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Dynamic risked equilibrium

Michael Ferris
Computer Sciences Department and Wisconsin Institute for Discovery, University of Wisconsin, Madison, WI 53706, ferris@cs.wisc.edu

Andy Philpott
Electric Power Optimization Centre, University of Auckland, New Zealand, a.philpott@auckland.ac.nz

We study a competitive partial equilibrium in markets where risk-averse agents solve multistage stochastic optimization problems formulated in scenario trees. The agents trade a commodity that is produced from an uncertain supply of resources. Both resources and the commodity can be stored for later consumption. Several examples of a multistage risked equilibrium are outlined, including aspects of battery and hydroelectric storage in electricity markets, distributed ownership of competing technologies relying on shared resources, and aspects of water control and pricing. The agents are assumed to have nested coherent risk measures based on one-step risk measures with polyhedral risk sets that have a non-empty intersection over agents. Agents can trade risk in a complete market of Arrow-Debreu securities. In this setting we define a risk-trading competitive market equilibrium and establish two welfare theorems: competitive equilibrium will yield a social optimum (with a suitably defined social risk measure) when agents have strictly monotone one-step risk measures. Conversely, a social optimum with an appropriately chosen risk measure will yield a risk-trading competitive market equilibrium when all agents have strictly monotone risk measures. The paper also demonstrates versions of these theorems when risk measures are not strictly monotone.

Key words: coherent risk measure, partial equilibrium, perfect competition, welfare theorem

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1. Introduction

In many competitive situations, manufacturers of a product that is sold over several periods use storage to improve their profits. Storage of the finished product enables the manufacturers to store the product in periods when prices are low for later sale in periods when they are high. In practice, prices are uncertain and so the optimal storage policy becomes the solution to a stochastic control problem in which manufacturers seek to maximize expected profits if risk neutral, or some risk-adjusted profit if they are risk averse.

In some cases it is also possible to store the raw materials used in production. An example arises in renewable electricity production in which intermittent generation (wind or photovoltaic energy) can be stored in a battery for later sale. Similarly hydroelectric reservoirs can store energy for later conversion to electricity, or farmers can store pasture (or silage, its harvested form) for later conversion into milk by dairy cows. The process by which the storage is replenished has a random element (e.g. wind, sunlight, catchment inflows, and beneficial weather, in the respective examples we cite). Storage of raw materials enables the producer to maximize their capacity utilization when sale prices are high, while possibly holding back production during low-priced periods.

Our current interest focuses on a situation in which prices of the finished product are determined by an equilibrium of several competing producers, where the total sales of product from the manufacturers equals the demand from consumers in each period. Demand is defined in terms of price by a known decreasing demand function.

The simplest case occurs when the future is known with certainty and producers have convex costs. Then an equilibrium time-varying price can be derived from a Lagrangian decomposition of a social planning model that seeks to maximize the consumer and producer surplus summed over all periods. The Second Welfare Theorem (see e.g. Feldman and Serrano 2006) in this setting is a straightforward consequence of Lagrangian duality theory, and states that the optimal social plan can be interpreted as a perfectly competitive equilibrium at the prices that solve the dual problem. The First Welfare Theorem, stating that any perfectly competitive equilibrium maximizes the consumer and producer surplus in the social plan is also immediate from this duality.

When the parameters of the model are uncertain, but governed by a known stochastic process, the social planning problem becomes a multistage stochastic programming problem. Stages are linked by storage, and decisions on operations must be taken at different (intermediate) times. Multistage stochastic optimization models have been well studied (see e.g. Birge and Louveaux 2011, Shapiro et al. 2014). If all agents act as price takers, and seek to maximize expected operating profits, then the first and second welfare theorems translate naturally into the stochastic setting. When the stochastic supply process (or its approximation) is represented by a scenario tree (see Birge and Louveaux 2011) the Lagrangian theory can be applied to the extensive form of the deterministic equivalent social planning problem and its dual to yield versions of these theorems.

Multistage stochastic optimization becomes more complicated when agents are risk-averse. As in the risk-neutral case, decisions are measurable with respect to the filtration defined by the random parameters. For a scenario tree this means that decisions are made in each node given the history of information accrued in previously visited nodes. When combined with realizations of the random parameters, these decisions lead to a stochastic process of payoffs defined at the nodes of the scenario tree. A risk-averse optimizer then requires a preference relation over these random payoff processes to be able to compare different policies. In a two-stage setting, risk preferences can be approached through a wide variety of models including but not limited to utility theory (Von Neumann and Morgenstern 2007), mean-variance optimization (Markowitz 1952), value-at-risk (Jorion 2000), stochastic dominance (Levy 1992), prospect theory (Tversky and Kahneman 1992), dual utility theory (Yaari 1987), and coherent risk measures (Artzner et al. 1999). For a summary and comparison of these and other approaches to optimzation under risk see e.g. Anderson (2013) or Shapiro et al. (2014).

A theory of extending one-step risk preferences to a multistage setting using conditional risk mappings is described in Shapiro et al. (2014). Conditional risk mappings add current costs to risk-adjusted uncertain future costs expressed as a certainty-equivalent value defined in terms of a single-step coherent risk measure as defined by Artzner et al. (1999). The translation equivariance and monotonicity axioms of coherent risk measures then enable the evaluation of the risk-adjusted cost of a random cost sequence using a recursive formula. When information is revealed over time, and agents make optimal decisions given their current history of observations of random outcomes, this enables the solution of a risk averse dynamic optimization problem using dynamic programming. Mean-variance measures of risk do not extend like this, and following the same approach using utility theory is only possible for a limited set of utility functions (e.g. linear or exponential, Howard and Matheson 1972) that have translation-equivariant certainty-equivalent forms. For example the one-step exponential utility function leads to the class of entropic risk measures (Föllmer and Knispel 2011) which are translation-equivariant and monotone.

When one-step risk measures are translation-equivariant, monotone, convex and positively homogeneous (i.e. coherent), the certainty-equivalent value of future costs at a node of the scenario tree can be expressed using duality theory as the conditional expectation of future costs with respect to a probability measure that is chosen to be the worst in a convex *risk set* of conditional probability distributions (see e.g. Shapiro et al. 2014). There is assumed to be a risk set defined for every node of the scenario tree.

Asset pricing models with risk-averse agents have been widely studied in a two-stage equilibrium setting. The classical economics literature has many examples of this including the capital asset pricing model (CAPM) (Sharpe 1964) and asset-pricing models of Arrow and Debreu (Arrow 1973) in complete markets. The CAPM model has been successfully applied to forward contracting in electricity markets (a key application area) by Bessembinder and Lemmon (2002). However, since it is based on a single-step mean-variance risk measure which is not translation equivariant, it is hard to see how to extend the CAPM model to a multistage setting.

Following Heath and Ku (2004) and Ralph and Smeers (2015), our work is more closely related to the asset-pricing models of Arrow (1973). In this setting, the welfare theorems rely heavily on the concept of *market completeness* achieved though a set of *Arrow-Debreu securities* that span all possible random future outcomes. In a classical two-stage setting, a complete market of Arrow-Debreu securities will ensure by a no-arbitrage argument that every collection of contingent payoffs in stage 2 can be priced at stage 1 using the state prices of the Arrow-Debreu instruments (see e.g. Varian 1987, for an elementary explanation of this principle). The results of Heath and Ku (2004) (assuming finite probability distributions) and Ralph and Smeers (2015) (for continuous distributions) provide asset prices for a complete market of Arrow-Debreu securities in exchange economies of agents with risk-averse coherent risk measures. They show that when the relative interiors of the risk sets of agents intersect, agents will trade Arrow-Debreu securities at equilibrium prices that are a probability measure lying in this intersection. These prices have an interpretation as risk-adjusted probabilities (state prices) that all agents agree on when evaluating their payoffs, and so provide risk-adjusted probabilities for a social planner to evaluate total system welfare. Using common probability distributions yields risk-adjusted welfare in equilibrium that has the same risk-adjusted value in the social plan.

Our goal in this paper is to extend the welfare theorems for partial equilibrium to a multistage setting with risk-averse agents. We assume perfect competition throughout the paper, so agents are assumed to be price takers. The recent paper Philpott et al. (2016) (building on the models of Heath and Ku (2004), Ralph and Smeers (2015)) studies a special case of this problem for multistage electricity markets when some producers operate hydroelectric reservoirs with uncertain inflows. Under an assumption that agents can trade risk using a complete set of Arrow-Debreu securities, Philpott et al. (2016) show that a risk-averse social planning solution with an appropriately chosen risk measure can be interpreted as a competitive equilibrium in which the agents trade risk. This result corresponds to the Second Welfare Theorem.

The result in Philpott et al. (2016) is specific to electricity systems with hydroelectric generators. In this paper we extend this theorem to systems that operate with storage in a more general setting. Like the hydroelectricity case, agents can store raw materials (water) for later electricity production, but we also admit the possibility of storing the commodity (corresponding to e.g. battery storage in the electricity setting). Agents might own and operate a single production or storage facility, or a collection of both production and storage facilities in different locations. The storage facilites could be a linked system of raw material storage sites (such as a river chain of hydro reservoirs) or a system of final product storage sites (e.g. warehouses linked by roads, or batteries linked by electricity transmission lines).

We also add to the theoretical results in Philpott et al. (2016), by giving new proofs of both first and second welfare theorems. Our second welfare theorem (Theorem 4 and Corollary 2) is an extension of Theorem 11 in Philpott et al. (2016) to the more general case. The proof in this general case is arguably simpler. It also illuminates the role of strict monotonicity of risk measures in stochastic risked equilibrium. Theorem 3 and Corollary 1 (which are both new) give our version of the First Welfare Theorem (which is not discussed in Philpott et al. 2016).

Our motivation in studying welfare theorems comes from a desire to understand imperfectly competitive markets. The analogue of the Second Welfare Theorem shows that a social planner could argue that their actions in solving a risk-averse social planning problem replicates what one might expect to see in a perfectly competitive market with a complete market for trading risk. A number of wholesale electricity markets (e.g. Brazil and Chile) operate on this principle, whereby regulated energy prices are computed using an agreed social planning model rather than emerging from a market trading process.

The (newly established) analogue of the First Welfare Theorem shows that if markets are perfectly competitive and endowed with a complete market for trading risk, and agents have sufficiently similar coherent risk measures, then one might expect them to arrive at an equilibrium using policies that maximize risk-adjusted social welfare. In other words, Theorem 3 and Corollary 1 provide a perfectly competitive benchmark against which real markets might be measured. In the real world, where markets are imperfect, the optimal value of a social planning model provides an upper bound on what might be achieved in welfare terms by reducing market imperfections.

It is worth remarking that the welfare results we establish suffer from some restrictive assumptions. Markets are not perfectly competitive, and nearly always incomplete. The assumption of a complete set of priced Arrow-Debreu securities to cover every possible random event is clearly impossible. A number of authors (see e.g. de Maere d'Aertrycke and Smeers 2013, Abada et al.

2017b, Kok et al. 2018) have explored the effect of replacing this assumption in two-stage models with a limited set of traded instruments. In some experiments this restriction can significantly reduce welfare, while in others it has only a minimal effect on welfare losses compared with outcomes from a risk-averse social plan.

Our welfare results provide appropriate competitive benchmarks for risk-averse market participants that can guide market oversight policy. In real electricity markets, observed outcomes diverge from these perfectly competitive benchmarks. If the losses in welfare from market imperfections are large, then it is important to identify market interventions that might reduce them. One might argue that market interventions that seek to complete the market for risk are a necessary first step before one tackles issues of potential market power abuse. At least if the market for risk is made complete, then participants acting as price takers will have incentives to make socially optimal decisions. If they do not do so in a complete market, then this suggests further market interventions to curb market power.

In summary, the contributions of the paper are as follows:

1. We extend the definition of multistage risked equilibrium given in Philpott et al. (2016) to a more general model.
2. We provide a simpler proof of our second welfare theorem as applied to multistage risked equilibrium with risk trading.
3. We state and give a proof of a first welfare theorem (which is new) as applied to multistage risked equilibrium with risk trading.
4. We illuminate the role that strict monotonicity of risk measures plays in multistage risked equilibrium.

The paper is laid out as follows. In the next section we describe the underlying model and its constituent stochastic, dynamic and optimizing agent components, and provide several motivating examples that can be cast into the framework. Section 3 provides a viewpoint of dynamic risk measures, with specific examples, introduces the notion of dynamic consistency, determines optimality conditions for a system optimization problem that incorporates a dynamic risk measure, and links this to a multistage risked equilibrium problem. Section 4 adds the notion of risk trading to these equilibria, and provides the main results, providing counterparts of the first and second welfare theorems in the multistage risked setting. We conclude the paper with a summary of the results and some suggestions for future research. The proofs of the main results of the paper are given in the appendices. We have split these into appendices A, B and C containing results related to coherent risk measures, some technical results linking conditional tree multipliers to unconditional multipliers, and the proofs of the main results, respectively.

2. Models

In our model, random events are defined by a discrete-time stochastic process, with a finite set of events in each stage. Such a process can be modeled using a *scenario tree* with nodes $n \in \mathcal{N}$ and leaves in $\mathcal{L}$. The probability of the event represented by node n is denoted $\phi(n)$. By convention we number the root node $n=0$. The unique predecessor of node $n \neq 0$ is denoted by n_-. We denote the set of children of node $n \in \mathcal{N} \setminus \mathcal{L}$ by n_+, and denote its cardinality by $|n_+|$. The set of predecessors of node n on the path from n to node 0 is denoted $\mathcal{P}(n)$ (so $\mathcal{P}(n) = \{n, n_-, n_{--}, \ldots, 0\}$), where we use the natural definitions for n_{--}. The set of successors of node n is $\mathcal{S}(n) = \{n\} \cup n_+ \cup n_{++} \cup \ldots.\}$ where n_{++} is defined in the obvious way. The depth $\delta(n)$ of node n is the number of nodes on the path to node 0, so $\delta(0) = 1$ and we assume that every leaf node has the same depth, say $\delta_{\mathcal{L}}$. The depth of a node can be interpreted as a time index $t = 1, 2, \ldots, T = \delta_{\mathcal{L}}$. A pictorial representation of a scenario tree with four time stages is given in Figure 1.

We assume that there are a number of agents in the model, indexed by $a \in \mathcal{A}$. At each node n in the scenario tree, the agents observe a realization of random parameters, and seek optimal actions $u_a(n)$ to minimize their current and risk-adjusted future disbenefit. The current disbenefit of agent a in node n consists of a cost $C_{an}(u_a(n))$, and expenses and rewards from trading with other agents. Ignoring these wealth transfers, the current system disbenefit in node n is the total cost $\sum_{a \in \mathcal{A}} C_{an}(u_a(n))$. We assume that each C_{an} is convex. For producer a, C_{an} measures production cost, and for consumer a, C_{an} measures consumption disbenefit.

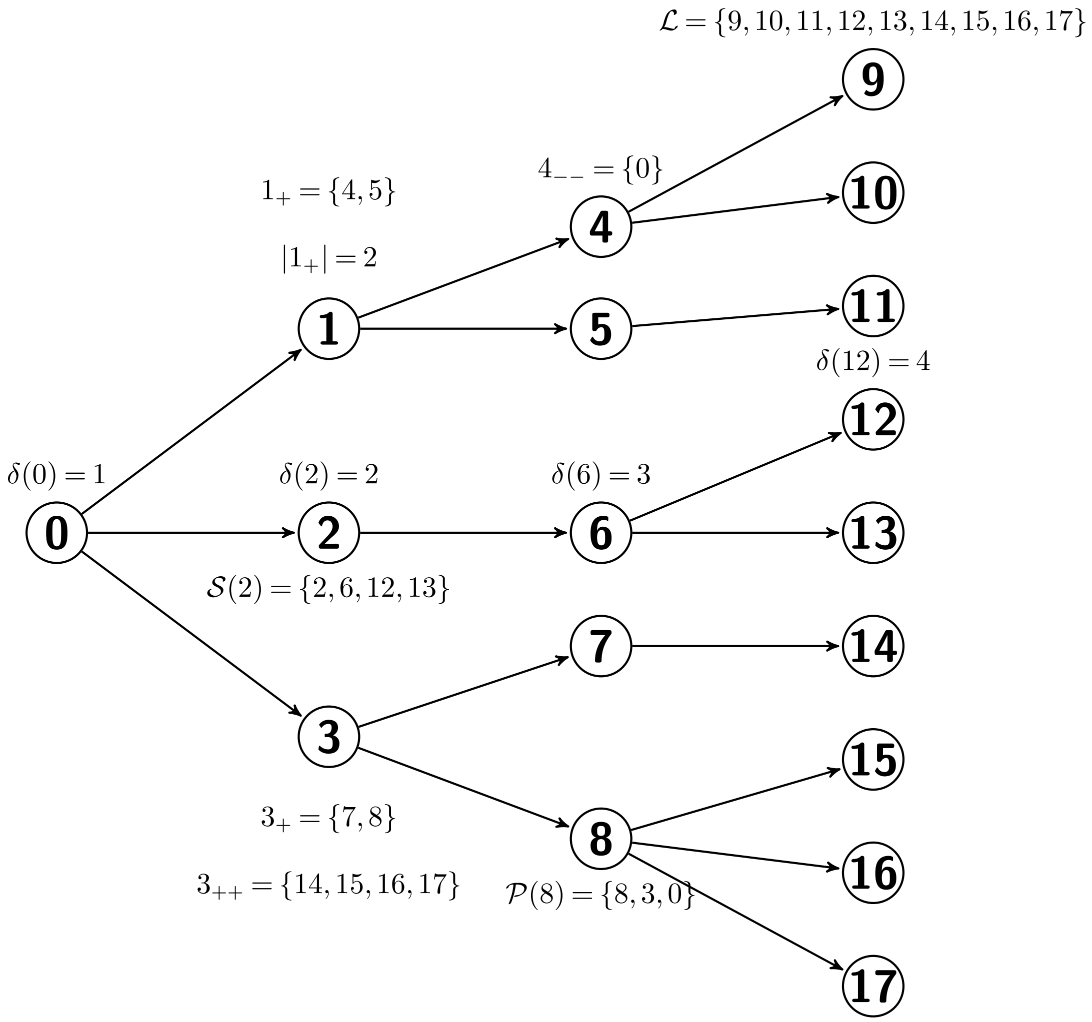


Figure 1 A scenario tree with nodes $\mathcal{N} = \{1,2,\ldots,17\}$, and $T = 4$

Each producing agent a consumes resources at scenario node n that are taken from the storage levels $x_a(n_-)$ and are released at rates defined by $u_a(n)$ yielding total production $g_{an}(u_a(n))$. Note that $x_a(n)$ and $u_a(n)$ are in fact vectors, indexed by locations. The storage is replenished by agent actions (such as charging a battery with purchased electricity) or by (possibly) random supplies (such as inflows or photovoltaic input). Denoting the latter by $\omega_a(n)$ gives a stochastic process defined by

$$x_a(n) \leq x_a(n_-) + \sum_{b \in \mathcal{A}} T_{ab} u_b(n) + \omega_a(n).$$

Note that the matrix T_{ab} in the dynamics allows for a network of connections between the locations of storage devices controlled by different agents, and the inequality allows for free disposal (or spilling) at the storage device location. The dynamics could be expressed a little more generally using a diagonal matrix S_a for gains or losses and making S and T dependent on node as

$$x_a(n) \leq S_a(n)x_a(n_-) + \sum_{b\in\mathcal{A}} T_{ab}(n)u_b(n) + \omega_a(n),$$

but since this does not change the subsequent analysis in any substantive way, we assume $S_a(n) \equiv I$ and $T_{ab}(n) \equiv T_{ab}$ in what follows. The actions u_a (water releases, battery charge or discharge) and storages x_a are constrained to lie in respective sets $\mathcal{U}_a$ and $\mathcal{X}_a$. Finally for each leaf node $n \in \mathcal{L}$, we define $V_{an}(x_a(n))$ to represent the value of residual storage $x_a(n)$ held by agent a at node n.

Given a scenario tree we can now formulate a risk-neutral model that seeks to minimize total expected social disbenefit.

$$\begin{aligned}
\text{SO:} \min_{u,x} \; & \sum_{n\in\mathcal{N}} \phi(n) \sum_{a\in\mathcal{A}} C_{an}(u_a(n)) - \sum_{n\in\mathcal{L}} \phi(n) \sum_{a\in\mathcal{A}} V_{an}(x_a(n)) \\
\text{s.t.} \; & x_a(n) \leq x_a(n_-) + \sum_{b\in\mathcal{A}} T_{ab}u_b(n) + \omega_a(n), \quad n\in\mathcal{N}, \quad a\in\mathcal{A}, && (1) \\
& \sum_{a\in\mathcal{A}} g_{an}(u_a(n)) \geq 0 \quad n\in\mathcal{N}, && (2) \\
& u_a(n) \in \mathcal{U}_a, \quad x_a(n) \in \mathcal{X}_a, \quad n\in\mathcal{N}, \quad a\in\mathcal{A}.
\end{aligned}$$

In order to demonstrate the applicability of our model, we now outline several examples that fit into this framework, and demonstrate the interplay between production units and storage devices, and the agents that own and operate them. This part of the paper can be skipped by the reader without losing the essence of the paper.

EXAMPLE 1. The first example involves a set of locations, each of which contains a single facility. Each facility is controlled by an agent $a \in \mathcal{A}$, where agents are either consumers, producers or storage operators. The facilities operate on a single good (e.g. gas) that can be produced, stored or consumed at the location, and transported from one location to another. The locations are connected via a network, an example of which is given in Figure 2. Note that in this case $x_a(n) \in \mathbb{R}$.

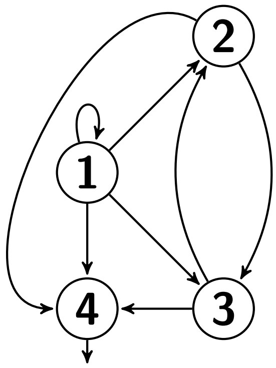


Figure 2 Network example with production at location 1, consumption at location 4 and storage at 2 and 3

Each agent controls the flows along the arcs emanating from the location (facility) that she controls. For example, agent $a = 2$ controls flows in arcs (2,3) and (2,4), so $u_2(n) = [u_{23}(n), u_{24}(n)]$ has two components. There are special arcs (1,1) corresponding to production at location 1 and (4,*) corresponding to consumption at location 4. Thus

$$u_1(n) = [u_{11}(n), u_{12}(n), u_{13}(n), u_{14}(n)],\ u_2(n) = [u_{23}(n), u_{24}(n)],\ u_3(n) = [u_{32}(n), u_{34}(n)],\ u_4(n) = [u_{4*}(n)].$$

The network in Figure 2 is represented by a collection of matrices T_{ab} with rows corresponding to the locations that a controls (in this example just 1), and $|u_b|$ columns. The *net flow into location* a is $\sum_{b\in\mathcal{A}} T_{ab}u_b(n)$, so when $a = 2$ we have $T_{a1} = [0 \quad 1 \quad 0 \quad 0]$, $T_{a2} = [-1 \quad -1]$, $T_{a3} = [1 \quad 0]$, and $T_{a4} = [0]$. While the representation of the network using these matrices is not as simple as it could be, it allows for generalizations in network descriptions and for ownership in formulations that we outline briefly below. Constraints (such as capacities or operational considerations) on flows and storage are captured by the sets $\mathcal{U}_a$ and $\mathcal{X}_a$.

In this example, the functions $g_{an}(u_k(n))$ are $u_k(n)$ if arc k emanates from location a and $-u_k(n)$ otherwise and thus they are separable over $u_a(n)$, $a \in \mathcal{A}$. The cost functions are production cost, consumption disbenefit, and 0 for storage devices, while $V_{an}(x_a(n))$ captures the value of residual storage $x_a(n)$ at any leaf node of the scenario tree. Situations that are covered by this type of formulation include a production/distribution network where storage devices are warehouses and arcs represent transportation links, and also the situation of a distributed system of batteries that could be used to store energy generated by fossil fuel or renewable energy production systems.

Example 2. The second example generalizes the previous situation to allow agents to be firms controlling a collection of production, storage and demand facilities in different locations. Thus $a \in \mathcal{A}$ now indexes firms, and $x_a(n) \in \mathbb{R}^{m(a)}$ is a vector of storage amounts at the locations controlled by firm a. The vector $u_a(n)$ again represents the controls emanating from locations that are controlled by a. Data representing costs, capacities, production and terminal values are suitably extended from the previous setting, but note that T_{ab} has $m(a)$ rows and $|u_b|$ columns. If we adapt the above example so that locations 3 and 4 are owned by ζ, then $u_\zeta(n) = [u_{34}(n), u_{4*}(n)]$ and

$$T_{\zeta 1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, T_{\zeta 2} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and } T_{\zeta\zeta} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

This example allows a modeler to look at the effects of plant ownership within a competitive equilibrium setting.

Example 3. The above examples do not involve raw materials. The third example extends the framework to differentiate between raw materials (think water) and a finished good (think electricity). Assume for simplicity at this time that we do not have a distribution/storage network for the finished good but simply have given demand for that good at a collection of locations, and ability to produce that good from raw materials at those locations. We can think of raw materials as being water flowing along a river network, or fuel stored in stockpile locations. In the first setting, the river network is modeled by a collection of trees and locations correspond to hydro generation facilities (dams). Water (raw material) flows through the tree (to a root representing exit from the system) and can be used by a hydro generator situated at a location to produce electricity, but that water continues to flow through the river network to the next reservoir where it could be used for additional generation. This fits naturally into the formulation above where $u_a(n)$ are the water releases on a given arc and $x_a(n)$ are the reservoir storages, except there are no production facilities (i.e. u_{11} disappears since water is only generated randomly using $\omega_1(n)$). The function

$g_{an}(u_a(n))$ encodes the production of electricity at the turbines to satisfy demand at that location. Water is not destroyed in this production process and continues to flow through the river network. Spillage is naturally handled by the inequality in the dynamics.

However, if instead we think of the raw resource as being fuel in a given stockpile, then the network (i.e. collection of T_{ab} matrices) models a transportation network for that fuel. Production locations correspond now to production of fuel. At each (electricity) demand location we add consumption arcs (similar to the arc (4,*) in Figure 2) that consumes the fuel to produce electricity. On these arcs k the electricity production function is $g_{an}(u_k(n))$, and the cost function represents the cost of using the fuel in electricity production.

Example 4. The final example extends the above to capture both a transportation network for the raw materials, and a distribution network for the final good. We assume for simplicity of exposition that the distribution network is acyclic. Consider the fuel and electricity example, and construct a network (i.e. a collection of T_{ab} matrices) that is the union of the fuel transportation network, and the electricity distribution network, combined with additional arcs that join a fuel node at a given location to an electricity production node at that location. Thus $u_k(n)$ represents flow of fuel on arc k of the transportation network, or flow of electricity along the distribution/storage network or the production of electricity from fuel on the arcs that join these two networks together. Flow along the (network joining) arcs represent the creation of electricity from fuel (in a linear fashion or using the slight generalization of $g_{an}(u_k(n))$) that can be incorporated as a loss or gain multiplier along that arc in the definition of T_{ab}. Thus, $\{T_{ab}\}$ contains the information of both networks, augmented with new generalized arcs to represent the conversion of raw quantities into finished goods. The vector $x_a(n)$ has components that correspond to the amount of raw material stored at a location (operated by a) in the fuel transportation network, or the amount of electricity stored at a location in the electricity distribution network. The flow around the electricity distribution network satisfies load-flow constraints that represent Kirchhoff's Laws. The consumption arcs in the fuel network of example 3 and the production arcs in the electricity distribution network of

Example 1 are replaced by these conversion arcs linking the two networks. The remainder of the cost and generation functions are unchanged.

The situation for hydroelectric generation is a little more involved since water is not consumed as it generates electricity. To model this, we consider the union of the river network and the electricity distribution network, augmented by arcs that join a generation location on the water network to a bus on the distribution network. The river network is effectively modeled as in Example 3 so there is only one water flow emanating from each location. However, flow out of a hydro production location generates water flow into the downstream river location and an amount of electricity at the bus (determined by the production function). We simply change the definition of the T_{ab} matrices corresponding to the hydro plants to ensure both flow of water and production of electricity.

If in Example 1 from Figure 2, water flows from location 1 to location 3 and then out of the system, and we add a hydro power station at location 3, then we update the controls u using

$$u_1(n) = [u_1^w(n), u_{12}^e(n), u_{13}^e(n), u_{14}^e(n)], \qquad u_2(n) = [u_{23}^e(n), u_{24}^e(n)],$$
$$u_3(n) = [u_3^w(n), u_{32}^e(n), u_{34}^e(n)], \qquad u_4(n) = [u_{4*}^e(n)].$$

where the superscripts correspond to arcs in the water network and the electricity network respectively. Agent $a = 1$ now controls both the water network at location 1 (row 1), and the electricity network at location 1 (row 2), and similarly for agent $a = 3$. The resulting collection of T_{ab} (net inflow into a) matrices is:

$$T_{11} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & -1 \end{bmatrix}, \quad T_{12} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad T_{13} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_{14} = \begin{bmatrix} 0 \end{bmatrix},$$
$$T_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad T_{22} = \begin{bmatrix} -1 & -1 \end{bmatrix}, \quad T_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad T_{24} = \begin{bmatrix} 0 \end{bmatrix},$$
$$T_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad T_{32} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad T_{33} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & -1 \end{bmatrix}, \quad T_{34} = \begin{bmatrix} 0 \end{bmatrix},$$
$$T_{41} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{42} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad T_{43} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_{44} = \begin{bmatrix} -1 \end{bmatrix}.$$

Pumped storage is an extension of this model that incorporates additional arcs from the electricity distribution network back into the river (or raw material) network. The essential idea is that energy can be converted back into a raw resource (at a given location) with pumping efficiency modeled via a multiplier factor on the additional arc.

3. Dynamic risk measures

The agents in our models are risk averse when contemplating a sequence of decisions that have random future consequences. To model this behavior we consider a single-stage model with finite sample space indexed by $m \in \mathcal{M}$. Each decision maker faced with a random disbenefit $Z(m)$, $m \in \mathcal{M}$, measures its risk using a *coherent risk measure* ρ as defined axiomatically by Artzner et al. (1999). Thus $\rho(Z)$ is a real number representing the risk-adjusted disbenefit of Z.

It is well-known that any coherent risk measure $\rho(Z)$ has a dual representation expressing it as

$$\rho(Z) = \sup_{\nu \in \mathcal{D}} \mathbb{E}_\nu[Z],$$

where $\mathcal{D}$ is a convex subset of probability measures on $\mathcal{M}$ (see e.g. Artzner et al. 1999, Heath and Ku 2004). $\mathcal{D}$ is called the *risk set* of the coherent risk measure. We use the notation $[p]_\mathcal{M}$ to denote any vector $\{p(m), m \in \mathcal{M}\}$. So any probability measure $\nu \in \mathcal{D}$ can be written $[\nu]_\mathcal{M}$, where $\nu(m)$ defines the probability of event m. The dual representation using a risk set plays an important role in the analysis we carry out in this paper. We refer to the case where the risk set is a singleton as *risk neutral*.

A number of examples of coherent risk measures are discussed in Shapiro et al. (2014) including worst-case and average value at risk (also known as conditional value at risk). Given a random disbenefit Z the *average value at risk* of Z at level $1-\alpha$ is defined as

$$\mathrm{AVaR}_{1-\alpha}(Z) = \inf_t \{t + \frac{1}{\alpha}\mathbb{E}[(Z-t)_+]\}. \tag{3}$$

Given a finite sample space indexed by $m \in \mathcal{M}$, with $\phi(m)$ the probability of m, $\mathrm{AVaR}_{1-\alpha}(Z)$ has a polyhedral risk set

$$\mathcal{D} = \{\nu : \sum_{m \in \mathcal{M}} \nu(m) = 1, \quad 0 \le \alpha\nu(m) \le \phi(m), \quad m \in \mathcal{M}\},$$

that can be derived by writing the dual of the optimization problem in (3).

In the rest of this paper we assume that risk sets are polyhedrons with known extreme points $\{[p^k]_{\mathcal{M}}, k \in \mathcal{K}\}$, where $\mathcal{K}$ is a finite index set. This condition is not essential to the theory we derive, but it simplifies the analysis without losing much generality. There are risk sets that are not polyhedral. For example, the *good-deal* risk measure originated in the work of Cochrane and Saa-Requejo (2000) and has been widely applied in capacity planning equilibrium models (see e.g. Abada et al. 2017b).

Assuming a polyhedral risk set we write

$$\sup_{\nu\in\mathcal{D}} \mathbb{E}_{\nu}[Z] = \sup_{\nu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \nu(m)Z(m) = \max_{k\in\mathcal{K}} \sum_{m\in\mathcal{M}} p^k(m)Z(m),$$

since the maximum of a linear function over $\mathcal{D}$ is attained at an extreme point. By a standard dualization, this gives

$$\sup_{\nu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \nu(m)Z(m) = \begin{cases} \min\ \theta \\ \text{s.t. } \ \theta \geq \sum_{m\in\mathcal{M}} p^k(m)Z(m),\ k\in\mathcal{K}. \end{cases}$$

LEMMA 1. *Suppose $\mathcal{D}$ is a polyhedral risk set with extreme points $\{[p^k]_{\mathcal{M}},\ k\in\mathcal{K}\}$ and $Z(m)$, $m\in\mathcal{M}$ is given. Then*

$$\theta = \sup_{\nu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \nu(m)Z(m)$$

if and only if there is some γ^k, $k\in\mathcal{K}$, with

$$\sum_{k\in\mathcal{K}} \gamma^k = 1$$
$$0 \leq \gamma^k \perp \theta - \sum_{m\in\mathcal{M}} p^k(m)Z(m) \geq 0, \quad k\in\mathcal{K}.$$

Furthermore, $\bar{\nu}$, defined by $\bar{\nu}(m) = \sum_{k\in\mathcal{K}} \gamma^k p^k(m)$, is in $\mathcal{D}$ and attains the supremum.

By definition, a coherent risk measure is *monotone*. This means that

$$Z_a \geq Z_b \Rightarrow \rho(Z_a) \geq \rho(Z_b).$$

A stronger condition is *strict monotonicity*. This requires that

$$Z_a \geq Z_b \text{ and } Z_a \neq Z_b \Rightarrow \rho(Z_a) > \rho(Z_b).$$

If strictly monotone coherent risk measures have polyhedral risk sets then these lie strictly inside the positive orthant.

LEMMA 2. *Suppose ρ is a coherent risk measure with a polyhedral risk set $\mathcal{D}$. Then $\mathcal{D} \subset int(\mathbb{R}_+^{|\mathcal{M}|})$ if and only if ρ is strictly monotone.*

We incorporate the risk measures discussed above into a multistage setting in which agents make production and consumption decisions over several time stages to minimize risk-adjusted expected disbenefit.

For a multistage decision problem, we require a dynamic version of risk. The concept of coherent dynamic risk measures was introduced in Riedel (2004) and is described for general Markov decision problems in Ruszczyński (2010). Formally one defines a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\emptyset, \Omega\} = \mathcal{F}_1 \subset \mathcal{F}_2 \ldots \subset \mathcal{F}_T \subset \mathcal{F}$ of σ-fields where all data in node 0 is assumed to be deterministic and decisions at time t are $\mathcal{F}_t$-measurable random variables (see Ruszczyński 2010). Working with finite probability spaces defined by a scenario tree simplifies this description.

Given a tree defined by $\mathcal{N}$, suppose the random sequence of actions $\{u(n), n \in \mathcal{N}\}$ results in a random sequence of disbenefits $\{Z(n), n \in \mathcal{N}\}$. We seek to measure the risk of this disbenefit sequence when viewed by a decision maker at node 0. At node n the decision maker is endowed with a one-step risk set $\mathcal{D}(n)$ that measures the risk of random risk-adjusted costs accounted for in $m \in n_+$. Thus elements of $\mathcal{D}(n)$ are finite probability distributions of the form $[p]_{n_+}$.

The risk-adjusted disbenefit $\theta(n)$ of all random future outcomes at node $n \in \mathcal{N} \setminus \mathcal{L}$ can be defined recursively. We denote the future risk-adjusted disbenefit in each leaf node $n \in \mathcal{L}$ by $\bar{\theta}(n)$. Then $\theta(n)$ is defined recursively to be

$$\theta(n) = \begin{cases} \bar{\theta}(n), & n \in \mathcal{L}, \\ \sup_{\nu \in \mathcal{D}(n)} \sum_{m \in n_+} \nu(m)(Z(m) + \theta(m)), & n \in \mathcal{N} \setminus \mathcal{L}. \end{cases} \tag{4}$$

When viewed in node n, $\theta(n)$ can be interpreted to be the fair one-time charge we would be willing to incur instead of the sequence of random future costs $Z(m)$ incurred in all successor nodes of n. In other words the measure $\theta(n)$ is a *certainty equivalent cost* or *risk-adjusted expected cost* of all the future costs in the subtree rooted at node n.

Since we assume for $n\in\mathcal{N}\setminus\mathcal{L}$ that $\mathcal{D}(n)$ is a polyhedron with extreme points $\{[p^k]_{n_+}, k\in\mathcal{K}(n)\}$, the recursive structure defined by (4) can then be simplified to

$$\sup_{\nu\in\mathcal{D}(n)}\sum_{m\in n_+}\nu(m)(Z(m)+\theta(m))$$
$$=\begin{cases}\min\ \theta \\ \text{s.t.}\ \ \theta\geq\sum\limits_{m\in n_+}p^k(m)\left(Z(m)+\theta(m)\right),\ k\in\mathcal{K}(n).\end{cases}\tag{5}$$

We now recall the system optimization problem SO, and modify this by adding variables θ so that it minimizes *risk-adjusted* system disbenefit using a dynamic risk measure defined using the extreme points $[p^k]_{n_+}$ of the system risk set $\mathcal{D}(n)$. The risk-averse system optimization problem is then formulated as follows.

$$\begin{aligned}
\text{SO}(\mathcal{D}):\ \min_{u,x,\theta}\ & \sum_{a\in\mathcal{A}}C_{a0}(u_a(0))+\theta(0) \\
\text{s.t.}\ \theta(n)\geq & \sum_{m\in n_+}p^k(m)\left(\sum_{a\in\mathcal{A}}C_{am}(u_a(m))+\theta(m)\right),\quad [\lambda^k(n)] \\
& \qquad k\in\mathcal{K}(n),\ n\in\mathcal{N}\setminus\mathcal{L}, && (6)\\
& x_a(n)\leq x_a(n_-)+\sum_{b\in\mathcal{A}}T_{ab}u_b(n)+\omega_a(n),\quad a\in\mathcal{A}, n\in\mathcal{N},\quad [\tilde{\alpha}_a(n)] && (7)\\
& \sum_{a\in\mathcal{A}}g_{an}(u_a(n))\geq 0\quad n\in\mathcal{N},\quad [\tilde{\pi}(n)] && (8)\\
& \theta(n)=-\sum_{a\in\mathcal{A}}V_{an}(x_a(n)),\quad n\in\mathcal{L}, \\
& u_a(n)\in\mathcal{U}_a,\quad x_a(n)\in\mathcal{X}_a,\quad n\in\mathcal{N},\quad a\in\mathcal{A}.
\end{aligned}$$

The terms in square brackets are the Lagrange multipliers for the constraints. These are related to prices in competitive equilibrium as we show in Theorem 1 below.

3.1. Dynamic consistency

The solution of SO($\mathcal{D}$) gives a policy of decisions $\{\bar{u}_a(n), n \in \mathcal{N}\}$ and resulting stocks $\{\bar{x}_a(n), n \in \mathcal{N}\}$. We digress briefly here to discuss the notion of dynamic consistency as applied to such a solution. Recall the set of successors $\mathcal{S}(n)$ of node n is the maximal subtree in $\mathcal{N}$ with root node n. Following Carpentier et al. (2012) we make the following definition.

Definition 1. An optimal solution $\{\bar{u}_a(n), \bar{x}_a(n), n \in \mathcal{N}\}$ to SO($\mathcal{D}$) is called *dynamically consistent* if for every $\bar{n} \in \mathcal{N}$, $\{\bar{u}_a(n), \bar{x}_a(n), n \in \mathcal{S}(\bar{n})\}$ is an optimal solution to SO($\mathcal{D}$) formulated in $\mathcal{S}(\bar{n})$ where node 0 is replaced by node $\bar{n}$ and we choose initial endowments $x_a(\bar{n}_-) = \bar{x}_a(\bar{n}_-)$.

Dynamic consistency of solutions to SO($\mathcal{D}$) is guaranteed under the following assumption.

Assumption 1. *For every $n \in \mathcal{N} \setminus \mathcal{L}$, $\mathcal{D}(n) \subset$ int$(\mathbb{R}_+^{|n_+|})$.*

Under this assumption, Lemma 2 ensures that one-step risk measures are strictly monotone. As shown in Shapiro (2017), this implies that optimal solutions to the tree problem with risk sets $\mathcal{D}(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$ correspond to dynamic programming policies that compute optimal solutions by backwards recursion. In other words the optimal policy for SO($\mathcal{D}$) will be dynamically consistent.

To see that Assumption 1 is necessary, observe that if it does not hold then it is possible for

$$\bar{\nu} \in \arg\max_{\nu \in \mathcal{D}(n)} \sum_{m \in n_+} \nu(m)(Z(m) + \theta(m)) \tag{9}$$

to have $\bar{\nu}(\bar{m}) = 0$ for some $\bar{m}$. If so, then evaluating the risk at node 0 will ignore all disbenefits in the subtree of nodes in $\mathcal{N}$ rooted at $\bar{m}$. Decisions in these nodes will not affect the overall risk-adjusted disbenefit in node 0 unless they change nodal disbenefits enough to change $\bar{\nu}$ in (9). If these decisions are suboptimal given that the decision maker is in the state of the world defined by $\bar{m}$, then the policy defined by all the decisions is not dynamically consistent.

Of course it is true that one can construct a dynamically consistent policy (by dynamic programming) even though the decision maker assigns zero probability to events in some nodes. We will show that such policies correspond to optimality conditions defined over the whole scenario tree. These will be sufficient but may not be necessary conditions for an optimal solution to an instance of SO($\mathcal{D}$) that violates Assumption 1.

3.2. Optimality conditions

We now define optimality conditions for the problem SO($\mathcal{D}$). Recall for any set $\mathcal{X}$ we define the *normal cone* at $\bar{x}$ to be

$$N_{\mathcal{X}}(\bar{x}) = \{d : d^{\top}(x - \bar{x}) \leq 0 \text{ for all } x \in \mathcal{X}\},$$

and recall that $\bar{x}$ minimizes a convex function $f(x)$ over convex set $\mathcal{X}$ if and only if

$$0 \in \nabla_x f(\bar{x}) + N_{\mathcal{X}}(\bar{x}).$$

When the set $\mathcal{X}$ has a particular representation in terms of nonlinear functions, these optimality conditions have a specific form (often termed the KKT conditions) provided that a constraint qualification holds. To facilitate use of these conditions within our proofs, we will assume that the following condition is satisfied throughout this paper.

Assumption 2. *The functions C_{an} are convex, while g_{an} and V_{an} are concave, so SO($\mathcal{D}$) is a convex optimization problem (eliminating $\theta(n), n \in \mathcal{L}$ if necessary). The nonlinear constraints in SO($\mathcal{D}$) satisfy a constraint qualification that ensures that SO($\mathcal{D}$) is equivalent to its KKT conditions.*

The weakest condition Gould and Tolle (1971) that ensures the necessity of the KKT conditions is referred to as the Guinard constraint qualification, and the stronger Slater constraint qualification is often used since it is easier to verify.

Since SO($\mathcal{D}$) is a convex optimization problem and the constraint qualification Assumption 2 holds, Assumption 1 implies that the following set of conditions SE($\mathcal{D}$) are necessary and sufficient for optimality in SO($\mathcal{D}$).

SE($\mathcal{D}$):

$$0 = 1 - \sum_{k \in \mathcal{K}(n)} \gamma^k(n), \quad n \in \mathcal{N} \setminus \mathcal{L}$$

$$0 \leq \gamma^k(n) \perp \theta(n) - \sum_{m \in n_+} p^k(m) \left(\sum_{a \in \mathcal{A}} C_{am}(u_a(m)) + \theta(m) \right) \geq 0, \quad k \in \mathcal{K}(n), n \in \mathcal{N} \setminus \mathcal{L}$$

$$\theta(n) = -\sum_{a\in\mathcal{A}} V_{an}(x_a(n)), \quad n\in\mathcal{L}$$

$$0 \in \nabla_{u_a(n)}\left[C_{an}(u_a(n)) - \pi(n)g_{an}(u_a(n)) - \sum_{b\in\mathcal{A}}\alpha_b(n)T_{ba}u_a(n)\right] + N_{\mathcal{U}_a}(u_a(n)), \quad a\in\mathcal{A}, n\in\mathcal{N}$$

$$0 \in \alpha_a(n) - \sum_{m\in n_+}\sum_{k\in\mathcal{K}(n)}\gamma^k(n)p^k(m)\alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad a\in\mathcal{A}, n\in\mathcal{N}\setminus\mathcal{L}$$

$$0 \in \alpha_a(n) - \nabla_{x_a(n)}V_{an}(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad a\in\mathcal{A}, n\in\mathcal{L}$$

$$0 \le \alpha_a(n) \perp -x_a(n) + x_a(n_-) + \sum_{b\in\mathcal{A}} T_{ab}u_b(n) + \omega_a(n) \ge 0, \quad a\in\mathcal{A}, n\in\mathcal{N}$$

$$0 \le \pi(n) \perp \sum_{a\in\mathcal{A}} g_{an}(u_a(n)) \ge 0, \quad n\in\mathcal{N}.$$

The variables $\pi(n)$ and $\alpha(n)$ are prices for energy and raw materials respectively. They are related to the Lagrange multipliers $\tilde{\pi}(n)$ and $\tilde{\alpha}(n)$ for equations (7) and (8) through division by a factor $\sigma(n)$. In the risk-neutral case $\sigma(n)$ is simply $\phi(n)$, the probability of being in node n. In the risk-averse case $\sigma(n)$ is defined by the *risk-adjusted probabilities*

$$\sigma(n) = \begin{cases} 1, & n = 0, \\ \sum_{j\in\mathcal{K}(n_-)} \lambda^j(n_-)p^j(n), & n\in\mathcal{N}\setminus\{0\}. \end{cases} \tag{10}$$

THEOREM 1. *(i) Any solution to SE($\mathcal{D}$) provides (u,x,θ) that solves SO($\mathcal{D}$) and satisfies*

$$\begin{aligned}\theta(n) &= \max_{\nu\in\mathcal{D}(n)} \sum_{m\in n_+}\nu(m)\left(\sum_{a\in\mathcal{A}} C_{am}(u_a(m)) + \theta(m)\right) \\ &= \sum_{m\in n_+}\bar{\nu}(m)\left(\sum_{a\in\mathcal{A}} C_{am}(u_a(m)) + \theta(m)\right),\end{aligned}$$

where $\bar{\nu}(m) = \sum_{k\in\mathcal{K}(n)}\gamma^k(n)p^k(m)$.

(ii) Under Assumption 1 for any solution (u,x,θ) of SO($\mathcal{D}$), there exist γ, π, α such that $(u,x,\theta,\gamma,\pi,\alpha)$ satisfy SE($\mathcal{D}$), and

$$\gamma^k(n) = \lambda^k(n)/\sigma(n),\ \alpha_a(n) = \tilde{\alpha}_a(n)/\sigma(n),\ \pi(n) = \tilde{\pi}(n)/\sigma(n) \tag{11}$$

defined using the dual variables of SO($\mathcal{D}$) and σ defined by (10).

3.3. Equilibrium

Given a set of agents $a \in \mathcal{A}$ we can define a risk-averse competitive equilibrium as follows. We first define an agent optimization problem that minimizes their risk-adjusted disbenefit at given prices.

$$\begin{aligned}
\mathrm{P}_a(\pi, \alpha, \mathcal{D}_a): \min_{u_a, x_a, \theta_a} \; & Z_a(0; u, x) + \theta_a(0) \\
\text{s.t. } & \theta_a(n) \geq \sum_{m \in n_+} p_a^k(m)(Z_a(m; u, x) + \theta_a(m)), \\
& \qquad k \in K_a(n), \; n \in \mathcal{N} \setminus \mathcal{L}, \\
& \theta_a(n) = -V_{an}(x_a(n)), \quad n \in \mathcal{L}, \\
& u_a(n) \in \mathcal{U}_a, \; x_a(n) \in \mathcal{X}_a, \quad n \in \mathcal{N},
\end{aligned}$$

where we use the shorthand notation

$$\begin{aligned}
Z_a(n; u, x) = & \; C_{an}(u_a(n)) - \pi(n) g_{an}(u_a(n)) + \alpha_a(n)\left(x_a(n) - x_a(n_-) - \omega_a(n)\right) \\
& - \sum_{b \in \mathcal{A}} \alpha_b(n) T_{ba} u_a(n), \quad n \in \mathcal{N}.
\end{aligned} \tag{12}$$

Here $\pi(n)$ is the commodity price at node n and $\alpha_a(n)$ is the resource price at node n at a's location. Recall that agents are assumed throughout this paper to behave as price takers, so prices will be determined in equilibrium by market clearing rather than anticipated by agents behaving as Cournot players. This means that $\alpha_a(n)$ is the price paid by *every* agent for resource at a's location, rather than individual agent prices that could emerge from a generalized Nash equilibrium in the Cournot setting.

For a producer, the first two terms in (12) are the production cost minus sales revenue. The third term is the cost incurred in node n in retaining extra resources for later use, and the final term $\sum_{b \in \mathcal{A}} \alpha_b(n) T_{ba} u_a(n)$ is the payment received from downstream beneficiaries for releases of resources. Observe that they pay at price $\alpha_b(n)$ that will typically be less than $\alpha_a(n)$ as agent a has extracted value from the resource en route to b. In the hydroelectric setting $\alpha_b(n)$ is a payment received for released water from downstream reservoirs. In the case where a single agent owns both reservoirs (i.e. a and b identify the same agent) the payment can be viewed as the loss in risk-adjusted expected water value incurred by the release.

The formulation $\mathrm{P}_a(\pi,\alpha,\mathcal{D}_a)$ exploits a particular form (12) of Z_a that arises in our storage applications. Alternative forms of SO and $\mathrm{P}_a(\pi,\alpha,\mathcal{D}_a)$ are possible, where Z_a represents, for example, the costs of capacity expansion and operation. In a capacity expansion model the variables $x_a(n)$ denote capacities, expansion is $y_a(n)$, and operation is $u_a(n)$. This gives constraints in SO of the form

$$x_a(n) = x_a(n_-) + y_a(n), \quad a \in \mathcal{A}, n \in \mathcal{N},$$

$$u_a(n) \le x_a(n_-), \quad a \in \mathcal{A}, n \in \mathcal{N}.$$

Optimality conditions involving these constraints involve Lagrange multipliers that apply to each agent individually, so the optimality conditions are arguably simpler for agent a than those in the constraints indexed by a in $\mathrm{SE}(\mathcal{D})$. These Lagrange multipliers (scaled by $\sigma(n)$) then feature in an expression for Z_a corresponding to (12).

DEFINITION 2. A *multistage risked equilibrium* $\mathrm{RE}(\mathcal{D}_\mathcal{A})$ is a stochastic process of prices $\{\pi(n), n \in \mathcal{N}\}$, $\{\alpha_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, and a corresponding collection of actions $\{(u_a(n), x_a(n), \theta_a(n)), a \in \mathcal{A}, n \in \mathcal{N}\}$, with the property that (u_a, x_a, θ_a) solves the problem $\mathrm{P}_a(\pi,\alpha,\mathcal{D}_a)$ and

$$\begin{aligned} 0 \le \pi(n) \quad &\perp \quad \sum_{a\in\mathcal{A}} g_{an}(u_a(n)) \ge 0, \quad n \in \mathcal{N}, \\ 0 \le \alpha_a(n) \quad &\perp \quad -x_a(n) + x_a(n_-) + \sum_{b\in\mathcal{A}} T_{ab} u_b(n) + \omega_a(n) \ge 0, \quad a \in \mathcal{A}, n \in \mathcal{N}. \end{aligned}$$

In a multistage risked equilibrium, the system clearing agent announces a set of prices $\{\pi(n), n \in \mathcal{N}\}$, $\{\alpha_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, and each agent chooses a sequence of actions adapted to the filtration defined by the scenario tree that minimizes their risk-adjusted disbenefit with these prices as viewed in node 0 of the tree. Since agents are price takers they do not anticipate possible responses of rival agents in later periods when making decisions now, although these responses have an implicit effect through future market clearing prices.

The existence of multistage risked equilibrium depends on the formulation of each problem $\mathrm{P}_a(\pi, \alpha, \mathcal{D}_a)$. Existence proofs for particular formulations typically invoke general results (see e.g. Rosen 1965, Arrow and Debreu 1954) based on fixed-point theorems that require bounds on the set of actions and convex disbenefit functions. Existence results for risked equilibrium models for capacity expansion can be found in de Maere d'Aertrycke and Smeers (2013), Abada et al. (2017b), Kok et al. (2018), Ralph and Smeers (2015).

Uniqueness of multistage risked equilibrium is not easy to demonstrate. An approach for two-stage investment problems based on degree theory is proposed by Abada et al. (2017a), but the application of this technique in a practical setting is very restricted. In our setting, strict monotonicity of risk measures appears to be a necessary condition for uniqueness as shown by the following example.

Example 5. Consider a model with one producer and one consumer and two periods. The consumer has a solar panel and battery storage. The producer has a diesel generator and a battery. Solar generation ξ_1 is zero in period 1 and has two possible outcomes in period 2, namely $\xi_2(\omega_1) = 1$ and $\xi_2(\omega_2) = 3$. Suppose the producer and consumer are endowed with the worst-case risk measure (denoted $\mathbb{W}$). This is not strictly monotone.

The cost of generating u in the diesel generator is $\frac{1}{2}u^2$ in period 1 and u^2 in period 2. The cost of storing x in the generator battery is 0, but it costs $\frac{v^2}{8}$ for the consumer to store v. The consumer uses power only in period 2 where she has utility for consumption z_2 defined by

$$U(z_2(\omega)) = 18 z_2(\omega) - z_2(\omega)^2 .$$

In a risked equilibrium we choose prices π_1, $\pi_2(\omega_1)$ and $\pi_2(\omega_2)$ so that markets clear in each state of the world and the producer and consumer each minimize their worst-case disbenefit over the two scenarios. Using variables θ and φ to identify the worst outcome, these problems can be formulated as

$$
\begin{aligned}
\text{P: } & \min\ \tfrac{1}{2}u_1^2 - \pi_1(u_1 - x) + \theta \\
& \text{s.t. } \theta \geq -\pi_2(\omega)(u_2(\omega) + x) + u_2(\omega)^2,\ \omega = \omega_1, \omega_2 \\
& \quad x \leq u_1, \\
& \quad u_1 \geq 0, x \geq 0, u_2(\omega) \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
\text{C: } & \min\ \pi_1 v_1 + \tfrac{v_1^2}{8} + \varphi \\
& \text{s.t. } \varphi \geq \pi_2(\omega) z_2(\omega) - 18(z_2(\omega) + v_1 + \xi_2(\omega)) + (z_2(\omega) + v_1 + \xi_2(\omega))^2 \\
& \quad v_1 \geq 0, z_2(\omega) \geq 0
\end{aligned}
$$

with market clearing conditions

$$
\begin{aligned}
& 0 \leq u_1 - x - v_1 \perp \pi_1 \geq 0, \\
& 0 \leq u_2(\omega) + x - z_2(\omega) \perp \pi_2(\omega) \geq 0.
\end{aligned}
$$

This system has the following multistage risked equilibria:

1. $\pi_1 = 4,\ \pi_2(\omega_1) = 4,\ \pi_2(\omega_2) = 5$. An optimal solution to P (where ω_1 yields the worst case outcome) is

   $$u_1 = 4, \quad x = 4, \quad u_2(\omega_1) = 2, \quad u_2(\omega_2) = 0,\ \theta = -20$$

   An optimal solution to C (where ω_1 also yields the worst case outcome) is

   $$v_1 = 0, \qquad z_2(\omega_1) = 6, \quad z_2(\omega_2) = 4,\ \varphi = -53$$

   The market-clearing conditions are easily verified.

2. $\pi_1 = 4,\ \pi_2(\omega_1) = 5,\ \pi_2(\omega_2) = 3$. An optimal solution to P (where ω_2 is worst case) is

   $$u_1 = 4, \quad x = 0, \quad u_2(\omega_1) = 1.5, \quad u_2(\omega_2) = 1.5,\ \theta = -2.25$$

An optimal solution to C (where ω_1 is worst case) is

$$v_1 = 4, \quad z_2(\omega_1) = 1.5, \quad z_2(\omega_2) = 1.5, \varphi = -67.25.$$

The market-clearing conditions are again easily verified.

There is something mildly dissatisfying about the equilibria in Example 5, in the sense that some of the agents do not care about behaving suboptimally in some states of the world if these decisions have no effect on their risk-adjusted disbenefit evaluated at an earlier time. As defined, a multistage risked equilibrium consists of state-dependent prices, and a plan of action for each agent that is to be constructed to optimize the agent's risk-adjusted disbenefit *as viewed at time 1*. When the risk measure is not strictly monotone (such as the worst-case measure $\mathbb{W}$), such a plan devised at time 1 might fail to be dynamically consistent.

For example, in the first equilibrium the purchaser chooses $z_2(\omega_2) = 4$ in ω_2 when $\pi_2(\omega_2) = 5$. This gives disbenefit

$$5z_2(\omega_2) - 18(z_2(\omega_2) + \xi_2(\omega_2)) + (z_2(\omega_2) + \xi_2(\omega_2))^2 = -57$$

whereas the optimal choice in ω_2 would be $z_2(\omega_2) = 3.5$ giving disutility -57.25. Neither choice is as bad as the optimal outcome in ω_1 which gives disutility $\varphi = -53$, which determines the optimal plan with risk measure $\mathbb{W}$. Of course, if the consumer found herself in ω_2 with $\pi_2(\omega_2) = 5$, then she would rationally choose $z_2(\omega_2) = 3.5$, but that is not needed in our definition of multistage risked equilibrium.

Example 5 illustrates why we distinguish between formulations SO and SE above, and AO and AE for agents in what follows. The SE and AE formulations correspond to solutions that are dynamically consistent, even though some nodes in $\mathcal{N}$ might have zero risk-adjusted probability. Solutions to SO and AO need not be dynamically consistent when the risk measure is not strictly monotone. The formulations are equivalent when Assumption 1 holds, thus ensuring dynamic consistency, which avoids paradoxical situations as illustrated here.

We conclude this discussion by noting strict monotonicity of each agent's risk measure is not sufficient to ensure uniqueness of multistage risked equilibrium. A counterexample is presented in Gérard et al. (2018).

4. Risk trading

We now turn our attention to the situation where agents with polyhedral risk sets can trade financial contracts to reduce their risk. We will show that the system optimal solution to a social planning problem corresponds to a perfectly competitive equilibrium with risk trading.

We use the notation $Z_a(n)$, $n \in \mathcal{N}$ to denote the disbenefit of agent a, and $\mathcal{D}_a(n)$ to denote the risk set of agent a, which is a polyhedral set with extreme points $\{[p_a^k]_{n_+}, k \in \mathcal{K}_a(n)\}$. In order to get some alignment between the objectives of agents and a social planner, we establish a connection between their risk sets using the following assumption and definitions.

ASSUMPTION 3. *For* $n \in \mathcal{N} \setminus \mathcal{L}$

$$\bigcap_{a \in \mathcal{A}} \mathcal{D}_a(n) \neq \emptyset.$$

DEFINITION 3. For $n \in \mathcal{N} \setminus \mathcal{L}$ the *social planning risk set* is

$$\mathcal{D}_s(n) = \bigcap_{a \in \mathcal{A}} \mathcal{D}_a(n).$$

Note that extreme points of $\mathcal{D}_s$ are indexed by $\mathcal{K}_s(n)$ and corresponding extreme points have subscript s.

The financial instruments that are traded are assumed to take a specific form.

DEFINITION 4. Given any node $n \in \mathcal{N} \setminus \mathcal{L}$, an *Arrow-Debreu security* for node $m \in n_+$ is a contract that charges a price $\mu(m)$ in node n to receive a payment of 1 in node $m \in n_+$, and zero in other nodes $m' \neq m$, $m' \in n_+$.

We shall assume throughout this section that the market for risk is *complete*. Formally this means that the set of Arrow-Debreu securities traded at each node n spans the set of possible outcomes in n_+. (Equivalently, the market would be complete if for every Arrow-Debreu security there was a portfolio of traded intruments that replicates its payoff in every state of the world.) It is important to emphasize that the trade in these instruments yields a common market price $\mu(m)$ that is paid by all agents in node n for each of the securites indexed by $m \in n_+$.

Assumption 4. *At every node $n \in \mathcal{N} \setminus \mathcal{L}$, there is an Arrow-Debreu security for each child node $m \in n_+$ that is traded in node n at an equilibrium price $\mu(m)$.*

To reduce its risk, suppose that each agent a in node n purchases $W_a(m)$ Arrow-Debreu securities for node $m \in n_+$. Each agent a's optimization problem with risk trading is then formulated as

$$
\begin{aligned}
&\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a): \\
&\qquad \min_{u_a, x_a, W_a, \theta_a} \; Z_a(0; u, x, W) + \theta_a(0) \\
&\qquad\qquad \text{s.t. } \theta_a(n) \geq \sum_{m \in n_+} p_a^k(m)(Z_a(m; u, x, W) - W_a(m) + \theta_a(m)), \\
&\qquad\qquad\qquad\qquad k \in K_a(n),\; n \in \mathcal{N} \setminus \mathcal{L}, \\
&\qquad\qquad\quad \theta_a(n) = -V_{an}(x_a(n)), \quad n \in \mathcal{L}, \\
&\qquad\qquad\quad u_a(n) \in \mathcal{U}_a,\; x_a(n) \in \mathcal{X}_a, \quad n \in \mathcal{N},
\end{aligned}
$$

where we use the shorthand notation

$$
\begin{aligned}
Z_a(n; u, x, W) = {} & C_{an}(u_a(n)) - \pi(n) g_{an}(u_a(n)) + \alpha_a(n)\left(x_a(n) - x_a(n_-) - \omega_a(n)\right) \\
& - \sum_{b \in \mathcal{A}} \alpha_b(n) T_{ba} u_a(n) + \sum_{m \in n_+} \mu(m) W_a(m), \quad n \in \mathcal{N}.
\end{aligned} \tag{13}
$$

Here the agent minimizes immediate cost plus the (insurance) cost of the security along with future costs, in the understanding that the security will pay back in the next period according to the situation realized. The interpretation of the notation in (13) is the same as in (12), with the exception of variables $W_a(m)$ that denote the number of Arrow-Debreu securities of type m bought by agent a at node n. The agent pays a market price $\mu(m)$ for each of these. The payoff for security m only occurs in scenario m as reflected in the first inequality of $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$. Observe that $W_a(m)$ can be negative (if the security is sold) and is an unbounded variable in this formulation. In equilibrium $W_a(m)$ will be traded at an equilibrium price $\mu(m)$.

The formulation $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ can be adapted to settings where risk is traded but the market for risk is not complete. For example, one could replace the set of Arrow-Debreu securities by a forward contract that costs $f(n)$ and pays out $\pi(m), m \in n_+$. This would give a multistage risked equilibrium as defined in the previous section which could be solved using complementarity software like PATH (Ferris and Munson 2000).

Assumption 3 above (which is a form of no-arbitrage condition) ensures that the trade in Arrow-Debreu securities is bounded at equilibrium prices. To see this, consider polyhedral risk sets $\mathcal{D}_a, a \in \mathcal{A}$, where we denote the extreme points of $\mathcal{D}_a$ by p_a^k, $k \in \mathcal{K}_a$. Consider the problem

$$\begin{aligned}
\text{P: } \min\ & \textstyle\sum_{a\in\mathcal{A}} v_a \\
\text{s.t. } & -\textstyle\sum_m p_a^k(m) W_a(m) + v_a \geq 0,\ k \in \mathcal{K}_a, a \in \mathcal{A},\ [\lambda_a^k] \\
& \textstyle\sum_a W_a(m) = 0, \qquad\qquad m = 1, \ldots, M.\ [\mu(m)]
\end{aligned}$$

The problem P constructs a set of security trades W_a, $a \in \mathcal{A}$, with $\sum_m p_a^k(m) W_a(m) \leq v_a$, $k \in \mathcal{K}_a, a \in \mathcal{A}$. Observe that P always has a feasible solution. The dual problem for P is

$$\begin{aligned}
\text{D: } \max\ & 0 \\
\text{s.t. } & -\textstyle\sum_{k\in\mathcal{K}_a} p_a^k(m) \lambda_a^k(m) + \mu(m) = 0,\ a \in \mathcal{A},\ m = 1, \ldots, M, \\
& \textstyle\sum_{k\in\mathcal{K}_a} \lambda_a^k = 1, \qquad\qquad a \in \mathcal{A}, \\
& \lambda_a^k \geq 0.
\end{aligned}$$

If $\cap_{a\in\mathcal{A}} \mathcal{D}_a \neq \emptyset$ then D is feasible and bounded, so P must have an optimal solution by virtue of the duality theorem of linear programming, and the payoffs for agents trading Arrow-Debreu securities must be bounded.

Conversely if $\cap_{a\in\mathcal{A}} \mathcal{D}_a = \emptyset$ then D is infeasible, so P must be unbounded. This means that for every $a \in \mathcal{A}$,

$$\rho_a(W_a) = \max_{\mathbb{P}\in\mathcal{D}_a} \mathbb{E}_{\mathbb{P}}[W_a] \leq v_a,$$

and there is at least one v_a in P that takes arbitrarily negative values. Thus there is a set of security trades W_a that leads to unbounded (negative) disbenefit for at least one agent. Observe that if this agent holds any other fixed positions with disbenefits Z_a then by subadditivity

$$\rho_a(W_a + Z_a) \leq \rho_a(W_a) + \rho_a(Z_a)$$

so agent a will still have unbounded disbenefit.

We can define a complementarity form of $\mathrm{AO}_a(\pi,\alpha,\mu,\mathcal{D}_a)$ as follows.

$\mathrm{AE}_a(\pi,\alpha,\mu,\mathcal{D}_a)$:

$$0 = 1 - \sum_{k\in\mathcal{K}_a(n)} \gamma_a^k(n), \quad n\in\mathcal{N}\setminus\mathcal{L} \tag{14a}$$

$$0 \le \gamma_a^k(n) \perp \theta_a(n) - \sum_{m\in n_+} p_a^k(m)\Big(Z_a(m;u,x,W) - W_a(m) + \theta_a(m)\Big) \ge 0,$$
$$k\in\mathcal{K}_a(n), n\in\mathcal{N}\setminus\mathcal{L} \tag{14b}$$

$$\theta_a(n) = -V_{an}(x_a(n)), \quad n\in\mathcal{L} \tag{14c}$$

$$0 \in \nabla_{u_a(n)} Z_a(n;u,x,W) + N_{\mathcal{U}_a}(u_a(n)), \quad n\in\mathcal{N} \tag{14d}$$

$$0 \in \alpha_a(n) - \sum_{m\in n_+} \mu(m)\alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad n\in\mathcal{N}\setminus\mathcal{L} \tag{14e}$$

$$0 \in \alpha_a(n) - \nabla_{x_a(n)} V_{an}(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad n\in\mathcal{L} \tag{14f}$$

$$0 = \mu(m) - \sum_{k\in\mathcal{K}_a(n)} \gamma_a^k(n) p_a^k(m), \quad m\in n_+, n\in\mathcal{N}\setminus\mathcal{L}, \tag{14g}$$

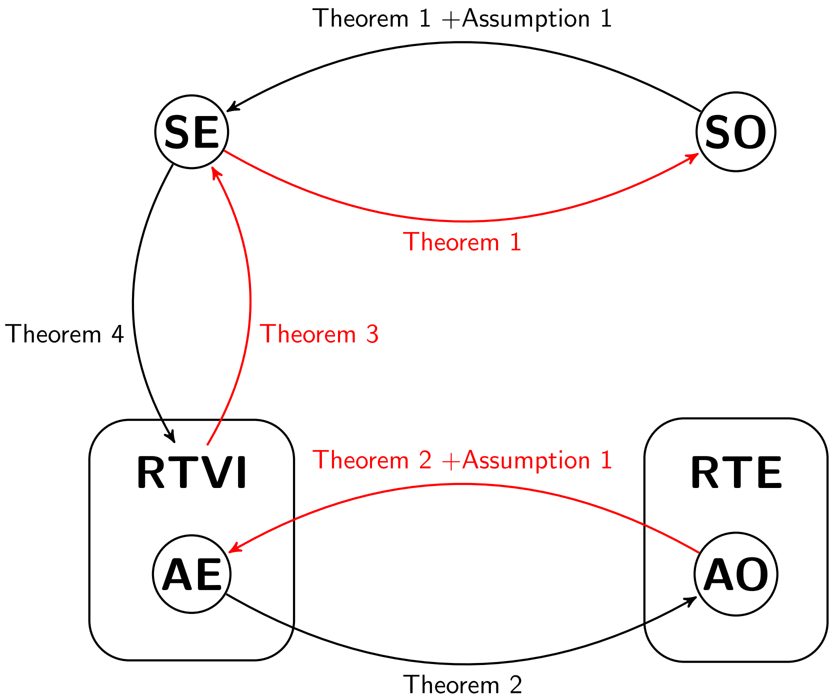
where $Z_a(n;u,x,W)$ is defined by (13).

Theorem 2 provides a link between the solution to the optimization problem AO faced by an agent at node 0 and the optimality conditions AE that a dynamically consistent optimal solution would satisfy at each node. Any solution to AE will solve AO. The converse is true when Assumption 1 holds.

Theorem 2. *(i) Any solution to $AE_a(\pi,\alpha,\mu,\mathcal{D}_a)$ provides a solution (u_a,x_a,W_a,θ_a) to the optimization problem $AO_a(\pi,\alpha,\mu,\mathcal{D}_a)$, and satisfies*

$$\theta_a(n) = \max_{\nu\in\mathcal{D}_a(n)} \sum_{m\in n_+} \nu(m)\left(Z_a(m;u,x,W) - W_a(m) + \theta_a(m)\right)$$
$$= \sum_{m\in n_+} \mu(m)\left(Z_a(m;u,x,W) - W_a(m) + \theta_a(m)\right).$$

(ii) If Assumption 1 holds, then any solution of $AO_a(\pi,\alpha,\mu,\mathcal{D}_a)$ provides a solution to $AE_a(\pi,\alpha,\mu,\mathcal{D}_a)$ for some γ_a.



Corollary 1: RTE + Assumption 1 solves RTVI and hence SO

Corollary 2: SO + Assumption 1 solves RTVI and hence RTE

Figure 3 An outline of the interplay of the main results.

The remainder of the paper seeks to connect competitive equilibrium in a market where agents trade risk to the solution of a social optimization problem. We do this by linking system optimization (SO) to a complementarity problem (SE) that is equivalent to a system of variational inequalities (RTVI). This system is in turn linked to the competitive equilibrium with risk trading (RTE). A broad outline of our proof strategy is given in Figure 3.

Suppose each agent solves the optimization problem $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ taking prices π, α, and μ as given. If these prices clear the markets for respective quantities, then we have a competitive equilibrium with risk trading.

DEFINITION 5. A *multistage risk-trading equilibrium* $\mathrm{RTE}(\mathcal{D}_\mathcal{A})$ is a stochastic process of prices $\{\pi(n), n \in \mathcal{N}\}$, $\{\alpha_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, $\{\mu(n), n \in \mathcal{N} \setminus \{0\}\}$, and a corresponding collection of actions for each $a \in \mathcal{A}$, $\{(u_a(n), x_a(n), \theta_a(n)), n \in \mathcal{N}\}, \{W_a(n), n \in \mathcal{N} \setminus \{0\}\}$ with the property that

$(u_a, x_a, W_a, \theta_a)$ solves the problem $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ and

$$0 \leq \pi(n) \quad \perp \quad \sum_{a \in \mathcal{A}} g_{an}(u_a(n)) \geq 0, \quad n \in \mathcal{N}, \tag{15}$$

$$0 \leq \alpha_a(n) \quad \perp \quad -x_a(n) + x_a(n_-) + \sum_{b \in \mathcal{A}} T_{ab} u_b(n) + \omega_a(n) \geq 0,$$
$$a \in \mathcal{A}, n \in \mathcal{N}, \tag{16}$$

$$0 \leq \mu(n) \quad \perp \quad -\sum_{a \in \mathcal{A}} W_a(n) \geq 0, \quad n \in \mathcal{N} \setminus \{0\}. \tag{17}$$

In the absence of Assumption 1, the solution set of $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ might strictly contain that of $\mathrm{AE}_a(\pi, \alpha, \mu, \mathcal{D}_a)$. We can then define a constrained form of $\mathrm{RTE}(\mathcal{D}_{\mathcal{A}})$ as follows.

DEFINITION 6. A *multistage risk-trading variational inequality* $\mathrm{RTVI}(\mathcal{D}_{\mathcal{A}})$ is a stochastic process of prices $\{\pi(n), n \in \mathcal{N}\}$, $\{\alpha_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, $\{\mu(n), n \in \mathcal{N} \setminus \{0\}\}$, and a corresponding collection of actions for each $a \in \mathcal{A}$, $\{(u_a(n), x_a(n), \theta_a(n)), n \in \mathcal{N}\}, \{W_a(n), n \in \mathcal{N} \setminus \{0\}\}$ with the property that for some γ_a, $(u_a, x_a, W_a, \theta_a, \gamma_a)$ solves the problem $\mathrm{AE}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ and (15), (16), and (17) are satisfied.

Note that RTVI is a more restrictive form of RTE, that is equivalent when Assumption 1 holds. We write $-\sum_{a \in \mathcal{A}} W_a(n) \geq 0$ rather than an equation, so no more Arrow-Debreu securities $W_a(m)$ are to be bought than sold. Under Assumption 1, the markets for Arrow-Debreu securities will clear with prices $\mu(m) > 0$, so (17) ensures that $\sum_a W_a(m) = 0$ in equilibrium. If Assumption 1 does not hold then it is possible to have $\mu(m) = 0$ in equilibrium and $\sum_a W_a(m) < 0$.

To see this, consider the second equilibrium in Example 5, where the worst-case risk measure $\mathbb{W}$ violates Assumption 1. Here the optimal solution to P has disbenefit

$$Z_P(\omega) = \frac{1}{2} u_1^2 - \pi_1(u_1 - x) + \theta,$$

so $Z_P(\omega_1) = -13.25$, $Z_P(\omega_2) = -10.25$, and the optimal solution to C has disbenefit

$$Z_C(\omega) = \pi_1 v_1 + \frac{1}{8} v_1^2 + \varphi,$$

so $Z_C(\omega_1) = -49.25$, $Z_C(\omega_2) = -58.25$.

We can include purchase of Arrow-Debreu securities in this model, for example $W_P(\omega_1) = 0$, $W_P(\omega_2) = 4$, $W_C(\omega_1) = 0$, and $W_C(\omega_2) = -5$, where positive numbers decrease disbenefit so that after trade $Z_P(\omega_1) = -13.25$, $Z_P(\omega_2) = -14.25$, $Z_C(\omega_1) = -49.25$, $Z_C(\omega_2) = -53.25$. Here after trading, C's disbenefit increases in ω_2 by 5, and P's disbenefit decreases in ω_2 by 4. Note that as long as $W_C(\omega_2) > -9$, increasing disbenefit in ω_2 does not alter the risk-adjusted disbenefit of C which is determined by its worst case of -49.25 in ω_1. So, unless its position changes in ω_1, C would be indifferent to selling up to 9 units of ω_2 Arrow-Debreu securities. Thus the price $\mu(2)$ in ω_2 is 0, and the price $\mu(1)$ in ω_1 is 1. Matching these values we have

$$W_P(\omega_1) + W_C(\omega_1) = 0$$

and

$$W_P(\omega_2) + W_C(\omega_2) = -1 < 0.$$

First and second welfare theorems can be derived for RTVI($\mathcal{D}_\mathcal{A}$).

Theorem 3. *Consider a set of agents $a \in \mathcal{A}$, each endowed with polyhedral node-dependent risk sets $\mathcal{D}_a(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$ satisfying Assumption 3. Suppose $\{\bar{\pi}(n), n \in \mathcal{N}\}$, $\{\bar{\alpha}_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, and $\{\bar{\mu}(n), n \in \mathcal{N} \setminus \{0\}\}$ form a multistage risk-trading variational inequality RTVI($\mathcal{D}_\mathcal{A}$) in which agent a solves $AE_a(\bar{\pi}, \bar{\alpha}, \bar{\mu}, \mathcal{D}_a)$ with a policy defined by $(\bar{u}_a(\cdot), \bar{x}_a(\cdot), \bar{\theta}_a(\cdot))$ together with a policy of trading Arrow-Debreu securities defined by $\{\bar{W}_a(n), n \in \mathcal{N} \setminus \{0\}\}$. For every $n \in \mathcal{N}$ define $\bar{\theta}(n) = \sum_{a \in \mathcal{A}} \bar{\theta}_a(n)$. Then*

(i) $\bar{\mu} \in \mathcal{D}_a$ for all $a \in \mathcal{A}$, and hence $\bar{\mu} \in \mathcal{D}_s$,

(ii)

$$\bar{\theta}(n) = \sum_{m \in n_+} \bar{\mu}(m) \left(\sum_{a \in \mathcal{A}} C_{am}(\bar{u}_a(m)) + \bar{\theta}(m) \right), \quad n \in \mathcal{N} \setminus \mathcal{L}. \tag{18}$$

(iii) there exist multipliers $\hat{\gamma}$ such that $(\bar{u}, \bar{x}, \bar{\theta}, \hat{\gamma}, \bar{\pi}, \bar{\alpha})$ is a solution to SE($\mathcal{D}_0$) with $\mathcal{D}_0 = \{\bar{\mu}\}$,

(iv) there exist multipliers $\bar{\gamma}$ such that $(\bar{u}, \bar{x}, \bar{\theta}, \bar{\gamma}, \bar{\pi}, \bar{\alpha})$ is a solution to SE($\mathcal{D}_s$)

and $\bar{\mu}(n) = \sum_{k \in \mathcal{K}_s(n)} \bar{\gamma}^k(n) p_s^k(m)$.

Theorem 3 shows that in equilibrium the prices $\bar{\mu}(n)$ of Arrow-Debreu securities give a set of risk-adjusted probabilities that scale prices to yield Lagrange multipliers, i.e. $\sigma(n) = \bar{\mu}(n), n \in \mathcal{N} \setminus \{0\}$. Moreover these risk-adjusted probabilities can be used for risk-neutral valuation; if all agents minimize expected disbenefit using these probabilities then this gives decisions that minimize expected social disbenefit using these probabilities. The values of $\bar{\mu}(n)$ emerge from trading in a complete risk market in the same way that an equivalent martingale measure emerges from observed asset prices in a complete financial market.

We are also able to establish a converse result to Theorem 3, the proof of which is in Appendix C.

Theorem 4. *Consider a set of agents $a \in \mathcal{A}$, each endowed with a polyhedral node-dependent risk set $\mathcal{D}_a(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$ satisfying Assumption 3. Let $(\bar{u}, \bar{x}, \bar{\theta}^s, \bar{\gamma}, \bar{\pi}, \bar{\alpha})$ be a solution to SE($\mathcal{D}_s$) with risk sets $\mathcal{D}_s(n) = \bigcap_{a\in\mathcal{A}} \mathcal{D}_a(n)$ and define $\bar{\mu}$ by*

$$\bar{\mu}(m) = \sum_{k\in\mathcal{K}_s(n)} \bar{\gamma}^k(n) p_s^k(m), \quad m \in n_+, \quad n \in \mathcal{N} \setminus \mathcal{L}.$$

Then there exist $\{\bar{\theta}_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$ such that the prices $\{\bar{\pi}(n), n \in \mathcal{N}\}$, $\{\bar{\alpha}_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, $\{\bar{\mu}(n), n \in \mathcal{N} \setminus \{0\}\}$ and actions of each agent $a \in \mathcal{A}$ $\{(\bar{u}_a(n), \bar{x}_a(n), \bar{\theta}_a(n)), n \in \mathcal{N}\}$, $\{\bar{W}_a(n), n \in \mathcal{N} \setminus \{0\}\}$ form a multistage risk-trading variational inequality RTVI($\mathcal{D}_{\mathcal{A}}$).

Under Assumption 1 we can establish versions of the welfare theorems in which each agent solves a multistage optimization problem $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ to yield a risk-trading equilibrium. These are the following corollaries, the proofs of which are immediate from Theorem 3 and Theorem 4 and the fact that Assumption 1 gives the equivalence of $\mathrm{AE}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ and $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$, and $\mathrm{SE}(\mathcal{D}_s)$ and $\mathrm{SO}(\mathcal{D}_s)$.

Corollary 1. *Suppose Assumption 1 holds. Consider a set of agents $a \in \mathcal{A}$, each endowed with a polyhedral node-dependent risk set $\mathcal{D}_a(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$ satisfying Assumption 3. Suppose $\{\bar{\pi}(n), n \in \mathcal{N}\}$, $\{\bar{\alpha}_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, and $\{\bar{\mu}(n), n \in \mathcal{N} \setminus \{0\}\}$ form a multistage risk-trading equilibrium RTE($\mathcal{D}_{\mathcal{A}}$) in which agent a solves $AO_a(\bar{\pi}, \bar{\alpha}, \bar{\mu}, \mathcal{D}_a)$ with a policy defined by $(\bar{u}_a(\cdot), \bar{x}_a(\cdot), \bar{\theta}_a(\cdot))$ together with a policy of trading Arrow-Debreu securities defined by $\{\bar{W}_a(n), n \in \mathcal{N} \setminus \{0\}\}$. Then $(\bar{u}, \bar{x}, \bar{\theta})$ is a solution to SO($\mathcal{D}_s$) where $\mathcal{D}_s(n) = \bigcap_{a\in\mathcal{A}} \mathcal{D}_a(n)$ and $\bar{\theta}(n) = \sum_{a\in\mathcal{A}} \bar{\theta}_a(n)$.*

Corollary 2. *Suppose Assumption 1 holds. Consider a set of agents $a \in \mathcal{A}$, each endowed with a polyhedral node-dependent risk set $\mathcal{D}_a(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$ satisfying Assumption 3. Now let $(\bar{u}, \bar{x}, \bar{\theta}^s)$ be a solution to $SO(\mathcal{D}_s)$ with risk sets $\mathcal{D}_s(n) = \bigcap_{a\in\mathcal{A}} \mathcal{D}_a(n)$. There exist prices $\{\bar{\pi}(n), n \in \mathcal{N}\}$, $\{\bar{\alpha}_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$ and multipliers $\bar{\gamma}$, defined in (11) such that*

1. $(\bar{u}, \bar{x}, \bar{\theta}^s, \bar{\gamma}, \bar{\pi}, \bar{\alpha})$ *satisfies* $SE(\mathcal{D}_s)$,
2. *If* $\bar{\mu}(m) = \sum_{k\in\mathcal{K}(n)} \bar{\gamma}^k(n) p^k(m), \quad m \in n_+,\ n \in \mathcal{N} \setminus \mathcal{L}$ *then there exist* $\{\bar{\theta}_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$ *such that the prices* $\{\bar{\pi}(n), n \in \mathcal{N}\}$, $\{\bar{\alpha}_a(n), a \in \mathcal{A}, n \in \mathcal{N}\}$, $\{\bar{\mu}(n), n \in \mathcal{N} \setminus \{0\}\}$, *and actions for each* $a \in \mathcal{A}$ $\{(\bar{u}_a(n), \bar{x}_a(n), \bar{\theta}_a(n)), n \in \mathcal{N}\}$, $\{\bar{W}_a(n), n \in \mathcal{N} \setminus \{0\}\}$ *form a multistage risk-trading equilibrium* $RTE(\mathcal{D}_\mathcal{A})$.

5. Discussion

This paper provides a theory for multistage risked equilibria. Its main contributions are as follows.

Firstly, we extend the definition of multistage risked equilibrium given in Philpott et al. (2016) to a more general model that allows storage and pricing of transfers of shared resources, along with a number of examples that demonstrate the richness of the equilibrium framework that we propose. Although our focus in the paper has been on models with storage, the construction of equilibrium and its correspondence with social planning is quite general, and encompasses multistage capacity-expansion models for example.

Secondly, we have established versions of the first and second welfare theorems in a setting where agents can trade risk. We give a proof of the first welfare theorem (which is new) and a simpler proof of the second welfare theorem as applied to multistage risked equilibrium with risk trading. The First Welfare Theorem provides a perfectly competitive benchmark against which real markets might be measured. In the real world, where markets are imperfect, the optimal value of a social planning model provides an upper bound on what might be achieved in welfare terms by reducing market imperfections. Observe that the welfare results rely on Assumption 3. The risk sets of the agents must intersect to enable trade to be bounded. In a non-polyhedral setting we would require the stronger condition that the intersection of the relative interiors of the risk sets is nonempty (see e.g. Ralph and Smeers 2015). If one agent believes that the risk-adjusted price of a given Arrow-Debreu contract strictly exceeds that asked by a prospective seller, then an infinite trade will result.

Thirdly, we illuminate the role that strict monotonicity of risk measures plays in multistage risked equilibrium. Our optimization versions of the welfare theorems (Corollaries 1 and 2) rely on Assumption 1. This is equivalent to the assertion that the one-step risk measure is strictly monotone, thus guaranteeing a nested risk measure that yields a dynamically consistent optimal solution. Dynamic consistency of optimal actions is related to subgame perfection in Nash equilibrium (see Selten 1975). As shown by Example 5, our definition of multistage risked equilibrium does not guarantee subgame perfection if Assumption 1 does not hold. Ideally we would like competitive equilibrium to specify an optimal action for each agent in every state of the world, even if this is discounted in equilibrium to have zero risk-adjusted disbenefit. It is therefore necessary for a social plan to specify a set of actions for the agents in such states. This can be done either by constraining it to be dynamically consistent using the formulation SE($\mathcal{D}_s$) in the absence of Assumption 1, or by imposing strict monotonicity on each agent's one-step risk measure.

The existence of a complete market for Arrow-Debreu securities that we require for our welfare theorems is an unrealistic assumption in practice. In some markets such as those for electricity there are typically very few traded securities that can be used to hedge risk. Forward contracts (often called contracts for differences) are the most popular instrument. Multistage risked equilibrium models with incomplete risk markets can be formulated as complementarity problems and solved (at least for small instances) using standard software such as PATH (Ferris and Munson 2000). Examples of such models for electricity can be found in Abada et al. (2017b), Kok et al. (2018). The results of these models show that equilibria in incomplete markets give welfare losses in comparison with a risk-averse social plan that vary when different instruments are used for hedging risk. The models can also be used to demonstrate the effects of ownership of generation plant or vertical integration of generation and retailing on welfare in incomplete perfectly competitive markets, where aggregation tends to reduce welfare losses due to risk pooling.

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Appendices

Appendix A: Coherent risk measures: proofs of lemmas

Proof of Lemma 1. For the forward implication, just choose $\gamma^k = 1$ for the term involving extreme point k that achieves the supremum. For the reverse implication, since $\theta \geq \sum_{m\in\mathcal{M}} p^k(m)Z(m)$ for each extreme point, it follows that $\theta \geq \sum_{m\in\mathcal{M}} \nu(m)Z(m)$ for each $\nu \in \mathcal{D}$ and hence $\theta \geq \sup_{\nu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \nu(m)Z(m)$. But complementary slackness shows that

$$\theta = \sum_{m\in\mathcal{M}} \bar{\nu}(m)Z(m),$$

where $\bar{\nu}$ is defined in the statement of the theorem and is clearly in $\mathcal{D}$ so $\theta \leq \sup_{\nu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \nu(m)Z(m)$ and thus equality holds. $\square$

Proof of Lemma 2. Suppose $\mathcal{D}$ lies in $\text{int}(\mathbb{R}_+^{|\mathcal{M}|})$. To show strict monotonicity, we suppose $Z_a \geq Z_b$ and $Z_a(\bar{m}) > Z_b(\bar{m})$ for some $\bar{m} \in \mathcal{M}$. Let $\rho(Z_a) = \sum_{m\in\mathcal{M}} \nu_a^*(m) Z_a(m)$, and $\rho(Z_b) = \sum_{m\in\mathcal{M}} \nu_b^*(m) Z_b(m)$. Then strict monotonicity follows from $\nu_b^*(\bar{m}) > 0$ since

$$\begin{aligned}\rho(Z_a) &= \sum_{m\in\mathcal{M}} \nu_a^*(m) Z_a(m) \\ &\geq \sum_{m\in\mathcal{M}} \nu_b^*(m) Z_a(m) \\ &> \sum_{m\in\mathcal{M}} \nu_b^*(m) Z_b(m) \\ &= \rho(Z_b).\end{aligned}$$

Conversely, suppose $\mathcal{D}$ does not lie in $\text{int}(\mathbb{R}_+^{|\mathcal{M}|})$, thus containing some point $\bar{\nu}$ with a zero component, say $\bar{\nu}(m_1) = 0$. Choose $Z(m) = 0$, $m = m_2, m_3, \ldots, m_{|\mathcal{M}|}$, and $Z(m_1) < 0$. Then $\bar{\nu} \in \arg\max_{\nu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \nu(m) Z(m)$, since $\sum_{m\in\mathcal{M}} \nu(m) Z(m) \leq 0$ for every $\nu \in \mathcal{D}$. Let

$$Z'(m) = \begin{cases} Z(m_1) - 1, & m = m_1 \\ Z(m), & \text{otherwise} \end{cases}$$

so $Z' \leq Z$ with $Z' \neq Z$. But $\bar{\nu} \in \arg\max_{\nu\in\mathcal{D}} \sum_{m\in\mathcal{M}} \nu(m) Z'(m)$, so

$$\rho(Z') = \sum_{m\in\mathcal{M}} \bar{\nu}(m) Z'(m) = \sum_{m\in\mathcal{M}} \bar{\nu}(m) Z(m) = \rho(Z),$$

violating the strict monotonicity of ρ. □

Appendix B: Tree multipliers

Consider a scenario tree with polyhedral risk sets $\mathcal{D}(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$, each having a finite set of extreme points $\{[p^k]_{n_+}, k \in \mathcal{K}(n)\}$. Any set of nonnegative numbers of the form $\{\gamma^k(n), k \in \mathcal{K}(n), n \in \mathcal{N} \setminus \mathcal{L}\}$ is called a set of *tree multipliers*. A set of tree multipliers is *conditional* if for every $n \in \mathcal{N} \setminus \mathcal{L}$, $\sum_{k\in\mathcal{K}(n)} \gamma^k(n) = 1$. A set of tree multipliers $\{\lambda^k(n), k \in \mathcal{K}(n), n \in \mathcal{N} \setminus \mathcal{L}\}$ is *unconditional* if

$$0 = 1 - \sum_{k\in\mathcal{K}(0)} \lambda^k(0), \tag{19}$$

$$0 = -\sum_{k\in\mathcal{K}(n)} \lambda^k(n) + \sum_{j\in\mathcal{K}(n_-)} \lambda^j(n_-) p^j(n), \quad n \in \mathcal{N} \setminus \mathcal{L}, n \neq 0. \tag{20}$$

Observe that any set of conditional tree multipliers γ corresponds to a unique set of unconditional tree multipliers λ defined recursively by setting $\lambda(0) = \gamma(0)$, and defining

$$\lambda(n) = \gamma(n) \sum_{j\in\mathcal{K}(n_-)} \lambda^j(n_-) p^j(n), \quad n \in \mathcal{N} \setminus \{0\}. \tag{21}$$

Since $\lambda(0) \geq 0$, repeated application of (21) implies $\lambda(n) \geq 0$ for every $n \in \mathcal{N} \setminus \{0\}$, so $\lambda^k(n)$ are well-defined tree multipliers. These are easily verified to be unconditional since for $n \in \mathcal{N} \setminus \mathcal{L}$

$$\sum_{k \in \mathcal{K}(n)} \lambda^k(n) = \sum_{j \in \mathcal{K}(n_-)} \lambda^j(n_-)p^j(n),$$

giving (20) and

$$\sum_{k \in \mathcal{K}(0)} \lambda^k(0) = \sum_{k \in \mathcal{K}(0)} \gamma^k(0) = 1,$$

giving (19). Conversely any unconditional set of tree multipliers corresponds to a unique set of conditional tree multipliers as long as Assumption 1 holds. To see this define $\gamma(0) = \lambda(0)$, and

$$\gamma^k(n) = \lambda^k(n)/(\sum_{\ell \in \mathcal{K}(n_-)} \lambda^\ell(n_-)p^\ell(n)) \quad k \in \mathcal{K}(n), \quad n \in \mathcal{N} \setminus \{0\}. \tag{22}$$

By Assumption 1 every component of $p^k(m)$, $m \in 0_+$ is strictly positive, and the vector $(\lambda(0))$ is nonnegative and nonzero by (19), so $\gamma^k(m)$ is well defined by (22) for $m \in 0_+$. However, (20) implies that the vector $(\lambda^k(m))$ is nonnegative and nonzero, and hence recursively that

$$\sum_{j \in \mathcal{K}(n_-)} \lambda^j(n_-)p^j(n) > 0, \quad n \in \mathcal{N} \setminus \{0\}. \tag{23}$$

Finally (20) and (22) imply $\sum_{k \in \mathcal{K}(n)} \gamma^k(n) = 1$, showing that $\{\gamma^k(n),\ k \in \mathcal{K}(n),\ n \in \mathcal{N} \setminus \mathcal{L}\}$ is conditional.

Given any unconditional tree multipliers λ recall (10) that defines risk-adjusted probabilities σ using

$$\sigma(n) = \begin{cases} 1, & n = 0, \\ \sum_{j \in \mathcal{K}(n_-)} \lambda^j(n_-)p^j(n), & n \in \mathcal{N} \setminus \{0\}. \end{cases} \tag{24}$$

Observe by (21) that (24) implies

$$\lambda^k(n) = \gamma^k(n)\sigma(n), \quad k \in \mathcal{K}(n), n \in \mathcal{N}, \tag{25}$$

whence multiplying by $p^k(m)$ and summing gives

$$\sum_{k \in \mathcal{K}(n)} \lambda^k(n)p^k(m) = \sigma(m) = (\sum_{k \in \mathcal{K}(n)} \gamma^k(n)p^k(m))\sigma(n), \quad m \in n_+, \quad n \in \mathcal{N} \setminus \mathcal{L}. \tag{26}$$

Conditional and unconditional multipliers satisfy the following lemma.

Lemma 3. *If $\theta(n), n \in \mathcal{N}$ and a conditional set of tree multipliers $\{\gamma^k(n),\ k \in \mathcal{K}(n),\ n \in \mathcal{N} \setminus \mathcal{L}\}$ satisfies*

$$0 \leq \gamma^k(n) \perp \theta(n) - \sum_{m \in n_+} p^k(m)\,(C(m) + \theta(m)) \geq 0, \quad k \in \mathcal{K}(n), n \in \mathcal{N} \setminus \mathcal{L}, \tag{27}$$

then there exist unconditional multipliers λ satisfying

$$0 \le \lambda^k(n) \perp \theta(n) - \sum_{m\in n_+} p^k(m)\,(C(m)+\theta(m)) \ge 0, \quad k\in\mathcal{K}(n), n\in\mathcal{N}\setminus\mathcal{L}. \tag{28}$$

Conversely, if (λ,θ) satisfies (19),(20),(28), and Assumption 1 holds, then $\sigma(n)$ defined by (24) is strictly positive for every $n\in\mathcal{N}$, and there exists conditional tree multipliers $\gamma^k(n)=\frac{\lambda^k(n)}{\sigma(n)}$, $n\in\mathcal{N}\setminus\mathcal{L}$ satisfying (27).

Proof. Given a set of conditional tree multipliers γ construct unconditional multipliers λ from (21) and $\lambda(0)=1$. Given these values, $\sigma\ge 0$ is defined by (24), so

$$0\le \sigma(n)\gamma^k(n)\perp \theta(n)-\sum_{m\in n_+} p^k(m)\,(C(m)+\theta(m))\ge 0,$$

yielding (28) via (25). Conversely, Assumption 1 implies (23), so we have $\sigma(n)>0$. The relationship (27) then follows from (28) by dividing through by $\sigma(n)>0$. $\square$

Appendix C: Proofs of main results

Proof of Theorem 1. The following Karush-Kuhn-Tucker conditions for SO($\mathcal{D}$) are necessary and sufficient for optimality in SO($\mathcal{D}$).

KKT($\mathcal{D}$):

$$
\begin{aligned}
0 &= 1-\sum_{k\in\mathcal{K}(0)}\lambda^k(0),\\
0 &= -\sum_{k\in\mathcal{K}(n)}\lambda^k(n)+\sum_{j\in\mathcal{K}(n_-)}\lambda^j(n_-)p^j(n), \quad n\in\mathcal{N}\setminus\mathcal{L}, n\neq 0\\
0 &\le \lambda^k(n)\perp\theta(n)-\sum_{m\in n_+}p^k(m)\left(\sum_{a\in\mathcal{A}}C_{am}(u_a(m))+\theta(m)\right)\ge 0, \quad k\in\mathcal{K}(n), n\in\mathcal{N}\setminus\mathcal{L}\\
\theta(n) &= -\sum_{a\in\mathcal{A}}V_{an}(x_a(n)), \quad n\in\mathcal{L}\\
0 &\in \nabla_{u_a(0)}\left[C_{a0}(u_a(0))-\tilde{\pi}(0)g_{a0}(u_a(0))-\sum_{b\in\mathcal{A}}\tilde{\alpha}_b(0)T_{ba}u_a(0)\right]+N_{\mathcal{U}_a}(u_a(0)), \quad a\in\mathcal{A}\\
0 &\in \nabla_{u_a(n)}\left[\sum_{k\in\mathcal{K}(n_-)}\lambda^k(n_-)p^k(n)C_{an}(u_a(n))-\tilde{\pi}(n)g_{an}(u_a(n))-\sum_{b\in\mathcal{A}}\tilde{\alpha}_b(n)T_{ba}u_a(n)\right]\\
&\quad +N_{\mathcal{U}_a}(u_a(n)), \quad a\in\mathcal{A}, n\in\mathcal{N}\setminus\{0\}\\
0 &\in \tilde{\alpha}_a(n)-\sum_{m\in n_+}\tilde{\alpha}_a(m)+N_{\mathcal{X}_a}(x_a(n)), \quad a\in\mathcal{A}, n\in\mathcal{N}\setminus\mathcal{L}\\
0 &\in \tilde{\alpha}_a(n)-\sum_{k\in\mathcal{K}(n_-)}\lambda^k(n_-)p^k(n)\nabla_{x_a(n)}V_{an}(x_a(n))+N_{\mathcal{X}_a}(x_a(n)), \quad a\in\mathcal{A}, n\in\mathcal{L}\\
0 &\le \tilde{\alpha}_a(n)\perp -x_a(n)+x_a(n_-)+\sum_{b\in\mathcal{A}}T_{ab}u_b(n)+\omega_a(n)\ge 0, \quad a\in\mathcal{A}, n\in\mathcal{N}
\end{aligned}
$$

$$0 \le \tilde{\pi}(n) \perp \sum_{a\in\mathcal{A}} g_{an}(u_a(n)) \ge 0, \quad n \in \mathcal{N}.$$

The proof proceeds to show the equivalence of these conditions to a solution of SE($\mathcal{D}$) under the hypotheses of the theorem.

(i) Suppose $(u,x,\theta,\gamma,\pi,\alpha)$ is a solution of SE($\mathcal{D}$). Since γ are conditional multipliers, and $\theta(n) = -\sum_{a\in\mathcal{A}} V_{an}(x_a(n))$, $n\in\mathcal{L}$, and (θ,γ) satisfies (27), Lemma 3 provides unconditional multipliers λ (and therefore σ from (24)) such that (28) holds for $C(m) = \sum_{a\in\mathcal{A}} C_{am}(u_a(m))$. Using these observations, the problem SE($\mathcal{D}$) leads to the conditions:

$$0 = 1 - \sum_{k\in\mathcal{K}(0)} \lambda^k(0),$$
$$0 = -\sum_{k\in\mathcal{K}(n)} \lambda^k(n) + \sum_{j\in\mathcal{K}(n_-)} \lambda^j(n_-)p^j(n), \quad n\in\mathcal{N}\setminus\mathcal{L}, n\neq 0$$
$$0 \le \lambda^k(n) \perp \theta(n) - \sum_{m\in n_+} p^k(m) \sum_{a\in\mathcal{A}} (C_{am}(u_a(m)) + \theta(m)) \ge 0, \quad k\in\mathcal{K}(n), n\in\mathcal{N}\setminus\mathcal{L}$$
$$\theta(n) = -\sum_{a\in\mathcal{A}} V_{an}(x_a(n)), \quad n\in\mathcal{L}$$
$$0 \in \nabla_{u_a(n)} \left[\sigma(n)C_{an}(u_a(n)) - \sigma(n)\pi(n)g_{an}(u_a(n)) - \sigma(n)\sum_{b\in\mathcal{A}} \alpha_b(n)T_{ba}u_a(n)\right] + N_{\mathcal{U}_a}(u_a(n)), \quad a\in\mathcal{A}, n\in\mathcal{N}$$
$$0 \in \sigma(n)\alpha_a(n) - \sum_{m\in n_+}\sum_{k\in\mathcal{K}(n)} \lambda^k(n)p^k(m)\alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad a\in\mathcal{A}, n\in\mathcal{N}\setminus\mathcal{L}$$
$$0 \in \sigma(n)\alpha_a(n) - \sigma(n)\nabla_{x_a(n)}V_{an}(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad a\in\mathcal{A}, n\in\mathcal{L}$$
$$0 \le \sigma(n)\alpha_a(n) \perp -x_a(n) + x_a(n_-) + \sum_{b\in\mathcal{A}} T_{ab}u_b(n) + \omega_a(n) \ge 0, \quad a\in\mathcal{A}, n\in\mathcal{N}$$
$$0 \le \sigma(n)\pi(n) \perp \sum_{a\in\mathcal{A}} g_{an}(u_a(n)) \ge 0, \quad n\in\mathcal{N}.$$

The relationships involving normal cones follow from multiplication by $\sigma(n)$ and (25), while the complementarity conditions follow from Lemma 3 and multiplication by $\sigma(n)$. If we let $\tilde{\alpha}_a(n) = \sigma(n)\alpha_a(n)$ and $\tilde{\pi}(n) = \sigma(n)\pi(n)$ then recalling (24) these conditions yield KKT($\mathcal{D}$), the KKT conditions for SO($\mathcal{D}$). Since any solution of SE($\mathcal{D}$) satisfies (27) in Lemma 3, (25) and Lemma 1 imply that

$$\theta(n) = \sup_{\nu\in\mathcal{D}(n)} \sum_{m\in n_+} \nu(m) \left(\sum_{a\in\mathcal{A}} C_{am}(u_a(m)) + \theta(m)\right)$$

is attained by $\bar{\nu}(m) = \sum_{k\in\mathcal{K}(n)} \gamma^k(n)p^k(m)$, which gives the last statement of (i).

(ii) For the converse result, suppose that we have a solution $(u,x,\theta,\lambda,\tilde{\pi},\tilde{\alpha})$ of the KKT conditions of SO($\mathcal{D}$) as shown above. Then Assumption 1 and Lemma 3 provide $\sigma(n) > 0$ and a conditional set of multipliers $\gamma^k(n) = \lambda^k(n)/\sigma(n)$ satisfying (27) for $C(m) = \sum_{a\in\mathcal{A}} C_{am}(u_a(m))$. Substituting

$\alpha_a(n) = \tilde{\alpha}_a(n)/\sigma(n)$ and $\pi(n) = \tilde{\pi}(n)/\sigma(n)$ into the KKT conditions of SO($\mathcal{D}$) and using (27) and (26) leads to

$$1 = \sum_{k\in\mathcal{K}(n)} \gamma^k(n), \quad n \in \mathcal{N}\setminus\mathcal{L}$$
$$0 \le \gamma^k(n) \perp \theta(n) - \sum_{m\in n_+} p^k(m) \sum_{a\in\mathcal{A}} (C_{am}(u_a(m)) + \theta(m)) \ge 0, \quad k \in \mathcal{K}(n), n \in \mathcal{N}\setminus\mathcal{L}$$
$$\theta(n) = -\sum_{a\in\mathcal{A}} V_{an}(x_a(n)), \quad n \in \mathcal{L}$$
$$0 \in \nabla_{u_a(n)} \left[\sigma(n)C_{an}(u_a(n)) - \sigma(n)\pi(n)g_{an}(u_a(n)) - \sigma(n)\sum_{b\in\mathcal{A}} \alpha_b(n)T_{ba}u_a(n)\right] + N_{\mathcal{U}_a}(u_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{N}$$
$$0 \in \sigma(n)\alpha_a(n) - \sum_{m\in n_+} \sigma(n) \sum_{k\in\mathcal{K}(n)} \gamma^k(n)p^k(m)\alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{N}\setminus\mathcal{L}$$
$$0 \in \sigma(n)\alpha_a(n) - \sigma(n)\nabla_{x_a(n)} V_{an}(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad a \in \mathcal{A}, n \in \mathcal{L}$$
$$0 \le \sigma(n)\alpha_a(n) \perp -x_a(n) + x_a(n_-) + \sum_{b\in\mathcal{A}} T_{ab}u_b(n) + \omega_a(n) \ge 0, \quad a \in \mathcal{A}, n \in \mathcal{N}$$
$$0 \le \sigma(n)\pi(n) \perp \sum_{a\in\mathcal{A}} g_{an}(u_a(n)) \ge 0, \quad n \in \mathcal{N}.$$

Dividing through by $\sigma(n)$ appropriately leads to a solution of SE($\mathcal{D}$) as required. □

Proof of Theorem 2. As we outlined above for the system optimization problem, $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ is equivalent to its KKT conditions, which are derived by applying nonnegative Lagrange multipliers $\lambda_a^k(n)$ to the inequality constraints. Since θ is unconstrained, λ_a satisfies (19) and (20), so they are unconditional tree multipliers. This enables us to substitute $\sigma_a(n)$ for $\sum_{j\in\mathcal{K}_a(n_-)} \lambda_a^j(n_-)p_a^j(n)$ to give the following KKT conditions for $\mathrm{AO}_a(\pi, \alpha, \mu, \mathcal{D}_a)$.

KKT_a:

$$0 = 1 - \sum_{k\in\mathcal{K}_a(0)} \lambda_a^k(0), \tag{29a}$$
$$0 = -\sum_{k\in\mathcal{K}_a(n)} \lambda_a^k(n) + \sigma_a(n), \quad n \in \mathcal{N}\setminus\mathcal{L}, n \neq 0 \tag{29b}$$
$$0 \le \lambda_a^k(n) \perp \theta_a(n) - \sum_{m\in n_+} p_a^k(m)\left(Z_a(m; u, x, W) - W_a(m) + \theta_a(m)\right) \ge 0, \quad k \in \mathcal{K}_a(n), n \in \mathcal{N}\setminus\mathcal{L} \tag{29c}$$
$$\theta_a(n) = -V_{an}(x_a(n)), \quad n \in \mathcal{L}. \tag{29d}$$
$$0 \in \nabla_{u_a(0)} Z_a(0; u, x, W) + N_{\mathcal{U}_a}(u_a(0)), \tag{29e}$$
$$0 \in \sigma_a(n)\nabla_{u_a(n)} Z_a(n; u, x, W) + N_{\mathcal{U}_a}(u_a(n)), \quad n \in \mathcal{N}\setminus\{0\} \tag{29f}$$
$$0 \in \sigma_a(n)\alpha_a(n) - \sum_{m\in n_+} \sigma_a(m)\alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{N}\setminus\mathcal{L} \tag{29g}$$

$$0 \in \sigma_a(n)\alpha_a(n) - \sigma_a(n)\nabla_{x_a(n)} V_{an}(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{L} \tag{29h}$$

$$0 = \mu(m) - \sum_{k\in\mathcal{K}_a(0)} \lambda_a^k(0) p_a^k(m), \quad m \in 0_+ \tag{29i}$$

$$0 = \sigma_a(q_-)\mu(q) - \sigma_a(q), \quad q \in n_{++} \cap \mathcal{N}, \quad n \in \mathcal{N}, \tag{29j}$$

with $Z_a(n;u,x,W)$ defined by (13).

(i) First suppose that $(u_a, x_a, W_a, \theta_a, \gamma_a)$ is a solution of $\text{AE}_a(\pi, \alpha, \mu, \mathcal{D}_a)$. Observe that (14b) implies that

$$\theta_a(n) \geq \sum_{m\in n_+} \nu(m)\left(Z_a(m;u,x,W) - W_a(m) + \theta_a(m)\right)$$

for every $\nu \in \mathcal{D}_a(n)$. Summing the complementarity condition (14b) over k and combining with (14a) gives

$$\begin{aligned}
\theta_a(n) &= \sum_{k\in\mathcal{K}_a(n)} \gamma_a^k(n)\theta_a(n) \\
&= \sum_{m\in n_+} \sum_{k\in\mathcal{K}_a(n)} \gamma_a^k(n) p_a^k(m)\left(Z_a(m;u,x,W) - W_a(m) + \theta_a(m)\right) \\
&= \sum_{m\in n_+} \mu(m)\left(Z_a(m;u,x,W) - W_a(m) + \theta_a(m)\right)
\end{aligned}$$

after substituting using (14g). It also follows from Lemma 3 that there exists λ_a which satisfies (28) with $C(m) = Z_a(m;u,x,W) - W_a(m)$ and $\theta_a(m)$ replacing $\theta(m)$. Given λ_a we can define σ_a using (24).

Putting these relationships together and substituting into the $\text{AE}_a(\pi, \alpha, \mu, \mathcal{D}_a)$ conditions (observing that $\sigma_a(n) \geq 0$) gives

$$\begin{aligned}
&0 = 1 - \sum_{k\in\mathcal{K}_a(0)} \lambda_a^k(0), \\
&0 = -\sum_{k\in\mathcal{K}_a(n)} \lambda_a^k(n) + \sum_{j\in\mathcal{K}_a(n_-)} \lambda_a^j(n_-) p_a^j(n), \quad n \in \mathcal{N}\setminus\mathcal{L}, n \neq 0 \\
&0 \leq \lambda_a^k(n) \perp \theta_a(n) - \sum_{m\in n_+} p_a^k(m)\Big(Z_a(m;u,x,W) - W_a(m) + \theta_a(m)\Big) \geq 0, \quad k \in \mathcal{K}_a(n), n \in \mathcal{N}\setminus\mathcal{L} \\
&\theta_a(n) = -V_{an}(x_a(n)), \quad n \in \mathcal{L} \\
&0 \in \sigma_a(n)\nabla_{u_a(n)} Z_a(n;u,x,W) + N_{\mathcal{U}_a}(u_a(n)), \quad n \in \mathcal{N} \\
&0 \in \sigma_a(n)\alpha_a(n) - \sum_{m\in n_+} \sigma_a(m)\alpha_a(m) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{N}\setminus\mathcal{L} \\
&0 \in \sigma_a(n)\alpha_a(n) - \sigma_a(n)\nabla_{x_a(n)} V_{an}(x_a(n)) + N_{\mathcal{X}_a}(x_a(n)), \quad n \in \mathcal{L} \\
&0 = \sigma_a(n)\mu(m) - \sum_{k\in\mathcal{K}_a(n)} \lambda_a^k(n) p_a^k(m), \quad m \in n_+, n \in \mathcal{N}\setminus\mathcal{L},
\end{aligned}$$

with $Z_a(n,u,x,W)$ defined by (13).

Clearly we recover (29a)–(29h). It simply remains to show that λ_a satisfies (29i) and (29j). Since $\lambda_a^k(0)=\gamma_a^k(0)$, (29i) is immediate from (14g). Since $\sigma_a(n)=\sum_{j\in\mathcal{K}_a(n_-)}\lambda_a^j(n_-)p_a^j(n)$, (14g) is equivalent to $\sigma_a(n)\mu(m)=\sigma_a(m)$ for $m\in n_+$, $n\in\mathcal{N}\setminus\mathcal{L}$, which gives (29j) if we identify q with m.
(ii) For the converse, suppose that we have a solution of (29), then Lemma 3 coupled with Assumption 1 provides $\sigma_a(n)>0$ and conditional multipliers $\gamma_a^k(n)=\lambda_a^k(n)/\sigma_a(n)$ that satisfy (27) for $C(m)=Z_a(m;u,x,W)-W_a(m)$ and $\theta(n)=\theta_a(n)$. Thus (14a), (14b) and (14c) are satisfied in the definition of the $\mathrm{AE}_a(\pi,\alpha,\mu,\mathcal{D}_a)$ problem. Now (14g) follows by dividing (29j) by $\sigma_a(q_-)$ and using (24) and (25). Noting (26) and then dividing (29g) and (29h) by $\sigma_a(n)$ then gives (14e) and (14f) respectively. The relationship (14d) follows from the definition of σ_a and (29e) and (29f). $\square$

Proof of Theorem 3. (i) If we have a solution of $\mathrm{AE}_a(\bar\pi,\bar\alpha,\bar\mu,\mathcal{D}_a)$ for each $a\in\mathcal{A}$, it follows from (14g) that $[\bar\mu]_{n_+}\in\mathcal{D}_a(n)$ for each n and thus $\bar\mu\in\mathcal{D}_a$ for all a, and hence $\bar\mu\in\mathcal{D}_s$ by Definition 3.
(ii) For each $a\in\mathcal{A}$ it follows from Theorem 2 that for $n\in\mathcal{N}\setminus\mathcal{L}$,

$$\bar\theta_a(n)=\sum_{m\in n_+}\bar\mu(m)\left(Z_a(m;\bar u,\bar x,\bar W)-\bar W_a(m)+\bar\theta_a(m)\right). \tag{30}$$

Summing over $a\in\mathcal{A}$ and invoking (17) gives

$$\bar\theta(n)=\sum_{a\in\mathcal{A}}\bar\theta_a(n)=\sum_{m\in n_+}\bar\mu(m)\left(\sum_{a\in\mathcal{A}}Z_a(m;\bar u,\bar x,\bar W)+\bar\theta(m)\right).$$

Recalling the definition of $Z_a(m;\bar u,\bar x,\bar W)$ from (13), summing over $a\in\mathcal{A}$, and invoking (15), (16) and (17) gives (18).
(iii) Suppose $\mathcal{D}_0=\{\bar\mu\}$. It follows that $\mathcal{K}_0(n)=\{1\}$ for $n\in\mathcal{N}\setminus\mathcal{L}$ where $p_0^1(m)=\bar\mu(m)$, for $m\in n_+$. Define $\hat\gamma^1(n)=1$, $n\in\mathcal{N}\setminus\mathcal{L}$. It then follows that the first, second and fifth conditions of $SE(\mathcal{D}_0)$ simplify to

$$\begin{aligned}
&\hat\gamma^1(n)=1,\quad n\in\mathcal{N}\setminus\mathcal{L},\\
&\theta(n)=\sum_{m\in n_+}\bar\mu(m)\left(\sum_{a\in\mathcal{A}}C_{am}(u_a(m))+\theta(m)\right),\quad n\in\mathcal{N}\setminus\mathcal{L},\\
&\quad 0\in\alpha_a(n)-\sum_{m\in n_+}\bar\mu(m)\alpha_a(m)+N_{\mathcal{X}_a}(x_a(n)),\quad n\in\mathcal{N}\setminus\mathcal{L}.
\end{aligned}$$

Combining these with the other conditions in (14), (15) and (16) shows that $(\bar u,\bar x,\bar\theta,\hat\gamma,\bar\pi,\bar\alpha)$ solves $SE(\mathcal{D}_0)$.
(iv) Suppose $(\bar u_a,\bar x_a,\bar W_a,\bar\theta_a,\bar\gamma_a)$ solves $\mathrm{AE}_a(\bar\pi,\bar\alpha,\bar\mu,\mathcal{D}_a)$. Let $Z_a(n;\bar u,\bar x,\bar W)$ be defined using (13) so that it follows from (15), (16) and (17) that

$$\begin{aligned}
&\sum_{a\in\mathcal{A}}\sum_{m\in n_+}\bar\mu(m)\left(C_{am}(\bar u_a(m))+\bar\theta_a(m)\right)\\
&\qquad=\sum_{a\in\mathcal{A}}\sum_{m\in n_+}\bar\mu(m)\left(Z_a(m;\bar u,\bar x,\bar W)-\bar W_a(m)+\bar\theta_a(m)\right),
\end{aligned}$$

which by (18)

$$= \sum_{a \in \mathcal{A}} \bar{\theta}_a(n)$$

and by (14a) and (14b) and Lemma 1

$$= \sum_{a \in \mathcal{A}} \sup_{\nu \in \mathcal{D}_a(n)} \sum_{m \in n_+} \nu(m) \left(\bar{Z}_a(m; \bar{u}, \bar{x}, \bar{W}) - \bar{W}_a(m) + \bar{\theta}_a(m) \right)$$

so that Assumption 3 and Definition 3 imply

$$\geq \sum_{a \in \mathcal{A}} \sup_{\nu \in \mathcal{D}_s(n)} \sum_{m \in n_+} \nu(m) \left(\bar{Z}_a(m; \bar{u}, \bar{x}, \bar{W}) - \bar{W}_a(m) + \bar{\theta}_a(m) \right)$$

and interchanging supremum and summation

$$\geq \sup_{\nu \in \mathcal{D}_s(n)} \sum_{m \in n_+} \nu(m) \sum_{a \in \mathcal{A}} \left(\bar{Z}_a(m; \bar{u}, \bar{x}, \bar{W}) - \bar{W}_a(m) + \bar{\theta}_a(m) \right)$$

since feasibility implies $-\sum_{a \in \mathcal{A}} \bar{W}_a(m) \geq 0$

$$\geq \sup_{\nu \in \mathcal{D}_s(n)} \sum_{m \in n_+} \nu(m) \sum_{a \in \mathcal{A}} \left(\bar{Z}_a(m; \bar{u}, \bar{x}, \bar{W}) + \bar{\theta}_a(m) \right)$$

by (15), (16) and (17)

$$= \sup_{\nu \in \mathcal{D}_s(n)} \sum_{m \in n_+} \nu(m) \sum_{a \in \mathcal{A}} \left(C_{am}(\bar{u}_a(m)) + \bar{\theta}_a(m) \right)$$

by (i)

$$\geq \sum_{m \in n_+} \bar{\mu}(m) \sum_{a \in \mathcal{A}} \left(C_{am}(\bar{u}_a(m)) + \bar{\theta}_a(m) \right).$$

Hence equality holds throughout and thus $[\bar{\mu}]_{n_+}$ solves

$$\sup_{\nu \in \mathcal{D}_s(n)} \sum_{m \in n_+} \nu(m) \left(\sum_{a \in \mathcal{A}} C_{am}(\bar{u}_a(m)) + \bar{\theta}(m) \right).$$

Lemma 1 then shows that these conditions are equivalent to the first two conditions of SE($\mathcal{D}_s$), which combined with the other conditions in $\mathrm{AE}_a(\bar{\pi}, \bar{\alpha}, \bar{\mu}, \mathcal{D}_a)$ gives the remaining conditions of SE($\mathcal{D}_s$). □

To prove Theorem 4, we will require a preliminary lemma that uses the following formulations. For each $n \in \mathcal{N} \setminus \mathcal{L}$, suppose $Z_a^s(m)$, $\theta^s(m)$ and $\theta_a^s(m)$ are given for each $m \in n_+$ and satisfy $\theta^s(m) = \sum_{a \in \mathcal{A}} \theta_a^s(m)$. Consider the problems:

$$\mathrm{R}(n, \mathcal{D}_s)\colon \max_{\nu \in \mathcal{D}_s(n)} \sum_{m \in n_+} \nu(m) \left(\sum_{a \in \mathcal{A}} Z_a^s(m) + \theta^s(m) \right) \tag{31}$$

$$\begin{aligned} &\mathrm{T}(n, \mathcal{D}_\mathcal{A})\colon \\ &\min_{[[W_a]_{n_+}]_{a \in \mathcal{A}}, \theta_a(n)} \sum_{a \in \mathcal{A}} \theta_a(n) \\ &\qquad \text{s.t. } \theta_a(n) \geq \sum_{m \in n_+} p_a^k(m) \left(Z_a^s(m) - W_a(m) + \theta_a^s(m) \right), \quad k \in \mathcal{K}_a(n), a \in \mathcal{A} \\ &\qquad -\sum_{a \in \mathcal{A}} W_a(m) \geq 0, \quad m \in n_+ \end{aligned}$$

$$\begin{aligned} &\mathrm{TD}(n, \mathcal{D}_\mathcal{A})\colon \\ &\max_{\mu, \phi} \sum_{m \in n_+} \sum_{a \in \mathcal{A}} \left(\sum_{k \in \mathcal{K}_a(n)} p_a^k(m) \phi_a^k(n) \right) \left(Z_a^s(m) + \theta_a^s(m) \right) \\ &\qquad \sum_{k \in \mathcal{K}_a(n)} \phi_a^k(n) = 1, \quad a \in \mathcal{A}, \\ &\qquad \mu(m) = \sum_{k \in \mathcal{K}_a(n)} p_a^k(m) \phi_a^k(n), \quad m \in n_+, a \in \mathcal{A} \\ &\qquad \mu(m) \geq 0, \quad m \in n_+, \quad \phi_a^k(n) \geq 0, \quad k \in \mathcal{K}_a(n), a \in \mathcal{A} \end{aligned}$$

and

$$\begin{aligned} &\mathrm{TOC}(n, \mathcal{D}_\mathcal{A})\colon \\ &0 = 1 - \sum_{k \in \mathcal{K}_a(n)} \phi_a^k(n), \quad a \in \mathcal{A} \\ &0 = \mu(m) - \sum_{k \in \mathcal{K}_a(n)} \phi_a^k(n) p_a^k(m), \quad m \in n_+, a \in \mathcal{A} \\ &0 \leq \phi_a^k(n) \perp \theta_a(n) - \sum_{m \in n_+} p_a^k(m) \left(Z_a^s(m) - W_a(m) + \theta_a^s(m) \right), \quad k \in \mathcal{K}_a(n), a \in \mathcal{A} \\ &0 \leq \mu(m) \perp -\sum_{a \in \mathcal{A}} W_a(m) \geq 0, \quad m \in n_+ \end{aligned}$$

The formulation R evaluates the one-stage risk of the random disbenefit $\sum_{a \in \mathcal{A}} Z_a$ using the coherent risk measure with risk set $\mathcal{D}_s(n) = \bigcap_{a \in \mathcal{A}} \mathcal{D}_a(n)$. The problem T on the other hand accumulates the risk measure of each agent a in a setting where they can exchange welfare W (constrained so that it cannot be created out of nothing). If the model has a variable $W_a(m)$ defined for each outcome $m \in n_+$, then the following analysis demonstrates that an exchange exists in node n that will yield the risk-adjusted value of the total social disbenefit faced by all agents if evaluated with risk set $\mathcal{D}_s(n)$.

LEMMA 4. *Let $n \in \mathcal{N}$ and suppose $\mathcal{D}_{\mathcal{A}}$ satisfies Assumption 3. The problems T, TD, TOC and R all have optimal solutions with the same optimal value. Any solution to one of these problems yields a solution to all of the others.*

Proof. Observe that T and TD are dual linear programs, and TOC gives the optimality conditions for T. The constraints of TD entail that $\mu(m), m \in n_+$ is a finite probability distribution that is constrained to lie in each $\mathcal{D}_a(n)$. Definition 3 means that TD is equivalent to R. So any optimal solution of one of these four formulations yields solutions to all the others. Observe that the feasible region of TD is compact and nonempty by Assumption 3, so T, TD, TOC and R all have optimal solutions with the same optimal value. □

Proof of Theorem 4. Suppose $(\bar{u}, \bar{x}, \bar{\theta}^s, \bar{\gamma}, \bar{\pi}, \bar{\alpha})$ is a solution of SE$(\mathcal{D}_s)$. It follows from Theorem 1 that defining $\bar{\mu}(m) = \sum_{k \in \mathcal{K}_s(n)} \bar{\gamma}^k(n) p_s^k(m) \in \mathcal{D}_s$ for each $m \in n_+$ we have

$$\begin{aligned}
\bar{\theta}^s(n) = \sum_{m \in n_+} \bar{\mu}(m) \Bigg(\sum_{a \in \mathcal{A}} \Bigg(& C_{am}(\bar{u}_a(m)) - \bar{\pi}(m) g_{am}(\bar{u}_a(m)) \\
& + \bar{\alpha}_a(m) \Big(\bar{x}_a(m) - \bar{x}_a(m_-) - \sum_{b \in \mathcal{A}} T_{ab} \bar{u}_b(m) - \omega_a(m) \Big) \Bigg) + \bar{\theta}^s(m) \Bigg) \\
= \sum_{m \in n_+} \bar{\mu}(m) \Bigg(\sum_{a \in \mathcal{A}} \Big(& C_{am}(\bar{u}_a(m)) - \bar{\pi}(m) g_{am}(\bar{u}_a(m)) \\
& + \bar{\alpha}_a(m) \left(\bar{x}_a(m) - \bar{x}_a(m_-) - \omega_a(m) \right) - \sum_{b \in \mathcal{A}} \bar{\alpha}_b(m) T_{ba} \bar{u}_a(m) \Big) + \bar{\theta}^s(m) \Bigg),
\end{aligned}$$

and $[\bar{\mu}]_{n_+}$ solves (31) where

$$\begin{aligned}
Z_a^s(m) = & C_{am}(\bar{u}_a(m)) - \bar{\pi}(m) g_{am}(\bar{u}_a(m)) \\
& + \bar{\alpha}_a(m) \left(\bar{x}_a(m) - \bar{x}_a(m_-) - \omega_a(m) \right) - \sum_{b \in \mathcal{A}} \bar{\alpha}_b(m) T_{ba} \bar{u}_a(m).
\end{aligned}$$

Consider now the leaf nodes $m \in \mathcal{L}$. At these nodes $\bar{\theta}^s(m) = -\sum_{a \in \mathcal{A}} V_{an}(\bar{x}_a(m))$ so defining $\bar{\theta}_a^s(m) = -V_{an}(\bar{x}_a(m))$ for each $a \in \mathcal{A}$ we have $\sum_{a \in \mathcal{A}} \bar{\theta}_a^s(m) = \bar{\theta}^s(m)$. Lemma 4 now shows there are values $[\bar{\phi}_a^k(n)]_{a \in \mathcal{A}, k \in \mathcal{K}_a(n)}$, $[[\bar{W}_a]_{n_+}]_{a \in \mathcal{A}}, \bar{\theta}_a(n)$ which combined with $[\bar{\mu}]_{n_+}$ solve TOC$(n, \mathcal{D}_{\mathcal{A}})$ for each node $n = m_-$, and that the solution value of R$(n, \mathcal{D}_s)$ (namely $\bar{\theta}^s(n)$) is equal to $\sum_{a \in \mathcal{A}} \bar{\theta}_a(n)$.

We now recursively apply this argument. For each node n in the penultimate stage, we let $\bar{\theta}_a^s(n) = \bar{\theta}_a(n)$, the above computed solution value, so that $\sum_{a\in\mathcal{A}} \bar{\theta}_a^s(n) = \bar{\theta}^s(n)$. Further, we define

$$Z_a^s(n) = C_{an}(\bar{u}_a(n)) - \bar{\pi}(n)g_{an}(\bar{u}_a(n)) + \bar{\alpha}_a(n)\left(\bar{x}_a(n) - \bar{x}_a(n_-) - \omega_a(n)\right)$$
$$- \sum_{b\in\mathcal{A}} \bar{\alpha}_b(n) T_{ba} \bar{u}_a(n) + \sum_{m\in n_+} \bar{\mu}(m)\bar{W}_a(m)$$

for the previously computed solution values for $\bar{W}_a(m)$, $m \in n_+$. For each node $q = n_-$, Lemma 4 constructs values $[\bar{\phi}_a^k(q)]_{a\in\mathcal{A},k\in\mathcal{K}_a(q)}$, $[\bar{\theta}_a(q), [\bar{W}_a]_{q_+}]_{a\in\mathcal{A}}$ which along with $[\bar{\mu}]_{q_+}$ solve $\mathrm{TOC}(q, \mathcal{D}_{\mathcal{A}})$ with $\bar{\theta}^s(q) = \sum_{a\in\mathcal{A}} \bar{\theta}_a(q)$. This argument can then be repeated until we reach the root node of $\mathcal{N}$.

This process generates $\bar{\mu}$ and values of $(\bar{u}, \bar{x}, \bar{\alpha}, \bar{\pi})$ that satisfy (14c), (14d), (14e) and (14f) for every $a \in \mathcal{A}$ since they are solutions to $\mathrm{SE}(\mathcal{D}_s)$. Furthermore, for each $a \in \mathcal{A}$, defining $\bar{\gamma}_a^k(n) = \bar{\phi}_a^k(n)$ and $Z_a(n; \bar{u}, \bar{x}, \bar{W}) = Z_a^s(n)$ from the solutions of $\mathrm{TOC}(n, \mathcal{D}_{\mathcal{A}})$, it follows from the definition of $\mathrm{TOC}(n, \mathcal{D}_{\mathcal{A}})$ that (14a), (14b) and (14g) are also satisfied. Thus we have constructed solutions for each problem $\mathrm{AE}_a(\bar{\pi}, \bar{\alpha}, \bar{\mu}, \mathcal{D}_a)$.

Since for each $n \in \mathcal{N} \setminus \mathcal{L}$, $\mathrm{TOC}(n, \mathcal{D}_{\mathcal{A}})$ includes the condition that

$$0 \le \bar{\mu}(m) \perp -\sum_{a\in\mathcal{A}} \bar{W}_a(m) \ge 0, \quad m \in n_+,$$

it follows that (17) holds. The final conditions (15) and (16) follow as they are part of the original solution of $\mathrm{SE}(\mathcal{D}_s)$. $\square$