in \mathcal{D}_s$ for each $m \in n_+$ we have

$$\begin{aligned}\theta^s(n) &= \sum_{m\in n_+} \mu(m)\Bigg(\sum_{a\in\mathcal{A}}\Bigg(C_{an}(u_a(m)) - \pi(m)g_{an}(u_a(m)) \\ &\quad + \alpha_a(m)\Big(x_a(m) - x_a(m_-) - \sum_{b\in\mathcal{A}} T_{ab}u_b(m) - \omega_a(m)\Big)\Bigg) + \theta^s(m)\Bigg) \\ &= \sum_{m\in n_+} \mu(m)\Bigg(\sum_{a\in\mathcal{A}}\Big(C_{an}(u_a(m)) - \pi(m)g_{an}(u_a(m)) \\ &\quad + \alpha_a(m)\left(x_a(m) - x_a(m_-) - \omega_a(m)\right) - \sum_{b\in\mathcal{A}} \alpha_b(m)T_{ba}u_a(m)\Big) + \theta^s(m)\Bigg).\end{aligned}$$

Consider the leaf nodes $m \in \mathcal{L}$. At these nodes $\theta^s(m) = -\sum_{a\in\mathcal{A}} V_{an}(x_a(m))$ so defining $\theta_a^s(m) = -V_{an}(x_a(m))$ for each $a \in \mathcal{A}$ we have $\sum_{a\in\mathcal{A}} \theta_a^s(m) = \theta^s(m)$. Letting

$$\begin{aligned}Z_a^s(m) &= C_{an}(u_a(m)) - \pi(m)g_{an}(u_a(m)) \\ &\quad + \alpha_a(m)\left(x_a(m) - x_a(m_-) - \omega_a(m)\right) - \sum_{b\in\mathcal{A}} \alpha_b(m)T_{ba}u_a(m)\end{aligned}$$

for the given solution values of $\mathrm{SE}(\mathcal{D}_s)$, Lemma 4 shows that $[\mu]_{n_+}$ and values $[\phi_a^k(n)]_{a\in\mathcal{A},k\in\mathcal{K}_a(n)}$, $[[W_a]_{n+}]_{a\in\mathcal{A}}, \theta_a(n)$ solves $\mathrm{TOC}(n, \mathcal{D}_\mathcal{A})$ for each node $n = m_-$, and that the solution value of $\mathrm{R}(n, \mathcal{D}_s)$ (namely $\theta^s(n)$) is equal to $\sum_{a\in\mathcal{A}} \theta_a(n)$.

We now recursively apply this argument. For each node n in the penultimate stage, we let $\theta_a^s(n) = \theta_a(n)$, the above computed solution value, so that $\sum_{a\in\mathcal{A}} \theta_a^s(n) = \theta^s(n)$. Further, we define

$$\begin{aligned}Z_a^s(n) &= C_{an}(u_a(n)) - \pi(n)g_{an}(u_a(n)) + \alpha_a(n)\left(x_a(n) - x_a(n_-) - \omega_a(n)\right) \\ &\quad - \sum_{b\in\mathcal{A}} \alpha_b(n)T_{ba}u_a(n) + \sum_{m\in n_+} \mu(m)W_a(m)\end{aligned}$$

for the given solution values of $\mathrm{SE}(\mathcal{D}_s)$ and the previously computed solution values for $W_a(m)$, $m \in n_+$. For each node $q = n_-$, Lemma 4 constructs solution values $[\mu]_{q+}$, $[\phi_a^k(q)]_{a\in\mathcal{A},k\in\mathcal{K}_a(q)}$, $[\theta_a(q), [W_a]_{q+}]_{a\in\mathcal{A}}$ for $\mathrm{TOC}(q, \mathcal{D}_\mathcal{A})$ such that $\theta^s(q) = \sum_{a\in\mathcal{A}} \theta_a(q)$. This argument can then be repeated until we reach the root node of $\mathcal{N}$.

This process generates μ and values of (u, x, α, π) that satisfy (11c), (11d), (11e) and (11f) for every $a \in \mathcal{A}$ since they are solutions to SE($\mathcal{D}_s$). Furthermore, for each $a \in \mathcal{A}$, extracting $\gamma_a^k(n) = \phi_a^k(n)$ and $Z_a(n; u, x, W) = Z_a^s(n)$ from the solutions of TOC($n, \mathcal{D}_{\mathcal{A}}$), it follows from the definition of TOC($n, \mathcal{D}_{\mathcal{A}}$) that (11a), (11b) and (11g) are also satisfied with $\gamma(n) = \gamma_a(n)$. Thus we have constructed solutions for each problem AE$_a(\pi, \alpha, \mu, \mathcal{D}_a)$.

Since for each $n \in \mathcal{N} \setminus \mathcal{L}$, TOC($n, \mathcal{D}_{\mathcal{A}}$) includes the condition that

$$0 \leq \mu(m) \perp -\sum_{a \in \mathcal{A}} W_a(m) \geq 0, \quad m \in n_+,$$

it follows that (14) holds. The final conditions (12) and (13) follow as they are part of the original solution of SE($\mathcal{D}_s$). $\square$