Deterministic vs. Stochastic Settlement Approaches to Market Clearing Mechanisms under Demand Uncertainty

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June 5, 2013

Abstract

Electricity markets face a substantial amount of uncertainty. Traditionally this uncertainty has been due to varying demand. With the integration of larger proportions of volatile renewable energy, this added uncertainty from generation must also be faced. Conventional electricity market designs cope with uncertainty by running two markets: a day ahead or pre-dispatch market that is cleared ahead of time, followed by a real-time balancing market to reconcile actual realizations of demand and available generation. In such markets, the day ahead market clearing process does not take into account the distribution of outcomes in the balancing market. Recently an alternative so-called stochastic settlement market has been proposed (see e.g. Pritchard et al. [5] and Bouffard et al. [2]). In such a market, the ISO co-optimizes pre-dispatch and spot in one single settlement market.

In this paper we consider simplified models for three types of market clearing mechanisms. We demonstrate that under the assumption of symmetry, our simplified stochastic programming market clearing is equivalent to a two period single settlement (TS) system that takes count of deviation penalties in the second stage. These however differ from a TS model that dispenses with deviation penalties and has been (and continues to be) in use in New Zealand (NZTS). Our models are targeted towards analyzing imperfectly competitive markets. We will construct Nash equilibria of the resulting games for the introduced market clearing mechanisms and compare them under the assumptions of symmetry and in an asymmetric example.

1 Introduction

Electricity markets face two key features that set them apart from other markets. The first is that electricity cannot be stored, so demand must equal supply at all times. This is particularly problematic given that demand for electricity is usually uncertain. Second, electricity is transported from suppliers to load over a transmission network with possible constraints. The combination of these two features means that in almost all electricity markets today an Independent System Operator (ISO) sets dispatch centrally and clears the market. Generators and demand-side users can make offers and bids, and the ISO will choose which are accepted according to a pre-determined settlement system.

The classic settlement system used in almost all existing electricity markets is one where the ISO sets dispatch to maximize social welfare. Effectively the ISO matches supply to meet (the uncertain) demand at every moment while maximizing welfare. This becomes particularly difficult in the short-run (up to 24 hours before actual market clearing) as some types of generator (e.g. steam turbines and to some extent gas turbines) need to ramp up their generation slowly, and it is costly to change their output rapidly. Different markets have approached this problem in different ways.

One common approach used is to run a deterministic two-settlement model. In the first period, usually run about 24 hours before clearing, generators make offers, and the ISO chooses a pre-dispatch. (Note that in the remainder of this paper, we use the terms generator and firm interchangeably.) This first market is run based on an estimate of what demand is expected to be, then a second 'balancing market' is run soon before the market actually clears. In this second market, new sets of offers are submitted and upon market clearing the dispatches can deviate from pre-dispatch levels. Both period's markets are based on that described above - maximizing social welfare, but they are run separately, and the result of one is not tied to the other. However the two markets are financially binding (hence the term two settlement). The results of the first pertain to pre-dispatch prices and quantities, while the results of the second are used for balancing prices and quantities.

Another option is used in New Zealand. Here generators can place offers for a given half hour period up to two hours prior to a designated period. During the actual half hour, the ISO will then run an optimization problem every five minutes, using the same bids each time, to figure out dispatch. Any generator may then be asked to deviate at 5 minutes notice. *Note that in this case, the same offer curves are used in the pre-dispatch phase as well as the actual half hour in question.* The predispatch market in New Zealand provides generators with an idea of what they might expect to be producing but it is not financially binding. Hence there is only a single settlement in the New Zealand system. In this deterministic two period single-settlement (NZTS) market, expected demand is used to clear the pre-dispatch quantities and the ISO has no explicit measure of any deviation costs for a generator.

An alternative to deterministic settlement systems is to use a stochastic settlement process. In a stochastic settlement, the ISO can choose both pre-dispatch and short-run deviations for each generator to maximize expected social welfare in one step. By co-optimizing both together, we might expect a stochastic settlement system to do better (on average) than two deterministic settlements. The idea of a stochastic settlement can be attributed to Bouffard et al., Wong and Fuller, and Pritchard et al. [2, 5, 6]. In these two-stage, single settlement models, the pre-dispatch clears with information about the future distribution of uncertainties in the system (e.g. demand and volatile renewable generation,) and information about deviation costs for each generator. These models assume that each firms' offers and deviation costs are truthful. In an imperfectly competitive market, this assumption is not valid. The question then remains: can the stochastic settlement auction give better expected social welfare when firms are behaving strategically? That is the question explored by this paper.

We start by introducing a simplified version of the NZTS market currently operated in New

Zealand. We will then introduce a simplified version of the stochastic programming mechanism for clearing electricity markets. We will establish that the stochastic program reduces to a two period single settlement model slightly different from the NZTS model were deviation penalties are explicitly considered by the ISO. We refer to this market clearing mechanism as ISOSP. We will present results pertaining to the existence of equilibria for the simplified NZTS and derive an analytical expression for a symmetric equilibrium. We then establish the key result that reduces the simplified stochastic market clearing mechanism to a NZTS type model, but with explicit deviation penalties. Here again we construct analytical expressions for symmetric equilibria. Finally we compare the symmetric equilibria of NZTS and ISOSP settlements and show that the ISOSP settlement with explicit deviation costs performs better in terms of expected social welfare. Section 6 concludes the paper.

2 The Market Environment

In this paper, we aim to compare different market designs for electricity. We begin by presenting assumptions that are common to all markets we consider, features of what we call the *market environment*. These include such considerations as the number of firms, the costs firms face, the structure of demand and so forth.

Assumption 2.1 *The market environment may be defined by the following features.*

- *Electricity is traded over a network with no transmission constraints and no line losses, thus we may consider all trading as taking place at a single node.* [1]
- *Demand for electricity is uncertain, and may realize in one of* $s \in \{1, \cdots, S\}$ *possible outcomes (scenarios), each with probability* θ_s. *Demand in state* s *is assumed to be linear, and defined by the inverse demand function* $p_s = Y_s - ZQ$, *where* Q *is the quantity of electricity and* p_s *is the market price in scenario* s. *Without loss of generality, assume* $Y_1 < Y_2 < ... < Y_S$. *We will denote the expected value of* Y_s *by* $Y = \sum_s \theta_s Y_s$.
- *There are* n *symmetric firms wishing to sell electricity. (In the penultimate section we dispense with the symmetry assumption.)*
- *For a given firm* i *in scenario* s, *we will denote the pre-dispatch quantity by* q_i, *and any short-run* change *in production by* $x_{i,s}$. *Thus a generator's actual production in scenario* s *is equal to* $q_i + x_{i,s}$, *which we denote by* $y_{i,s}$.
- *Each firm* i*'s* long-run *cost function is* $\alpha q_i + \frac{\beta}{2} q_i^2$, *where* q_i *is the quantity produced by firm* i, *and* $\beta > 0$.
- *Each firm's* short-run *cost function is* $\alpha\left(q_i + x_{i,s}\right) + \frac{\beta}{2}\left(q_i + x_{i,s}\right)^2 + \frac{\delta}{2} x_{i,s}^2$, *where* q_i *is the long-run expected dispatch of firm* i, *and* $q_i + x_{i,s}$ *is the actual short-run dispatch and* $\delta > 0$.

[1] This assumption may also shed light on any scenario where line capacities do not bind, even if in other scenarios they do bind.

- *As minimum marginal cost of generation should not be more than maximum price of electricity, we assume*

$$\alpha \le Y_s \qquad \forall s \in \{1, \ldots, S\}.$$

- *There is an Independent System Operator (ISO) who takes bids and determines dispatch and prices according to the given market design.*

- *All the above assumptions are common knowledge to all participants in the market.*

Our assumptions on generators' cost functions are particularly critical to the analysis that follows, and deserve further explanation. Generators face two distinct costs when generating electricity. If given sufficient advance notice of the quantity they are to dispatch, the generator can plan the allocation of turbines to produce that quantity most efficiently. This is what we mean by a *long-run* cost function. The interpretation of this is the lowest possible cost at which a generator can produce quantity q. In electricity markets, however, demand fluctuates at short notice, and the ISO may ask a generator to change its dispatch at short notice. In this case, generators may not have enough time to efficiently reallocate its turbines. For example, many thermal turbines take hours to ramp-up. Most likely, the generator will have to adopt a less efficient production method, such as running some turbines above their rated capacity which also increases the wear on the turbines. Thus there is some inherent cost in deviating from an expected pre-dispatch in the short-run. This cost can be incurred even if the requested deviation is negative. We assume that the generator will be unable to revert to the most efficient mode of producing this quantity $q_{i,s} + x_{i,s}$ in the short-run, so pays a penalty cost. Note that this imposes a positive penalty cost upon the generator for making the short-run change, even if the change is negative. This penalty cost is additively imposed on top of the 'efficient' cost of producing at the new level. We call this cost the *deviation cost*. Note that we assume the symmetric case in which cost of generation and deviation is determined through the same constant parameters (α, β, δ).

Our goal is to compare the outcomes of different markets imposed upon this environment. To be able to draw comparisons in different paradigms, we need to examine the steady state behaviour of participants under the different market clearing mechanisms. To this end, we need to compute equilibria arising under the different market clearing mechanisms. In order to make the computations tractable, we will restrict the firms to offer linear supply functions in the following sections of this paper.

3 Deterministic Two Period Settlement (NZTS) Model

In this section we will introduce a deterministic two period market which is inspired by the market clearing mechanism as it operates currently in New Zealand. In the New Zealand market, firms bid a step supply function for a given half hour period. The bid is made at least two hours in advance. The market will then be cleared six times, every five minutes during the given half hour period.[2] We simplify the situation by assuming the market clears only twice; once after the bids are submitted, but before demand is realized. This we call the 'pre-dispatch settlement' which tells the generators approximately how much they should produce. Once demand is realized, the same bids will be used to determine actual dispatch in what we call the 'spot settlement'. The difference between pre-dispatch and spot dispatch is a generator's short-run deviation, which is subject to potentially higher costs as we described earlier however the ISO has no knowledge of this cost and it is not explicitly stated in the generators' bids. This cost can be indirectly reflected in the supply functions the generators bid in.

3.1 Mathematical Model

Our simplified mathematical model for the NZTS market has two distinct stages; pre-dispatch and spot. Each generator i bids a supply function $a_i + b_i q_i$ before the pre-dispatch market to represent their quadratic costs. At this point, demand is uncertain. The ISO will then use the generator's bid twice: once to clear the pre-dispatch market, and once again after demand is realized to clear the spot market. The pre-dispatch market determines the pre-dispatch quantities each generator is asked to dispatch, and the spot market determines the final quantities the generators are asked to dispatch. As in reality, in both the pre-dispatch and spot markets, the ISO aims to maximize social welfare, assuming generators are bidding their true cost functions. Since demand is unknown in pre-dispatch, the ISO will nominate (and use) an expected demand (and will not consider the distribution of demand).

$$\begin{array}{rcll} \min z & = & \sum_{i=1}^{n} \left(a_i q_i + \frac{b_i}{2} q_i^2\right) - \left(YQ - \frac{Z}{2}Q^2\right) & \\ s.t. & & \sum_i q_i - Q = 0 & [f] \end{array} \tag{1}$$

From this first settlement, the ISO can extract a forward price f equal to the shadow price on the (expected demand balance) constraint. Recall that the pre-dispatch quantity for generator i is denoted by q_i. After pre-dispatch is determined, true demand is realized, and the ISO then clears the spot market (using the specific demand scenario that has been realized) to maximize welfare by solving (2).

$$\begin{array}{rcll} \min z & = & \sum_{i=1}^{n} \left(a_i y_{i,s} + \frac{b_i}{2} y_{i,s}^2\right) - \left(Y_s C_s - \frac{Z}{2}C_s^2\right) & \\ s.t. & & \sum_i y_{i,s} - C_s = 0 & [p_s] \end{array} \tag{2}$$

Here again the ISO computes a spot price p_s as the shadow price on the constraint. (Note that we can eliminate the constraint and substitute C_s in the objective, however imposing this constraint enables the easy introduction of the price as the shadow price attached to the constraint.) The generator is not permitted to change its bid after pre-dispatch, but does face the usual additional deviation cost δ for its short-run deviation.

[2]Within this five minute period a frequency keeping generator will match any small changes in demand. We ignore this aspect of the market, as frequency-keeping is purchased through a separate market and until recently was procured through annual contracts.

Note that in both ISO optimization problems (1, 2) we have dispensed with non-negativity constraints on the pre-dispatch and dispatch quantity respectively. We will demonstrate that the resulting equilibria of our NZTS market model will always have associated non-negative pre-dispatch and dispatch quantities. We have eliminated the non-negativity constraints following the convention of supply function equilibrium models (see e.g. [4, 1]) in order to enable the analytic computation of equilibrium supply offers.

In the last section of this paper, we return to this point, enforce non-negativity constraints, and present numerical experiments where the equilibrium offers (and associated dispatch quantities,) are computed using global optimization techniques.

Firm i's profit in scenario s in this market is then given by

$$u_{i,s}^{TS}\left(q_i, x_{i,s}\right) = fq_i + p_s\left(y_{i,s} - q_i\right) - \left(\alpha y_{i,s} + \frac{\beta}{2}y_{i,s}^2 + \frac{\delta}{2}\left(y_{i,s} - q_i\right)^2\right). \tag{3}$$

3.2 Equilibrium Analysis of the Deterministic Two Period Market

In this section we will present equilibria of the NZTS market model. We will first compute the optimal dispatch quantities from the ISO's optimal dispatch problems (1) and (2) for any number of players. We will then embed these quantities in the generator's expected profit function and allow the generators to simultaneously optimize over their (linear) supply function parameters to obtain equilibrium offers.

Proposition 3.1 *Problem (1) is a convex program with a strictly convex objective. Its unique optimal solution and the corresponding optimal dual f are given by*

$$\begin{aligned} f &= \frac{Y + ZA}{ZB + 1} \\ q_i &= fB_i - A_i \end{aligned}$$

where $A_i = \frac{a_i}{b_i}$, $B_i = \frac{1}{b_i}$, $A = \sum_i A_i$ and $B = \sum_i B_i$.

Proof Note that problem (1) has a single linear constraint and that its objective is a strictly convex quadratic as we have assumed that $b_i > 0$ and $Z > 0$. The problem therefore has a unique optimal solution delivered by the first order conditions provided below.

$$Q - \sum_i q_i = 0 \tag{4}$$

$$f - Y + ZQ = 0 \tag{5}$$

$$-f + a_i + b_i q_i = 0 \quad \forall i \tag{6}$$

Using equation (5) we can rewrite equation (6) as

$$Y - ZQ = a_i + b_i q_i \quad \forall i \tag{7}$$

Now summing over all i we obtain

$$\sum_i q_i = (\sum_i \frac{1}{b_i})(Y - ZQ) - (\sum_i \frac{a_i}{b_i}) \tag{8}$$

Note that $B = \sum_i \frac{1}{b_i}$ and $A = \sum_i A_i$. This together with equations (8) and (4) yields

$$Q = \frac{BY - A}{ZB + 1}.$$

Now substituting Q from the above into equation (5), we obtain

$$f = \frac{Y + ZA}{ZB + 1}.$$

Similarly substituting Q into (7) yields

$$q_i = B_i(Y - Z\frac{BY - A}{ZB + 1}) - A_i.$$

This equation simplifies to

$$q_i = fB_i - A_i,$$

and we obtain the expressions in the statement of the proposition.

Proposition 3.2 *For each scenario s, problem (2) is a convex program with a strictly convex objective. Its unique optimal solution and the corresponding optimal dual p_s, are given by*

$$\begin{aligned} p_s &= \frac{Y_s + ZA}{ZB + 1} \\ y_{i,s} &= p_s B_i - A_i \end{aligned}$$

where A_i, B_i, A and B are defined above in proposition (3.1).

Proof Note that problems (2) and (1) are structurally identical, therefore the simple proof of proposition (3.1) applies again here.

Remark Note from the above that the pre-dispatch price (and quantity) are equal to the expected spot market prices (and quantities respectively). That is

$$f = \sum_{s=1}^{S} \theta_s p_s. \tag{9}$$

We will now compute the linear supply functions resulting from the equilibrium of the TS market game laid out in (2.1). Before we begin with the firm computations, we will establish a technical lemma that we utilize in establishing the equilibrium results.

Lemma 3.3 *Assume that function $f(x,y): R^2 \to R$ is defined on a $D_x \times D_y$ with $D_x, D_y \subseteq R$. Furthermore assume that $x^*(y) \in D_x$, maximizes $f(x,y)$ for any arbitrary but fixed y. Also assume $g(y) = f(x^*(y), y)$ is maximized at $y^* \in D_y$. Then, $f(x,y)$ is maximized at $(x^*(y^*), y^*)$.*

Proof Note that for any $(x,y) \in D_x \times D_y$,

$$f(x,y) \leq f(x^*(y), y)$$

by the assumption on $x^*(y) \in D_x$. Furthermore $f(x^*(y), y) \leq f(x^*(y^*), y^*)$. Clearly then

$$f(x,y) \leq f(x^*(y^*), y^*) \quad \text{for any } (x,y) \in D_x \times D_y.$$

3.2.1 Firm i's computations

In this section we will focus on firm i's expected profit function. Note that using equation (9) we obtain

$$u_i^{TS} = E_s[u_{i,s}^{TS}] \;=\; \sum_{s=1}^{S} \theta_s \left(p_s y_{i,s} - \left(\alpha y_{i,s} + \frac{\beta}{2} y_{i,s}^2 + \frac{\delta}{2} \left(y_{i,s} - q_i \right)^2 \right) \right).$$

Using propositions (3.1) and (3.2), we can re-write u_i^{TS} as a function of a_i and b_i. In order to find a maximum of u_i^{TS} (for a fixed set of competitor offers) we appeal to a transformation that will yield concavity results for u_i^{TS}. We consider u_i^{TS} to be a function of A_i and B_i (instead of a_i and b_i). Note that the transformation ($A_i = \frac{a_i}{b_i}, \quad B_i = \frac{1}{b_i}$) is a one-to-one transformation.

Proposition 3.4 *Let all competitor (linear) supply function offers be fixed. The following maximizes u_i^{TS} (and is therefore firm i's best response).*

$$\begin{aligned} B_i &= \frac{1 + ZB_{-i}}{Z + \beta + \delta + Z(\beta + \delta)B_{-i}} \\ A_i &= \frac{\alpha + B_i \left(Z\alpha - \delta \left(Y + ZA_{-i} \right) \right) + Z\alpha B_{-i}}{2Z + \beta + Z\beta B_{-i}} \end{aligned}$$

Proof We can show that u_i^{TS} is a concave function of A_i, assuming B_i is a fixed parameter. Here we have dispensed with the expression for u_i^{TS} as a function of A_i and B_i as it is long and rather complicated. This expression can be found in the online technical companion [3].

We note that u_i^{TS} is a smooth function of A_i and B_i. Let $A_{-i} = \sum_{j \neq i} A_j$ and $B_{-i} = \sum_{j \neq i} B_j$. Then

$$\frac{\partial^2 u_i^{TS}}{\partial {A_i}^2} = -\frac{\left(1 + ZB_{-i}\right)\left(2Z + \beta + Z\beta B_{-i}\right)}{\left(1 + ZB\right)^2} \leq 0$$

Let B_i be arbitrary but fixed. As u_i^{TS} is a concave function of A_i the first order condition yields an expression for $A_i^*(B_i)$, the value of A_i that maximizes u_i^{TS} (for the fixed B_i).

$$A_i^*(B_i) = \frac{(1+ZB_{-i})(-Y+\alpha-ZA_{-i}+Z\alpha B_{-i})+B_i(Z(Y+ZA_{-i})+(Z\alpha+\beta Y+Z\beta A_{-i})(ZB_{-i}+1))}{(1+ZB_{-i})(2Z+\beta+Z\beta B_{-i})}.$$

We can embed $A_i^*(B_i)$ into u_i^{TS} and find the maximizer in terms of B_i. Lemma (3.3) then can be applied to demonstrate that the end result delivers the maximum of u_i^{TS}.

After embedding this value of A_i^* into the profit function, the derivative with respect to B_i of u_i^{TS} is.

$$\frac{du_i}{dB_i} = \frac{\left(Y^2 - \sum_s \theta_s Y_s^2\right)\left(-1 + (Z+\beta+\delta)B_i + Z\left(-1 + (\beta+\delta)B_i\right)B_{-i}\right)}{(1+ZB)^3}$$

$B_i^* = \frac{1+ZB_{-i}}{Z+\beta+\delta+Z(\beta+\delta)B_{-i}}$, is the zero of this derivative. Recall that $Y = \sum_s \theta_s Y_s$, therefore Jensen's inequality implies $Y^2 - \sum_s \theta_s Y_s^2 \leq 0$. Thus, $\frac{du_i}{dB_i} \geq 0$, when $B_i < B_i{}^*$, and $\frac{du_i}{dB_i} \leq 0$, when $B_i > B_i^*$. In other words, u_i is a quasi-concave function of B_i and is maximized at $B_i = B_i{}^*$.

Note that evaluating A_i^* at B_i^* yields

$$A_i^* = \frac{\alpha + B_i\left(Z\alpha - \delta\left(Y + ZA_{-i}\right)\right) + Z\alpha B_{-i}}{2Z + \beta + Z\beta B_{-i}}.$$

From the above, we can obtain the equilibrium of the NZTS model by solving all best responses simultaneously. This gives the unique and symmetric solution

$$\text{2S-EQM: } B_i = \frac{2}{-(n-2)Z + \beta + \delta + \sqrt{(n-2)^2Z^2 + 2nZ(\beta+\delta) + (\beta+\delta)^2}} \tag{10}$$

$$A_i = \frac{\alpha + (nZ\alpha - Y\delta)B_i}{2Z + \beta + (n-1)Z(\beta+\delta)B_i}, \tag{11}$$

or alternatively

$$\text{2S-EQM: } b_i = \frac{-(n-2)Z + \beta + \delta + \sqrt{(n-2)^2Z^2 + 2nZ(\beta+\delta) + (\beta+\delta)^2}}{2} \tag{12}$$

$$a_i = \frac{\alpha b_i + (nZ\alpha - Y\delta)}{2Zb_i + \beta b_i + (n-1)Z(\beta+\delta)}, \tag{13}$$

As we discussed earlier, these equilibrium offers yield non-negative pre-dispatch and dispatch quantities. Below we formally state this result, however the computations to show the non-negativity of these quantities can be found in the technical companion [3].

Proposition 3.5 *The equilibrium pre-dispatch and spot production quantities of the firms in the NZTS market are non-negative, i.e.*

$$\begin{aligned} q_i &\geq 0 && \forall i, \\ y_{i,s} &\geq 0 && \forall i, s. \end{aligned}$$

4 Stochastic Settlement Market

4.1 ISOSP Model

We now introduce the market model we will use to analyze a stochastic settlement market. As discussed in the introduction, the stochastic settlement market contains only a single stage of bidding, but the market clearing procedure takes into account the distribution of future demand when determining dispatch. The market works as follows. When the market opens, demand is uncertain. Firms are allowed to bid their 'normal' cost functions (the cost of producing a given output most efficiently) and a 'penalty' cost function that they would need to be paid to deviate in the short-run. Since firms have quadratic cost functions, they can bid their actual costs by submitting a linear supply function. Each firm i chooses a_i and b_i to bid the linear supply function $a_i + b_i q$, and d_i to bid the (marginal) penalty cost $d_i q$. Note that as with the NZTS model, these bids (a_i, b_i, d_i) need not be their true values (α, β, δ). The offered b_i is required to be positive and d_i should be non-negative.

After generators have placed their bids, the ISO computes the market dispatch according to the stochastic settlement model (outlined below). At this point demand is still uncertain. The ISO chooses two key variables. The first is the *pre-dispatch* quantity for each firm. This is the quantity the ISO asks each firm to prepare to produce, namely the pre-dispatch quantities q_i defined in Section 2. The second is the *short-run deviation* for generator i under each scenario s. This deviation is the variable $x_{i,s}$ defined in Section 2, representing the adjustment made to firm i's predispatch quantity in scenario s. The ISO can choose both pre-dispatch and short-run deviations simultaneously, while aiming to maximize expected social welfare. The ISO assumes that generators have bid their true costs.

In the final stage, demand is realized, and the ISO will ask generators to modify their pre-dispatch quantity according to the short-run deviation for the particular scenario. Each generator ends up with producing $q_i + x_{i,s}$. Two prices are calculated during the course of optimizing welfare. The first is the (shadow) price of the pre-dispatch quantities. We will denote this by f. The second are the prices of each of the deviations, for each of the scenarios. We will denote these by p_s for scenario s. Each generator is paid f per unit for its pre-dispatch quantity q_i, and p_s for its deviations $x_{i,s}$. Thus in realization s, generator i makes profit equal to

$$u_{i,s}^{SS}\left(q_i, x_{i,s}\right) = f q_i + p_s x_{i,s} - \alpha\left(q_{i,s} + x_{i,s}\right) + \frac{\beta}{2}\left(q_{i,s} + x_{i,s}\right)^2 + \frac{\delta}{2} x_{i,s}^2. \tag{14}$$

Mathematically, the stochastic optimization problem solved by the ISO can be represented as follows. [3]

ISOSP:

[3] This is a modified version of Pritchard et al.'s problem. There is only one node and thus no transmission constraints, and demand is elastic.

$$
\begin{array}{rcll}
\min z & = & \sum_{s=1}^{S} \theta_s \left(\sum_{i=1}^{n} \left[a_i(q_i + x_{i,s}) + \frac{b_i}{2}(q_i + x_{i,s})^2 + \frac{d_i}{2} x_{i,s}^2 \right] - (Y_s C_s - \frac{Z}{2} C_s^2) \right) & \\
s.t. & & \sum_i q_i - Q = 0 & [f] \\
 & & Q + \sum_i x_{i,s} - C_s = 0 \quad \forall s \in \{1, \ldots, S\} & [p_s]
\end{array}
$$

Q and C_s stand for the total contracted (or pre-dispatched) quantity and total consumption in scenario s respectively. Note that we could have eliminated the two equality constraints. However, their dual variables are the market prices f and p_s respectively, so for clarity we have left them in.

4.2 Characteristics of the Stochastic Optimization Problem

We begin by presenting a series of results that simplify the set of solutions to the ISOSP problem. These results drastically simplify the subsequent analysis of firms' behaviour in equilibrium under stochastic settlement.

Lemma 4.1 *In the stochastic settlement market clearing, the expected deviation of firm i from pre-dispatch quantity q_i^* is zero, that is, the optimal solution to ISOSP will always satisfy*

$$\sum_s \theta_s x_{i,s}^* = 0.$$

Proof Let us assume Q_i^* and $x_{i,s}^*$ form ISOSP's optimal solution. Let us define for each i and s the quantity $k_{i,s} := q_i^* + x_{i,s}^*$, the total production of firm i in scenario s. Note that $C_s = \sum_i q_i^* + \sum_i x_{i,s}^*$. Assume, on the contrary, that there exists at least one firm j such that $\sum_s \theta_s x_{j,s}^* \neq 0$. The optimal objective value of ISOSP is then given by

$$\sum_i \sum_s \theta_s (a_i k_{i,s} + \frac{b_i}{2}(k_{i,s})^2) + \sum_i \sum_s \theta_s \frac{d_i}{2}(x_{i,s}^*)^2 + Y_s \sum_i k_{i,s} - \frac{Z}{2}(\sum_i k_{i,s})^2. \tag{15}$$

Note that as $\sum_s \theta_s x_{j,s}^* \neq 0$, the term $\sum_i \sum_s \theta_s \frac{d_i}{2}(x_{i,s}^*)^2$ is positive. Now, for a fixed i and $k_{i,s}$ given from above, consider the problem

$$
\begin{array}{rcl}
\min\limits_{q_i, x_{i,s}} w & = & \dfrac{d_i}{2} \sum\limits_{s=1}^{S} \theta_s x_{i,s}^2 \\
\forall s : q_i + x_{i,s} & = & k_{i,s}.
\end{array} \tag{16}
$$

This problem clearly reduces to the univariate problem

$$\min_{q_i} w = \sum_{s=1}^{S} \theta_s (k_{i,s} - q_i)^2,$$

which is optimized at

$$q_i = \sum_{s=1}^{S} \theta_s k_{i,s}.$$

Define $\hat{q_i}$ and $\hat{x_{i,s}}$ by

$$\hat{q_i} = \begin{cases} q_i^*, & i \neq j \\ \sum_{s=1}^{S} \theta_s k_{j,s} & \text{otherwise,} \end{cases}$$

and

$$\hat{x_{i,s}} = \begin{cases} x_{i,s}^*, & i \neq j \\ k_{j,s} - \hat{q_j} & \text{otherwise.} \end{cases}$$

By definition, $\hat{q_i} + \hat{x_{i,s}} = q_i^* + x_{i,s}^*$ for all i and s. It is easy to see that the quantities $\hat{q_i}$ and $\hat{x_{i,s}}$ yield a feasible solution to ISOSP. Furthermore, the objective function evaluated at $\hat{q_i}$ and $\hat{x_{i,s}}$ is given by

$$\sum_i \sum_s \theta_s (a_i k_{i,s} + \frac{b_i}{2}(k_{i,s})^2) + Y_s \sum_i k_{i,s} - \frac{Z}{2}(\sum_i k_{i,s})^2.$$

This value is strictly less than the objective evaluated at q_i^* and $x_{i,s}^*$ (given by 15), as we have already established that $\sum_i \sum_s \theta_s \frac{d_i}{2}(x_{i,s}^*)^2 > 0$. This yields the contradiction that proves the result.

Corollary 4.2 *In the stochastic problem ISOSP, if $q_i^* + x_{i,s}^* \geq 0$ is satisfied $\forall s \in \{1, \ldots, S\}$ then $q_i^* \geq 0$ will hold.*

Proof In lemma 4.1 we established that $\sum_s \theta_s x_{i,s}^* = 0$. Therefore there exists a scenario s' such that $x_{i,s'}^* \leq 0$. clearly then $q_i^* + x_{i,s'}^* \geq 0$ implies $q_i^* \geq 0$.

Discussion Lemma 4.1 is the crucial result that drives the rest of our characterizations. This results hinges on the fact that we penalize quadratic deviation from the pre-dispatch quantity. This model penalizes the deviations upward and downward identically. Therefore the predispatch point is optimized based on the mean demand scenario. The reader may argue that allowing for different upward and downward penalties is more realistic. However as Pritchard et. al. (see [5]) show, such allowance of asymmetric penalties can lead to systematic arbitrage by the ISO, where a generator may be required to deviate upward "in every scenario" simply to increase expected welfare. This is undesirable for a market clearing mechanism. We have therefore confined our attention to the symmetric upward and downward penalty case for this paper, which guarantees systematic arbitrage will not occur. We now use the above results and intuition to prove that the ISO's optimization problem can be viewed as a deterministic two period settlement system where unlike NZTS, the deviation penalties are explicitly stated in the ISO's problem in the second period.

Lemma 4.3 *Problem ISOSP is equivalent to the following optimization problem which is separable in the pre-dispatch and the spot market variables*

$$
\begin{aligned}
z &= \sum_{i=1}^{n}\left(a_i q_i + \frac{b_i}{2} q_i^2\right) - Y\sum_{i=1}^{n} q_i + \frac{Z}{2}\left(\sum_{i=1}^{n} q_i\right)^2 \\
&\quad + \sum_{i=1}^{n}\left(\frac{b_i+d_i}{2}\sum_{s=1}^{S}\theta_s x_{i,s}^2\right) - \sum_{i=1}^{n}\sum_{s=1}^{S}\theta_s Y_s x_{i,s} + \frac{Z}{2}\sum_{s=1}^{S}\theta_s\left(\sum_{i=1}^{n} x_{i,s}\right)^2 .
\end{aligned}
$$

Proof Substituting for C_s from constraints into the objective function of ISOSP yield

$$
\begin{aligned}
z &= \sum_{s=1}^{S}\theta_s\Bigg(\sum_{i=1}^{n}\left(a_i\left(q_i + x_{i,s}\right) + \frac{b_i}{2}\left(q_i + x_{i,s}\right)^2 + \frac{d_i}{2}x_{i,s}^2\right) \\
&\quad - Y_s\sum_{i=1}^{n}\left(q_i + x_{i,s}\right) + \frac{Z}{2}\left(\sum_{i=1}^{n}\left(q_i + x_{i,s}\right)\right)^2\Bigg)
\end{aligned}
$$

Rearranging the above we obtain:

$$
\begin{aligned}
z &= \sum_{i=1}^{n}\left(a_i q_i + \frac{b_i}{2} q_i^2\right) - Y\sum_{i=1}^{n} q_i + \frac{Z}{2}\left(\sum_{i=1}^{n} q_i\right)^2 \\
&\quad + \sum_{i=1}^{n}\left(a_i\sum_{s=1}^{S}\theta_s x_{i,s}\right) + \sum_{i=1}^{n}\left(\frac{b_i+d_i}{2}\sum_{s=1}^{S}\theta_s x_{i,s}^2\right) - \sum_{i=1}^{n}\sum_{s=1}^{S}\theta_s Y_s x_{i,s} + \frac{Z}{2}\sum_{s=1}^{S}\theta_s\left(\sum_{i=1}^{n} x_{i,s}\right)^2 \\
&\quad + \sum_{i=1}^{n}\left(q_i b_i\sum_{s=1}^{S}\theta_s x_{i,s}\right) + \sum_{s=1}^{S}\theta_s Z\sum_{i=1}^{n}\sum_{j=1}^{n} q_i x_{j,s}
\end{aligned}
$$

Note that the first part of the objective above is a function of pre-dispatch quantities q_i and the second only a function of the spot dispatches $x_{i,s}$. Furthermore, observe that in both of the terms in the third part, the factor $\sum_{s=1}^{S}\theta_s x_{j,s}$ appears. We can therefore appeal to lemma (4.1) and eliminate this last part. This completes the proof.

Note: We have therefore established that ISOSP reduces to a deterministic two period single settlement model very similar to NZTS but with penalties explicitly present in the second period.

The rest of this section is devoted to deriving explicit expressions for the solution of ISOSP. In the next section we will use these expressions to arrive at best response functions for the firms and subsequently in constructing an equilibrium for the stochastic market settlement. In order to simplify the equations and arrive at explicit solutions, we will transform the space of the parameters of ISOSP (firm decision variables). We will use the following transformation

$$H : \begin{pmatrix} \mathbb{R} \\ \mathbb{R}^+ - \{0\} \\ \mathbb{R}^+ \end{pmatrix} \to \begin{pmatrix} \mathbb{R} \\ \mathbb{R}^+ - \{0\} \\ \mathbb{R}^+ - \{0\} \end{pmatrix} \text{ that is one-to-one and onto.}$$

$$\begin{pmatrix} A_i \\ B_i \\ R_i \end{pmatrix} = H \begin{pmatrix} a_i \\ b_i \\ d_i \end{pmatrix} := \begin{pmatrix} a_i/b_i \\ 1/b_i \\ 1/(b_i + d_i) \end{pmatrix}$$

If we further define

$$A = \sum_i A_i, \quad B = \sum_i B_i \quad \text{and } R = \sum_i R_i,$$

ISOSP reduces to minimizing the following:

$$\begin{aligned} z \quad = \quad & \sum_{i=1}^{n} \left(\frac{A_i}{B_i} q_i + \frac{1}{2B_i} q_i^2 \right) - Y \sum_{i=1}^{n} q_i + \frac{Z}{2} \left(\sum_{i=1}^{n} q_i \right)^2 \\ & + \sum_{s=1}^{S} \theta_s \left[\sum_{i=1}^{n} \frac{1}{2R_i} x_{i,s}^2 - (Y_s - Y) \sum_{i=1}^{n} x_{i,s} + \frac{Z}{2} \left(\sum_{i=1}^{n} x_{i,s} \right)^2 \right]. \end{aligned}$$

Note as before (lemma (4.3)) that the above is separable in q_i's and $x_{i,s}$'s, we can therefore solve the two stages separately. Note also that the two problems are convex optimization problems therefore KKT conditions will readily produce the optimal solution (for derivation please refer to the technical companion [3]).

Proposition 4.4 *If (q, x, f, p) represents the solution of ISOSP, then we have*

$$\begin{aligned} q_i &= \frac{(Y + ZA)B_i}{1 + ZB} - A_i && (17) \\ x_{i,s} &= \frac{(Y_s - Y)R_i}{1 + ZR} && (18) \\ f &= \frac{Y + ZA}{1 + ZB} \\ p_s &= \frac{Y + ZA}{1 + ZB} + \frac{Y_s - Y}{1 + ZR} \end{aligned}$$

Observe from the expression for f that this forward price (paid on pre-dispatch quantities) is independent of any deviation costs in the spot market.

Corollary 4.5 *In the solution of ISOSP, forward price is equal to the expected spot market price.*

Proof This is simply observed from proposition 4.4.

The fact that contract price is equal to the expected spot market price, implies that there is no systematic arbitrage.

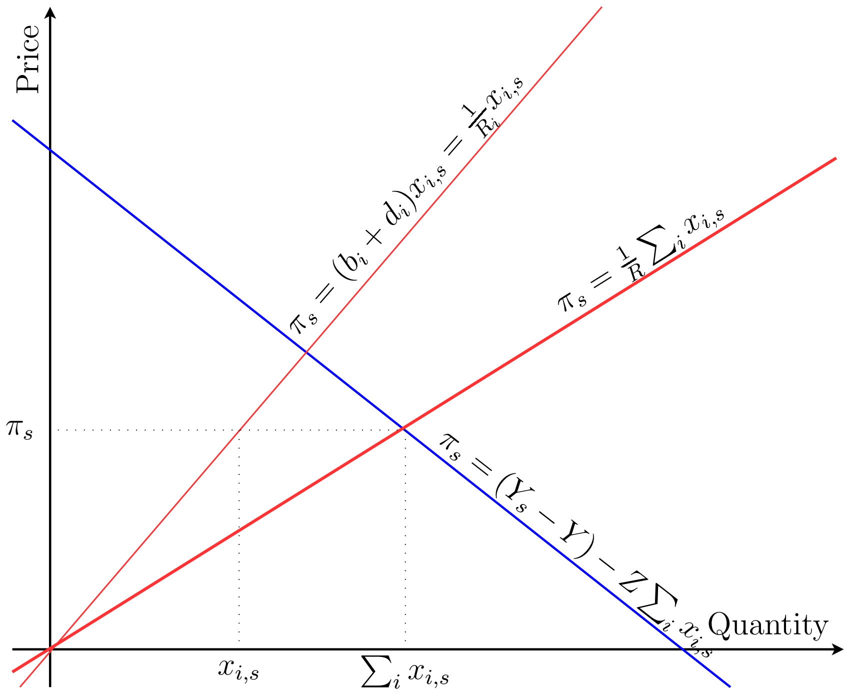

Figure 1: Market clearing of the spot market using firms' supply functions as an equivalent representation of ISOSP problem

4.3 Equilibrium Analysis of the Stochastic Settlement Market

In section (4.1) we presented firm i's profit under scenario s in equation (14). In our market model, we assume that all firms are risk neutral and therefore interested only in maximizing their expected profit. Firm i's expected profit is given by

$$u_i = fq_i + \sum_{s=1}^{S} \theta_s \left(p_s x_{i,s} - \left(\alpha(q_i + x_{i,s}) + \frac{\beta}{2}\ (q_i + x_{i,s})^2 + \frac{\delta}{2} x_{i,s}^2 \right) \right), \tag{19}$$

The above expression for u_i can be expanded and we can observe that

$$\begin{aligned} u_i \quad = \quad & fq_i - (\alpha q_i + \frac{\beta}{2} q_i^2) \\ & + \sum_{s=1}^{S} \theta_s \left(p_s x_{i,s} - \frac{\beta + \delta}{2} x_{i,s}^2 \right) \\ & - \alpha \sum_{s=1}^{S} \theta_s x_{i,s} - \beta q_i \sum_{s=1}^{S} \theta_s x_{i,s} \end{aligned}$$

Note that from lemma (4.1), the generator would know that for any admissible bid, the corresponding expected deviation from pre-dispatch quantities $\sum_{s=1}^{S} \theta_s x_{i,s} = 0$. Therefore the expected profit for the generator becomes

$$
\begin{aligned}
u_i &= f q_i - (\alpha q_i + \frac{\beta}{2} q_i^2) \\
&\quad + \sum_{s=1}^{S} \theta_s \left(p_s x_{i,s} - \frac{\beta+\delta}{2} x_{i,s}^2 \right).
\end{aligned}
$$

We can use the expressions obtained from proposition (4.4) to write u_i as follows.

$$
\begin{aligned}
u_i &= -\frac{1}{2}\beta A_i^2 + \frac{A_i\left(-ZA + \alpha + ZB\alpha + ZA\beta B_i + Y\left(-1+\beta B_i\right)\right)}{1+ZB} \\
&\quad + \frac{1}{2(1+ZB)^2(1+ZR)^2}\Bigg(\\
&\quad 2(1+ZR)^2(ZA+Y)(ZA+Y-(1+ZB)\alpha)B_i - (1+ZR)^2(ZA+Y)^2\beta B_i^2 \\
&\quad + (1+ZB)^2 R_i\left(-2+(\beta+\delta)R_i\right)\left(Y^2 - \sum_s \theta_s Y_s^2\right)\Bigg)
\end{aligned} \tag{20}
$$

Although this expression of the expected profit for the generator is rather ugly, it does have the advantage that upon differentiating with respect to R_i, all dependence on A_i and B_i drops and we are left with

$$
\frac{du_i}{dR_i} = \frac{\left(Y^2 - \sum_s \theta_s Y_s^2\right)\left(-1 + (Z+\beta+\delta)R_i + ZR_{-i}\left(-1+(\beta+\delta)R_i\right)\right)}{\left(1+ZR\right)^3}. \tag{21}
$$

Recall that $R_{-i} = \sum_{j\neq i} R_j$. (For verification of this derivative term see the technical companion [3].) The fact that this derivative is free of A_i and B_i indicates that u_i is separable in R_i and (A_i, B_i), that is

$$
u_i(A_i, B_i, R_i) = g_i(A_i, B_i) + h_i(R_i). \tag{22}
$$

Due to this natural separability, our equilibrium analysis will focus on finding best responses in terms of A_i, R_i and B_i very similar to the NZTS section.

Equation (22) enables us to maximize u_i by maximizing g_i and h_i over (A_i, B_i) and R_i respectively. This is helpful as we can establish quasi-concavity results for g_i and h_i separately.

We start our investigations by examining g_i. The full expression for g_i can be found in the technical companion [3]. Holding B_i fixed, note that

$$
\frac{d^2 g_i}{dA_i^2} = -\frac{\left(1+ZB_{-i}\right)\left(2Z+\beta+Z\beta B_{-i}\right)}{\left(1+ZB\right)^2}.
$$

This demonstrates that g_i is concave in A_i for any fixed B_i. Furthermore, for any fixed B_i, we can use the first order conditions to find $A_i^*(B_i)$, i.e. the value of A_i that maximizes $g_i(A_i, B_i)$ for the fixed B_i.

$$
A_i^*(B_i) = \frac{\left(1+ZB_{-i}\right)\left(\alpha - ZA_{-i} + Z\alpha B - Y\right) + \left(Y + ZA_{-i}\right)\left(Z+\beta+Z\beta B_{-i}\right)B_i}{\left(1+ZB_{-i}\right)\left(2Z+\beta+Z\beta B_{-i}\right)} \tag{23}
$$

To find the optimal value for g_i, we can now appeal to lemma (3.3) and substitute the expression for $A_i^*(B_i)$ in $g_i(A_i^*(B_i), B_i)$. Surprisingly, upon undertaking this substitution, it can be observed that $g_i(A_i^*(B_i), B_i)$ is a constant value. Figure 2 depicts g_i.

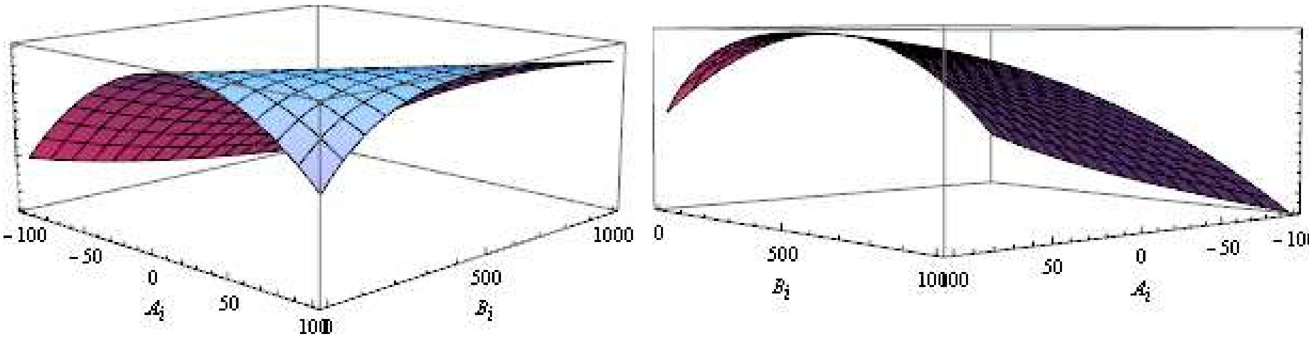

Figure 2: Two views of the function g_i. Note that the optimal value of g_i is obtained along a continuum, for any value of B_i.

To uncover the intuition behind this feature of g_i, we can offer the following mathematical explanation. We observe that

$$\begin{aligned}\frac{dg_i}{dA_i} &= \frac{-\left(1+ZB_{-i}\right)\left(Y-\alpha+ZA_{-i}+(2Z+\beta)A_i+ZB_{-i}\left(-\alpha+\beta A_i\right)\right)}{(1+ZB)^2} \\ &+\frac{\left(Z(Y+\alpha)+Y\beta+Y(Z\alpha+Y\beta)B_{-i}+ZA_{-i}\left(Z+\beta+Z\beta B_{-i}\right)\right)B_i}{(1+ZB)^2}\end{aligned}$$

and that

$$\frac{dg_i}{dB_i} = -\frac{Y+ZA}{1+ZB}.\frac{dg_i}{dA_i}.$$

Therefore, stationary conditions enforced in A_i will also imply stationarity in B_i.

As $g_i(A_i^*(B_i), B_i)$ is constant for any $B_i > 0$, for any value of $B_i > 0$, the tuple $(A_i^*(B_i), B_i)$ is an argmax of g_i for any positive B_i. The following analysis on h_i will explain how optimal R_i is constrained by the value of B_i.

Proposition 4.6 *Suppose that R_{-i} is fixed. Then h_i is optimized at*

$$R_i^* = \min\{B_i, \frac{1+ZR_{-i}}{Z+\beta+\alpha+Z(\beta+\delta)R_{-i}}\}$$

Proof Note that at

$$\hat{R}_i = \frac{1+ZR_{-i}}{Z+\beta+\alpha+Z(\beta+\delta)R_{-i}} \tag{24}$$

The derivative $\frac{dh_i}{dR_i} = \frac{du_i}{dR_i}$ vanishes. Also recall from Jensen's inequality that $Y^2 \leq \sum_s \theta_s Y_s^2$. It can therefore be seen from (21) that this derivative is positive for $R_i < \hat{R}_i$ and negative for $R_i > \hat{R}_i$. Recall further that the definition of B_i and R_i require $R_i \leq B_i$. Therefore, in optimizing h_i, we need to enforce this constraint and we obtain

$$R_i^* = \min\{B_i, \frac{1 + ZR_{-i}}{Z + \beta + \alpha + Z(\beta + \delta)R_{-i}}\}.$$

We now return to u_i, the expected profit function for firm i. As $u_i(A_i, B_i, R_i) = g_i(A_i, B_i) + h_i(R_i)$, we can start by obtaining the maximum value of g_i attained at a point $(A_i^*(B_i), B_i)$ for any positive B_i. Subsequently, we proceed to optimize $h_i(R_i)$. Proposition (4.6) readily delivers the optimal R_i. We have therefore proved the following theorem.

Theorem 4.7 *The best response of firm i, holding competitor offers fixed, is to offer in*

$$A_i = \frac{(1 + ZB_{-i})(\alpha - ZA_{-i} + Z\alpha B - Y) + (Y + ZA_{-i})(Z + \beta + Z\beta B_{-i}) B_i}{(1 + ZB_{-i})(2Z + \beta + Z\beta B_{-i})},$$

$$R_i = \frac{1 + ZR_{-i}}{Z + \beta + \alpha + Z(\beta + \delta)R_{-i}},$$

and any choice of B_i where

$$B_i \geq \frac{1 + ZR_{-i}}{Z + \beta + \alpha + Z(\beta + \delta)R_{-i}}.$$

Theorem 4.7 indicates that the game has multiple (infinite) symmetric equilibria. To prevent the problem of unpredictability, caused by multiple equilibria, from here on we assume that the ISO chooses B_i as a system parameter. This parameter is identical for and known to all participants. This also provides the ISO with the opportunity to choose B_i in a way to obtain a preferable equilibrium (i.e. an equilibrium that yields higher social welfare).

Proposition 4.8 *The unique symmetric equilibrium quantities of the stochastic settlement market are as follows.*

$$a_i = \frac{\alpha - Y + B_i\left(-Z(Y(n-2) - (2n-1)\alpha) + Y\beta + Z(n-1)(Zn\alpha + Y\beta)B_i\right)}{B_i\left(Z(n+1) + \beta + Y(n-1)(Zn + \beta)B_i\right)} \tag{25}$$

$$d_i = \max\{0, \frac{-Z(n-2) + \beta + \delta + \sqrt{Z^2(n-2)^2 + 2Zn(\beta+\delta) + (\beta+\delta)^2}}{2} - \frac{1}{B_i}\} \tag{26}$$

Proposition 4.9 *Let*

$$\hat{b} = \frac{-Z(n-2) + \beta + \delta + \sqrt{Z^2(n-2)^2 + 2Zn(\beta+\delta) + (\beta+\delta)^2}}{2}.$$

In a stochastic settlement market with $b_i \leq \hat{b}$ and large number of firms, firms tend to offer their true cost parameters. In other words,

$$\lim_{n\to\infty} a_i = \alpha$$
$$\lim_{n\to\infty} b_i + d_i = \beta + \delta.$$

When the fixed parameter b_i is chosen equal to β, $\lim_{n\to\infty} d_i = \delta$.

Proof The equations are simply derived from the equilibrium values of a_i and d_i given in proposition 4.8.

Proposition 4.9 shows that our market is behaving competitively in the sense that when number of firms increases, they offer their true cost parameters.

One important feature of the equilibrium values are the non-negativity of the pre-dispatch and dispatch. This is important, because we neglected the non-negativity constraints in ISOSP in the first place.

Theorem 4.10 *Let $(\mathbf{q}^*, \mathbf{x}^*)$ represent the equilibrium of the stochastic settlement market, then the following inequalities hold.*

$$\forall i, s : \ q_i^* + x_{i,s}^* \geq 0$$

$$\forall i : \ q_i^* \geq 0$$

The proof of the above theorem is contained in the technical companion.

Though, the equilibrium pre-dispatch and dispatch are non-negative, one might raise an objection that a game without the non-negativity constraints embedded in the ISO's optimization problem, is different from the original game. Therefore, there is no assurance the found equilibrium is also the equilibrium of the original game. The following theorem states that the obtained equilibrium values are also the equilibrium of the original game with non-negativity constraints. The proof of this theorem is quite lengthy and consists of several technical lemmas. This proof can be found in the technical companion [3].

Theorem 4.11 *The equilibrium of the symmetric stochastic settlement game without the non-negativity constraints in ISOSP is also the equilibrium of the stochastic settlement game with the non-negativity constraints.*

Thus far we established that under the assumption of symmetry, the stochastic settlement (ISOSP) market is equivalent to a two period deterministic settlement in which the deviation penalties are explicitly present in the second period (DTS). We then proceeded to derive an analytical symmetric equilibrium expression for ISOSP. In this process, we enhanced the definition of our game to avoid multiple equilibria and allow the ISO to choose the marginal cost parameters for the (symmetric) players in this game. The issue of multiple equilibria arises as there are multiple optimal solutions to the best response problem. Specifically, for any choice of b_i and d_i, so long as $R_i = \frac{1}{b_i+d_i} = \hat{R}_i$, we obtain an optimal solution (subject to boundary conditions of course).

While in the context of our computations, due to the natural decomposition of u_i, it was natural to treat b_i as the free variable, our intention has been to compare the NZTS mechanism with the ISOSP market clearing proposed. As we observed that ISOSP is equivalent to DTS, it would make sense to think of a game where the ISO imposes the deviation penalty on all participants by picking $d_i = d \geq 0$. If we think of the ISO choosing d, announcing d to all participants and imposing this value as the deviation penalty in the second stage, the resulting game, along with its symmetric equilibrium, is equivalent to the game where the ISO selects b for the range where $0 \leq d \leq \hat{b}$.

Here observe that if ISO selects $d = \delta$, then as the number of participants increases, in the symmetric equilibrium we obtain $b_i \to \beta$ and $a_i \to \alpha$. Furthermore, it is clear that if $d = 0$, then the equilibria for NZTS are recovered.

5 Comparison of the Two Markets

We are interested in the performance of the two market clearing mechanisms ISOSP and NZTS, under strategic behaviour. Our criterion for comparing the two models is social welfare. Social welfare is defined as the sum of the consumer and producer welfare and in our market environments this reduces to

$$\begin{aligned} \mathrm{W} = &\sum_{s=1}^{S} \theta_s \left(Y_s \left(\sum_{i=1}^{n} y_{i,s} \right) - \frac{Z}{2} \left(\sum_{i=1}^{n} y_{i,s} \right)^2 \right) \\ &- \sum_{s=1}^{S} \theta_s \left(\sum_{i=1}^{n} \left(\alpha y_{i,s} + \frac{\beta}{2} {y_{i,s}}^2 + \frac{\delta}{2} \left(y_{i,s} - q_i \right)^2 \right) \right). \end{aligned} \tag{27}$$

Note that the different social welfare values W^{SS} (for the ISOSP) and W^{NZTS} (for the NZTS mechanism) are found through the same formula, however with the different equilibrium $y_{i,s}$ variables.

Recall that following theorem 4.7 the choice of B_i (equivalently the choice of b_i,) was delegated to the ISO. The next proposition establishes that when firms are bidding strategically, the stochastic settlement market dominates the NZTS market provided the ISO chooses the slope of the supply function sufficiently low.

Proposition 5.1 *The social welfare of ISOSP is higher than the NZTS market provided the parameter b_i is chosen less than the threshold value $\hat{b}$, where*

$$\hat{b} = \frac{-Z(n-2) + \beta + \delta + \sqrt{Z^2(n-2)^2 + 2Zn(\beta+\delta) + (\beta+\delta)^2}}{2}.$$

Proof To prove the proposition, we show when $b_i = \hat{b}$, we can conclude $\mathrm{W}^{\mathrm{SS}} = \mathrm{W}^{\mathrm{NZTS}}$. Then, we demonstrate W^{SS} is a decreasing function of b_i, and therefore, $\mathrm{W}^{\mathrm{SS}} \geq \mathrm{W}^{\mathrm{NZTS}}$, when $b_i \leq \hat{b}$ (note that $\mathrm{W}^{\mathrm{NZTS}}$ is a constant and does not change with b).

When $b_i = \hat{b}$, equations (25), (11), and (10) yield that the equilibrium quantities are identical in the stochastic settlement and deterministic two period settlement markets. That is

$$\begin{aligned} B_i^{\mathrm{SS}} &= B_i^{\mathrm{NZTS}} \\ A_i^{\mathrm{SS}} &= A_i^{\mathrm{NZTS}} \\ R_i^{\mathrm{SS}} &= B_i^{\mathrm{SS}} \end{aligned}$$

Here we can simplify the expressions for $y_{i,s}$ and q_i (from propositions 3.1, 3.2, and 4.4) to obtain

$$\begin{aligned} q_i^{\mathrm{SS}} &= q_i^{\mathrm{NZTS}} = \frac{YB_i - A_i}{1 + ZB} \\ y_{i,s}^{\mathrm{SS}} &= y_{i,s}^{\mathrm{NZTS}} = \frac{Y_s B_i - A_i}{1 + ZB} \end{aligned}$$

Therefore social welfare of these models (equation 27) are the same provided $b_i = \hat{b}$.

We can rewrite the social welfare expression (27) as

$$W = \sum_{s=1}^{S} \theta_s \left(Y_s \left(\sum_{i=1}^{n} y_{i,s} \right) - \frac{Z}{2} \left(\sum_{i=1}^{n} y_{i,s} \right)^2 - \sum_{i=1}^{n} \left(\alpha y_{i,s} + \frac{\beta}{2} {y_{i,s}}^2 + \frac{\delta}{2} {x_{i,s}}^2 \right) \right). \tag{28}$$

Note that the expression for social welfare is the same for both models and only depends on the corresponding quantities dispatched from each model (i.e. $y_{i,s}^{SS}$ vs $y_{i,s}^{NZTS}$ etc).

Furthermore, note that $x_{i,s}^{SS}$ is independent of b, and therefore,

$$\frac{dW^{SS}}{db_i} = -\frac{1}{b_i^2} \sum_{i,s} \frac{dW^{SS}}{dy_{i,s}^{SS}} \frac{dy_{i,s}^{SS}}{dB_i}. \tag{29}$$

On the other hand, taking the derivative of $y_{i,s}^{SS}$ with respect to B_i (see the technical companion [3]) we obtain

$$\frac{dy_{i,s}^{SS}}{dB_i} = \frac{(Y - \alpha)(n-1)Z^2}{(Z + nZ + \beta + (n-1)Z(nZ + \beta)B_i)^2} \geq 0.$$

The right hand side is readily seen to be non-negative as $Y > \alpha$ and $n > 1$.

As $\frac{dy_{i,s}^{SS}}{dB_i}$ is independent of firm i and scenario s (note that B_i is chosen by the ISO and fixed to a single parameter for all firms,) we can re-arrange (29) and obtain

$$\frac{dW^{SS}}{db_i} = -\frac{1}{b_i^2}\frac{dy_{i,s}^{SS}}{dB_i}\sum_{i,s}\frac{dW^{SS}}{dy_{i,s}^{SS}}.$$

On the other hand, differentiating (28) yields

$$\frac{dW^{SS}}{dy_{i,s}^{SS}} = \theta_s\left(Y_s - \alpha - (Zn+\beta)y_{i,s}^{SS}\right).$$

Hence,

$$\begin{aligned}\sum_s \frac{dW^{SS}}{dy_{i,s}^{SS}} &= Y - \alpha - (Zn+\beta)\, q_i^{SS} \\ &= \frac{bZ(Y-\alpha)}{Z(n-1)(nZ+\beta)+b((n+1)Z+\beta)} \geq 0.\end{aligned}$$

Therefore we can conclude that,

$$\frac{dW^{SS}}{db_i} \leq 0.$$

Note that we can easily show that

$$\hat{b} \geq \beta + \delta,$$

and therefore, if the fixed b_i is chosen equal to β then $W^{\mathrm{SS}} \geq W^{\mathrm{NZTS}}$.

Example Consider a market with two symmetric generators as defined in table 1.

Parameter	Value
α, β, δ	$50, 1, 0.5$
Y_1, Y_2, Z	$100, 150, 1$
θ_1, θ_2	$0.5, 0.5$
n	2

Table 1: The market environment for the example

Figure 3 shows how the social welfare of the stochastic settlement mechanism is affected by the choice of b. It also demonstrates that for small enough bs, the stochastic settlement mechanism has a higher equilibrium social welfare in comparison with the NZTS mechanism. For the rest of this example, we assume the ISO chooses $b = \beta = 1$ which ensures higher equilibrium social welfare from the stochastic settlement in comparison with the conventional mechanism.

Another interesting experiment is to investigate the effect of δ on these mechanisms.

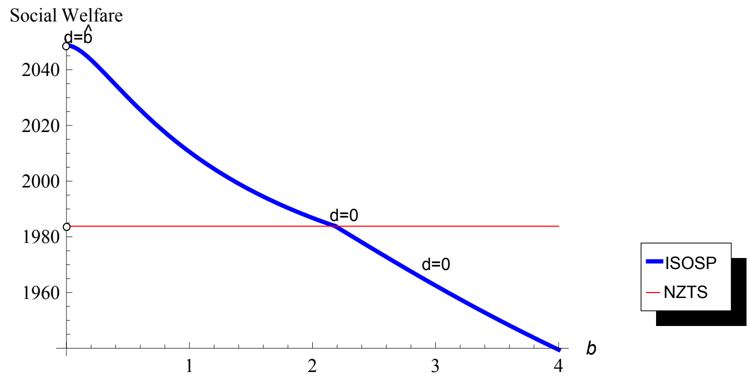

Figure 3: The effect of b on the social welfare of stochastic settlement model and an how it compares to the deterministic two period settlement mechanism

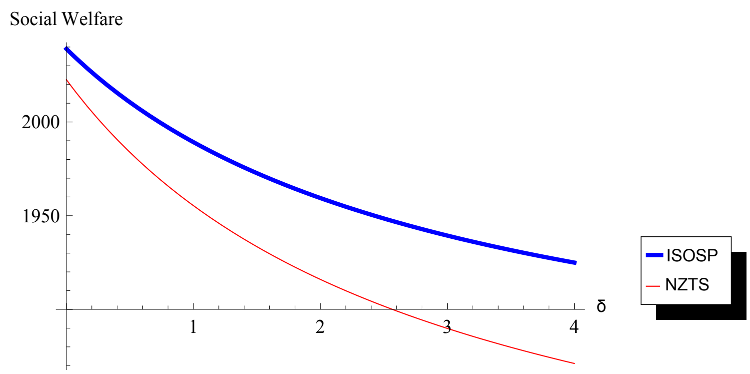

Figure 4: The SP mechanism yields higher social welfare for different δ values

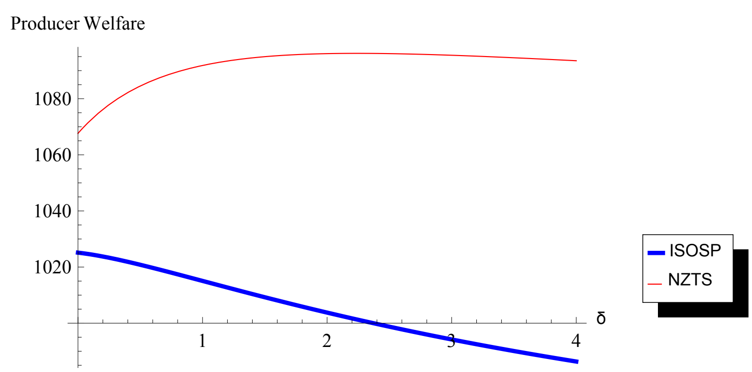

Figure 5: The SP mechanism yields lower producer welfare for a range of examples (i.e. different δ values)

Figures 4, 5, and 6 compare the stochastic settlement and the NZTS mechanisms for this market, however for different δ values. A first observation is the stochastic settlement mecha-

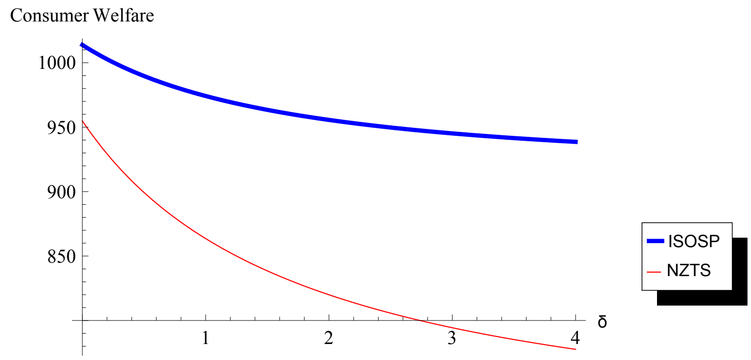

Figure 6: The SP mechanism yields higher consumer welfare for a range of examples (i.e. different δ values

nism increases social and consumer welfare and decreases producer welfare in comparison with the two settlement mechanism. Also, these effects are enhanced by increasing δ. This is an expected result that the strength of the stochastic settlement is bolder when cost of deviation is higher.

It is also interesting to investigate the effect of competition on these mechanisms. To do so, we can test the effect of number of firms on these mechanism.

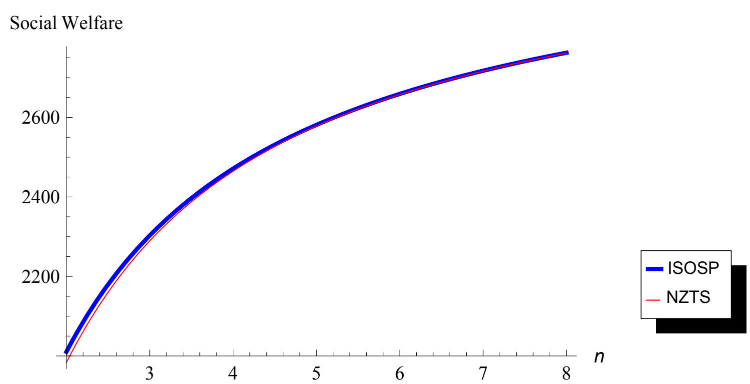

Figure 7: Social welfare of the deterministic two period settlement mechanism converges that of the stochastic mechanism when n increases. Competition increases with a bigger market.

Figure 7 shows the difference in the social welfare of our two mechanism as a function of n. It shows that when the number of generators increase, the performance of the stochastic and deterministic two period settlement mechanisms converges.

6 Conclusion

In this paper, we set up a simple modelling environment in which we were able to compare the New Zealand inspired deterministic two period single settlement market clearing mechanism against a stochastic settlement auction which reduces to another two period single settlement auction with explicit penalties of deviation, therefore different from the NZTS model. We were able to model firms' best responses in these markets, and so find equilibrium behaviour in each. We find that in our symmetric models, the ISOSP auction provably dominates the NZTS auction when measuring expected social welfare.

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