Unit Commitment in Electricity Pool Markets

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Abstract

We consider an electricity generator making offers of energy into an electricity pool market over a horizon of several trading periods (typically a single trading day). The generator runs a set of generating units with given start-up costs, shut-down costs and operating ranges. At the start of each trading period the generator must submit to the pool system operator a new supply curve defining quantities of offered energy and the prices at which it wants these dispatched. The amount of dispatch depends on the supply curve offered along with the offers of the other generators and market demand, both of which are random, but do not change in response to the actions of the generator we consider. After dispatch the generator determines which units to run in the current trading period to meet the dispatch. The generator seeks a supply function that maximizes its expected profit. We describe an optimization procedure based on dynamic programming that can be used to construct optimal offers in successive time periods over a fixed planning horizon.

1 Introduction

There has been much attention in the power-systems optimization literature to unit commitment problems. Traditional unit commitment addresses the scheduling of start-up/shut-down decisions and operation levels for power generation units such that fuel costs over some time horizon are minimal. This optimization problem has received extensive coverage in the literature, see for instance [18]. With the emergence of electricity trading, the prime objective in power optimization has shifted from cost minimization to profit maximization. Integrated optimization of power production and trading then becomes a crucial issue. This optimization problem is inherently uncertain due to the lack of information agents in power markets are facing about competitors' market behaviour.

Stochastic programming offers a variety of models and solution techniques for optimization under uncertainty. First stochastic programming contributions to unit commitment consider uncertain parameters as exogenous random variables independent of the decisions to be optimized. A prominent example is uncertainty of power demand as has been addressed in [6], [8], [12], [20] for generation systems including coal and gas fired thermal units as well as pumped-storage plants. Modelling leads to large-scale mixed-integer linear stochastic programs with inherent block structure. The latter is utilized for different decomposition algorithms, see [17] for an overview.

This paper aims at optimizing energy offers into an electricity pool market made by a utility who is running a power system consisting of thermal units. In electricity pool markets each generator submits offers of energy in the form of *supply curves* that specify how much they will offer to the market at each price. The central dispatching authority then dispatches generation to meet demand at least cost. In most pool markets the supply curves are submitted in the form of an *offer stack*, consisting of a finite set of price-quantity pairs, indicating that the generator is willing to produce those quantities at the corresponding prices. The number of different price-quantity pairs that can be offered in any offer period depends on the particular market rules.

It is important to be clear about the structure of the pool market we are studying. We assume that the generator offers a (possibly) different supply curve in each trading period over the planning horizon, and these are chosen immediately before the demand in that period is realized. This may be thought of as re-offering as the day unfolds, in contrast to the typical day-ahead supply function that must be chosen once for all trading periods in the next day. Our model therefore is intended to represent a balancing market, or the real-time spot markets that operate in New Zealand and Australia.

In this context the dispatch for each generator is somewhat under its control, since it can construct its offer in each trading period to tailor the dispatch. We distinguish here between the unit commitment, which determines what units should run, and the dispatch that determines the total output to be generated by these units. The supply curve offered by the generator, of course, influences the outcome of the random event of getting dispatched or not. This makes the unit commitment problem somewhat different from that in a centrally planned system, in which generators choose unit commitments and generation to meet (possibly uncertain, but exogenously random) demand at least cost, see again [18] and [6], [8], [12], [20]. For stochastic programming models dealing with exogenous price uncertainty in unit commitment we refer to [14], [21], [23].

The problem of constructing optimal supply curves for a generator with startup and shutdown costs has been discussed briefly in [11] in the case where the offers of generators do not affect the electricity price. A dynamic programming recursion is given, showing how an optimal unit commitment might be constructed in this setting.

In this paper we extend this model to the situation in which the offers by market participants have an effect on the clearing price in the current period. Our analysis is based on the concept of a market distribution function (see [1]), which represents for a given generator the impact of uncertainties in both demand and the behaviour of other market participants on its dispatch and clearing price. In this setting we assume that the market is not in equilibrium, and focus solely on the optimization problem for a single generator. We assume that the other generators choose offer curves that are fixed and do not respond to the choice that our generator makes, but our generator does not know a priori what those offers might be. So the generator's offers will affect the clearing price, but not in a perfectly predictable way. One interpretation of this setting might be a situation in which the market has undergone a change in structure, and we believe that other players still play their original equilibrium strategies, albeit with some random perturbation.

Formally, the market distribution function $\psi(q,p)$ for a particular generator is defined to be the probability that a single offer of (q,p) by the generator is not fully dispatched. So $\psi(q,p)$ incorporates information about the fixed (but probabilistic) offer curves of the other generators. Under the assumption that the other generators choose fixed offer curves, the market distribution function ψ for a given generator is independent of the offer curve made by this generator, but given ψ we can determine the effect of this generator's offer on its observed clearing price and dispatch quantity. More precisely, the market distribution function, when restricted to any offer curve, defines the cumulative probability distribution function of dispatch outcomes if this curve were to be offered. Thus, using a market distribution function, it is possible to derive general optimality conditions for supply-curve offers that maximize expected profit (see [1] and [4]).

Another approach to incorporating market power of electricity generators offering into a pool market has been taken in [13]. Uncertainty enters the model via scenarios reflecting foreign bids. Decisions to be optimized involve own bids and own generation. A linear stochastic program with mixed-integer recourse is set up to maximize expected profit. Solution relies on scenario decomposition as introduced in [7].

In this paper we assume that a market distribution function exists for each trading period over a single day of K trading periods, and we wish to construct an offer curve for each period to maximize expected profit. Let $\psi_k(q,p)$ be the market distribution function for trading period $k = 1, 2, \ldots, K$. We assume that all these functions are known at each point in the planning horizon. Methods for estimating market distribution functions from observed data are explored in [3] and [16].

A key assumption in our approach is that the market distribution functions in later trading periods do not alter in response to the offers of the generator we are considering in the current period. It is plausible in a single-period-offer optimization problem to model competitors' offers and demand as being drawn from known probability distributions, since in the short term, market outcomes in electricity pool markets rarely correspond to one-shot Nash equilibria of the type discussed in [2], [9], [10]. However when offers are repeated over K consecutive trading periods this assumption becomes more difficult to sustain, as in practice competing electricity traders will observe market outcomes and adjust their future offers accordingly. We shall return to discuss this issue in our concluding remarks.

The generator on whom we shall focus is assumed to be risk neutral and seeks to maximize its expected profit summed over each trading period. The profit function accounts for switching units on and off and can include a position in derivative contracts. For simplicity we assume that these amount to a single swap contract for q_c at strike price f. Observe that such a contract will have no effect on the optimal offer in a price-taking model but may have a significant effect on the offer curve when the generator has the ability to influence the clearing price.

The paper is laid out as follows. In section 2 we provide a model for the unit commitment problem for a generator offering a single unit to the market. Here the generator must decide in advance if it wants to run its unit, and if so to design an appropriate offer curve. In section 3 we turn our attention to a generator with several units. In this case the generator has a two-stage problem with recourse. The first stage determines an offer to make and the second stage determines which units to run to meet the dispatch. The second-stage cost accounts for future decisions using a dynamic programming value function. Our approach is to represent this value function as a cost function, to be used in the calculation of an optimal offer curve. In general this cost function will be discontinuous, and so section 4 is devoted to extending the optimality conditions to deal with such functions. In section 5 we apply the optimality conditions to an example problem in which all units are identical. In section 6 we discuss how our model might be extended to deal with a generator with non-identical units.

2 The single unit

We begin by studying the case of a single unit. Suppose the generator has a single unit with an operating range $q \in [a, b]$ and running cost $C(q)$. Suppose it costs U to switch the unit on, and D to switch the unit off. At each trading period $k = 1, 2, \ldots, K$ the generator must decide the offer curve for this unit.

Following [1] we define the offer by a parameterized curve $\mathbf{s} = \{(x(t), y(t)) \mid 0 \leq t \leq 1\}$, where $x(t)$ represents the quantity offered and $y(t)$ represents its offer price. We assume that x and y have continuous and bounded derivatives everywhere except (possibly) at a finite number of points. At these points we require the existence of left and right-hand derivatives of x and y defined as

$$x'_-(t) = \lim_{\delta \downarrow 0} \frac{x(t) - x(t-\delta)}{\delta},$$

$$x'_+(t) = \lim_{\delta \downarrow 0} \frac{x(t+\delta) - x(t)}{\delta},$$

with $y'_-(t)$ and $y'_+(t)$ defined similarly. This allows a variety of forms of supply curve to be modelled including the step function offer stacks that are used in many juristictions.

The supply curve offered will be dispatched a random amount Q by the market. Ideally the offer should be designed so that $Q \in \{0\} \cup [a, b]$. However it is not possible in a pool market with a convex dispatch mechanism to construct a single offer curve with this property. This means that the generator must decide before offering the supply curve whether to run the unit. If it is to run, then the generator must offer to ensure dispatch in $[a, b]$, but if the unit is not to run then the generator should not offer at all.

This gives a recursion that the generator can use to compute an optimal policy. Let $V_k(1)$ ($V_k(0)$) be the optimal expected profit the generator can make from the end of trading period k to the end of period K, if the unit is running (not running) at the end of period k. We assume that

$$V_K(0) = V_K(1) = 0.$$

Let

$$R_k = \max_s \int_s (pq - C(q) + q_c(f - p)) d\psi_k(q, p),$$

and

$$S_k = \int_0^\infty q_c(f - p) d\psi_k(0, p).$$

Here $\int_s R(q, p) d\psi(q, p)$ is interpreted as a Stieltjes line integral which can be shown to be well-defined for continuous R and ψ (see [1]). This gives the recursion:

$$V_{k-1}(0) = \max\{S_k + V_k(0), R_k - U + V_k(1)\},$$

$$V_{k-1}(1) = \max\{S_k - D + V_k(0), R_k + V_k(1)\}.$$

In the formula for R_k, the offer curve $\mathbf{s}$ must be chosen so as to ensure dispatch in $[a, b]$. In electricity pool markets the construction of so-called "must-run" offers poses some difficulties, since the presence of this hard constraint effectively prices that part of the offer that lies in $[0, a)$ at $-\infty$. Moreover, it may transpire (e.g. in the case of low demand) that too many generators wish to be dispatched above their lower bound, in which case some generator will be dispatched by the market (infeasibly) in $[0,a)$.

To overcome this problem, some pool markets (e.g. New Zealand) restrict offer prices to be strictly positive, except for generators who are successful bidders in an auxiliary "must-run auction". These generators are allowed to offer at price 0 (see [15]). This gives a two-stage structure to the generator's decision problem, in which the first-stage decision seeks the offer to make to the must-run auction, and then the second-stage decision computes the optimal offer to make to the pool market given the results of this auction.

Here we have assumed that the generator has obtained rights (possibly by auction) to offer all their generation at price zero, thereby guaranteeing full dispatch. Then in the above recursion we seek offer curves which have zero price where $q \in [0,a)$, and infinite price for $q > b$. In the interval $[a,b]$, the optimal offer can be computed using the optimality conditions of [1].

The model above can be extended to encompass the case where the unit has a minimum down-time t_D and minimum up-time t_U, both assumed to be strictly positive integers (see [20]). Let $V_k(1)$ ($V_k(0)$) be the optimal expected profit the generator can make from the end of trading period k to the end of period K, if the unit has been running for at least t_U periods (not running for at least t_D periods) at the end of period k. We adopt the convention that $V_K(\cdot) = 0$, even if the unit has not satisfied the minimum up-time or minimum down requirements.

This gives the recursion:

$$V_{k-1}(0) = \max\{S_k + V_k(0), \sum_{l=k}^{l=\min\{k+t_U-1,K\}} R_l - U + V_{\min\{k+t_U-1,K\}}(1)\},$$

$$V_{k-1}(1) = \max\{\sum_{l=k}^{l=\min\{k+t_D-1,K\}} S_l - D + V_{\min\{k+t_D-1,K\}}(0), R_k + V_k(1)\}.$$

We can also extend the model to include ramping, by assuming that it takes t_R periods to ramp up to operating range. This gives the recursion:

$$V_{k-1}(0) = \begin{cases} \max\{S_k + V_k(0), & \\ \sum_{l=k+t_R}^{l=\min\{k+t_R+t_U-1,K\}} R_l - U + V_{\min\{k+t_R+t_U-1,K\}}(1)\}, & k+t_R \leq K, \\ S_k + V_k(0), & k+t_R > K, \end{cases}$$

$$V_{k-1}(1) = \max\{\sum_{l=k}^{l=\min\{k+t_D-1,K\}} S_l - D + V_{\min\{k+t_D-1,K\}}(0), R_k + V_k(1)\}.$$

3 Unit dispatch by price

We now consider the case in which the generator has $N > 1$ units at its disposal. We shall assume that the generator offers all its generation in one supply curve. If it were a price-taker then it is easy to see that an offer curve for each unit can be computed individually and combined to give an overall offer (as long as this satisfies the relevant rules of the market). Here the N-unit problem decouples into N single-unit problems that can be attacked by the dynamic programming recursion of the previous section.

In the price-setting case, this decomposition is no longer possible, as the total amount offered by the generator will affect the clearing price. In this case the generator might decide in advance of a trading period what set of units to commit, and then once these are switched on and ready to run, determine an optimal offer curve for this set. The offer curve submitted must be constructed to account for the particular characteristics of the available units. In particular if each unit n has an operating range $[a_n, b_n]$ then the dispatch must be constrained to $[\sum_{n\in M} a_n, \sum_{n\in M} b_n]$ where M is the set of units that are running.

With an appropriate definition of state space that encapsulates the unit availability (e.g. using a Boolean vector $z = (z_1, z_2, \ldots, z_N)$) this model can be formulated as a dynamic program along the same lines as that proposed for the single unit in the previous section. This model requires an optimal curve calculation for each possible state vector z, to yield $R_k(z)$, say. Once computed $R_k(z)$ can be used in a recursion that tracks the admissible state transitions of the units to compute for each state z the optimal value function $V_k(z)$.

One drawback of this model is that the range of generation available is constrained by the units committed. To select the right level of unit commitment in advance requires some degree of anticipation of prices in the coming trading period. On the other hand, in a model without unit commitment constraints, the advantage of offering a supply curve is that the offer can anticipate the power price in the trading period and be dispatched by the market clearing mechanism an amount that is adapted to this price. Indeed under some special conditions on the form of offer of other generators, it is possible to construct a supply curve that defines an optimal quantity to offer for each possible realization of the market price (see [2, Theorem 4]). We would like to capture some of these benefits of a market dispatch in our unit commitment model.

Therefore in this section we explore a unit commitment model in which both the unit commitment and the dispatch are determined by the market outcome. In this model an offer curve is submitted to the market. This results in a (random) dispatch quantity, and corresponding clearing price. Given this quantity the generator must decide as a recourse action which units to run, and how much each unit should generate so as to deliver the dispatched quantity at least cost. The first-stage decision is to determine what supply curve to offer so that the generator's expected profit is maximized.

The model we consider is therefore substantially different from the unit commitment model in the previous section, in that we now ignore minimum up-times, minimum down-times or ramp rates. This makes it inapplicable to generators with large units that require several trading periods to warm up before generating. On the other hand, generators with a number of small thermal units that are dispatched as a block, and can be started quickly (albeit at some cost), will be faced with the problem we address here.

To simplify analysis we confine our attention to the case of N identical units, each with an operating range $q \in [a, b]$ and operating cost $C(q)$, assumed to be continuously differentiable. We assume that it costs U to switch each unit on, and D to switch it off. In contrast with the previous section we assume here that the unit commitment decision will be made after the dispatch is determined by the pool market. Since the unit commitment is a random outcome to be determined by the clearing price, we ignore minimum up-times or down-times or ramp rates.

For a dispatch of q the number of units to be run must lie in the set

$$J(q) = \{j \mid j \text{ units can generate the amount } q\}.$$

In our model, it is easy to see that

$$J(q) = \{\lceil \frac{q}{b} \rceil, \lceil \frac{q}{b} \rceil + 1, \ldots, \lfloor \frac{q}{a} \rfloor\}.$$

The curve to offer in each trading period can be computed by applying a dynamic programming recursion. We wish to choose an offer curve $\mathbf{s}$ with the property that it maximizes the generator's expected current revenue minus fuel costs plus expected future profit from optimally switching units on or off and offering generation in the next and future periods. Let $V_k(n)$ be the optimal expected profit that the generator can make from the end of period k to the end of period K if n units are running at the end of period k. We set $V_K(n) = 0$.

In our recursion it is helpful for each n to define the piecewise constant function

$$G_n(q) = \max_{j \in J(q)} \{V_k(j) - U[j-n]_+ - D[n-j]_+\},$$

representing the optimal future expected profit minus switching cost if the generator has n units running and is dispatched q in period k. Here $[y]_+ = \max\{y, 0\}$.

Now recall

$$S_k = \int_0^\infty q_c(f-p) d\psi_k(0, p).$$

This gives

$$V_{k-1}(n) = \max\{S_k - nD + V_k(0), R_k(n)\}, \tag{1}$$

where

$$R_k(n) = \max_s \int_s (pq + q_c(f-p) - C(q) + G_n(q)) d\psi_k(q, p). \tag{2}$$

A key calculation in this recursion is the computation of $R_k(n)$, the expected profit to be made from offering optimally in period $k, k+1, \ldots, K$. This is a single-period offer optimization problem with cost function

$$C_n(q) = C(q) - G_n(q),$$

and the requirement that the offer must guarantee a dispatch of more than a. Observe that since $G_n(q)$ is a piecewise constant function of q, the integrand in (2) will be piecewise smooth with a finite number of jump discontinuities. We thus need to verify that the Stieltjes integral in (2) is well defined for such integrands and represents the expected profit. This discussion will be deferred to the next section, where we also extend the optimality conditions of [1] to deal with this situation.

4 Optimality conditions

The function $C_n(q)$ defined in the previous section is the sum of a step function with a finite number of steps and a continuously differentiable function. In this section we extend the optimality conditions of [1] to deal with such integrands, so as to be able to derive optimal offer curves as part of a dynamic programming recursion. We observe that optimality conditions for market distribution functions with discontinuities in price have been obtained in [5]. In this work we continue to assume that ψ is continuous in both arguments, but that the profit function is smooth with a finite number of jump discontinuities in q. (For convenience, in this section we suppress the dependence of ψ on trading period.)

We first consider a profit function that has a jump discontinuity at a single point $\bar{q}$. Suppose that

$$R_g(q,p) = R(q,p) + g(q)$$

where $R(q,p)$ is continuously differentiable and

$$g(q) = \begin{cases} 0, & q < \bar{q}, \\ \bar{g}, & q \geq \bar{q}. \end{cases}$$

The expected profit from such a function will be the sum of expected profit from $R(q,p)$ and expected profit from $g(q)$.

For any offer curve $\mathbf{s} = \{(x(t), y(t)) \mid 0 \leq t \leq 1\}$, let

$$l = \inf\{t \mid x(t) = \bar{q}\}, \quad u = \sup\{t \mid x(t) = \bar{q}\}.$$

Then $\psi(x(l), y(l))$ is the probability that $\mathbf{s}$ will be dispatched at a quantity q less than $\bar{q}$. Thus the expected profit P_g from $g(q)$ when $\mathbf{s}$ is offered is

$$\begin{aligned} P_g &= \bar{g}\,\Pr(q \geq \bar{q}) \\ &= \bar{g}[1 - \psi(x(l), y(l))]. \end{aligned}$$

Thus we may write

$$\int_s R_g(q,p)d\psi(q,p) = \int_s R(q,p)d\psi(q,p) + \bar{g}[1 - \psi(x(l), y(l))].$$

A similar analysis applies to left-continuous functions $R_h(q,p) = R(q,p) + h(q)$ where

$$h(q) = \begin{cases} 0, & q \leq \bar{q}, \\ \bar{h}, & q > \bar{q}. \end{cases}$$

This gives

$$\int_s R_h(q,p)d\psi(q,p) = \int_s R(q,p)d\psi(q,p) + \bar{h}[1-\psi(x(u),y(u))].$$

Now let

$$g(q) = \sum_{i\in\mathcal{G}} g_i(q)$$

where each $g_i(q)$, $i \in \mathcal{G}$, is a right-continuous step function with a jump of $\bar{g}_i$ at $\bar{q}_i$, and $l_i = \inf\{t \mid x(t) = \bar{q}_i\}$, and let

$$h(q) = \sum_{i\in\mathcal{H}} h_i(q)$$

where each $h_i(q)$, $i \in \mathcal{H}$ is a left-continuous step function with a jump of $\bar{h}_i$ at $\bar{q}_i$, and $u_i = \sup\{t \mid x(t) = \bar{q}_i\}$. (Both $\mathcal{G}$ and $\mathcal{H}$ are assumed to be finite sets.) We let

$$R_{gh}(q,p) = R(q,p) + g(q) + h(q)$$

and define

$$V(\mathbf{s}) = \int_s R_{gh}(q,p)d\psi(q,p).$$

By the above discussion we have

$$V(\mathbf{s}) = \int_s R(q,p)d\psi(q,p) + \sum_{i\in\mathcal{G}} \bar{g}_i[1-\psi(x(l_i),y(l_i))] + \sum_{i\in\mathcal{H}} \bar{h}_i[1-\psi(x(u_i),y(u_i))].$$

The generator seeks an offer curve $\mathbf{s}$ to solve

$$\text{P: maximize} \quad V(\mathbf{s}).$$

We now proceed to derive optimality conditions for this problem. Offer curves that satisfy these conditions will have the property that expected profit will be reduced by feasible perturbations of the curve. Observe that variations to an offer curve $\mathbf{s}$ will have an effect on $\int_s R(q,p)d\psi(q,p)$ as well as the terms containing l_i and u_i. Our optimality conditions will separately identify the effects of these changes and combine them.

To identify the effect of curve perturbations on $\int_s R(q,p)d\psi(q,p)$ we follow [1] and define

$$Z(q,p) = \frac{\partial R(q,p)}{\partial q}\frac{\partial \psi(q,p)}{\partial p} - \frac{\partial R(q,p)}{\partial p}\frac{\partial \psi(q,p)}{\partial q}.$$

(Henceforth for notational convenience we shall denote partial derivatives by subscripts, so $\psi_p = \frac{\partial\psi(q,p)}{\partial p}$, for example). The effect of offer curve perturbations on the other terms will involve ψ_p. As shown in [1] the line integral of $R(q,p)d\psi(q,p)$ around any closed curve $\mathcal{C}$ enclosing region $\mathcal{S}$ has the same value as $\int\int_{\mathcal{S}} Z(q,p)dqdp$. When a candidate stack $\mathbf{s}$ forms part of $\mathcal{C}$, and when $Z(q,p)$ has the same sign everywhere in $\mathcal{S}$, we can use a variational argument to demonstrate the suboptimality of $\mathbf{s}$. The form of this argument is given in some detail in the proof of Lemma 4.2 below, but is only repeated in sketch form in other parts of the paper. A detailed exposition can be found in [1].

First recall from [1] the definition of the set

$$I = \{t \mid x'(t) > 0, \quad y'(t) > 0\},$$

and let

$$\bar{Q} = \{\bar{q}_i \mid i \in \mathcal{G} \cup \mathcal{H}\}.$$

We now have the following result.

Lemma 4.1 *If* $\mathbf{s} = \{(x(t), y(t)),\ 0 \le t \le 1\}$ *is a local optimum for* P *then for all* $t \in I$, $Z(x(t), y(t)) = 0$. *Moreover if for all* t *in some interval* $L \subseteq I$, Z *is differentiable at* $(x(t), y(t))$ *then we have* $\frac{\partial Z}{\partial p}(x(t), y(t)) \ge 0$, *and* $\frac{\partial Z}{\partial q}(x(t), y(t)) \le 0$, *for* $t \in L$.

Proof. See [1] ∎

Lemma 4.1 means that an optimal offer curve will be horizontal or vertical at all points except possibly where $Z(x(t), y(t)) = 0$. The horizontal sections of a locally optimal offer curve must be such that small vertical perturbations of the curve result in a stack with a profit that is no more than the original curve. This leads to the following lemma.

Following [1] we let $t_0 = \inf\{t \mid x(t) > 0, y(t) > 0, \psi(x(t), y(t)) > 0\}$, and define

$$w_x(t) = \int_0^t Z(x(\tau), y(\tau))x'(\tau)d\tau + \sum_{i \in \mathcal{G}(t)} \bar{g}_i \psi_p(x(l_i), y(l_i)) + \sum_{i \in \mathcal{H}(t)} \bar{h}_i \psi_p(x(u_i), y(u_i)),$$

where $\mathcal{G}(t) = \{i \in \mathcal{G} \mid l_i < t\}$, and $\mathcal{H}(t) = \{i \in \mathcal{H} \mid u_i < t\}$.

Lemma 4.2 *Suppose* $t_1 > t_0$, $t_1 \notin \{l_i \mid i \in \mathcal{G}\} \cup \{u_i \mid i \in \mathcal{H}\}$, *and* $x(t_0) \notin \bar{Q}$. *If* $w_x(t_1) - w_x(t_0) > 0$, *then* $\mathbf{s}$ *is not a local optimum for* P.

Proof. The proof is similar to that of Lemma 4.3 in [1]. Let

$$t(\epsilon) = \inf\{t \mid x(t) > 0,\ y(t) > \epsilon,\ \psi(x(t), y(t) - \epsilon) > 0\},$$

where we choose $\epsilon > 0$ small enough so that $[x(t_0), x(t(\epsilon))] \cap \bar{Q} = \emptyset$. ($x(t_0) \le x(t(\epsilon))$ by the monotonicity of ψ.) We construct the perturbed stack

$$\mathbf{r}(t) = (x^r(t), y^r(t)) = \begin{cases} (x(t), (y(t(\epsilon)) - \epsilon)t/t(\epsilon)), & 0 \le t \le t(\epsilon), \\ (x(t), y(t) - \epsilon), & t(\epsilon) < t \le t_1, \\ (x(t_1), y(t_1) + t - t_1 - \epsilon), & t_1 < t \le t_1 + \epsilon, \\ (x(t - \epsilon), y(t - \epsilon)), & t_1 + \epsilon < t \le 1 + \epsilon. \end{cases}$$

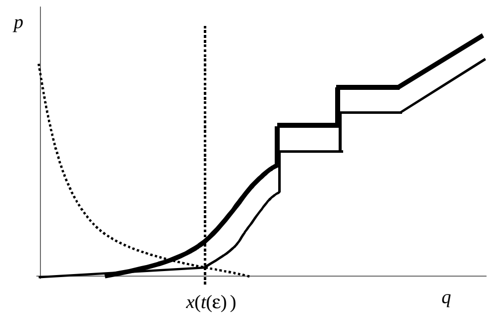

Figure 1: Plot of **s** (solid) and **r** (thin)

as shown in Figure 1. It is shown in [1, p.92] that if $\mathcal{S}$ is the region between **r** and **s** then

$$\begin{aligned}\int_r R(q,p)d\psi(q,p) - \int_s R(q,p)d\psi(q,p) &= \int\int_{\mathcal{S}} Z(q,p)dqdp \\ &= \epsilon \int_{t_0}^{t_1} Z(x(\tau),y(\tau))x'(\tau)d\tau + o(\epsilon).\end{aligned}$$

Now comparing **r** and **s** gives

$$\begin{aligned}V(\mathbf{r}) - V(\mathbf{s}) &= \int_r R_{gh}(q,p)d\psi(q,p) - \int_s R_{gh}(q,p)d\psi(q,p) \\ &= \int_r R(q,p)d\psi(q,p) - \int_s R(q,p)d\psi(q,p) \\ &+ \sum_{i\in\mathcal{G}} \bar{g}_i[1-\psi(x^r(l_i^r),y^r(l_i^r))] - \sum_{i\in\mathcal{G}} \bar{g}_i[1-\psi(x(l_i),y(l_i))] \\ &+ \sum_{i\in\mathcal{H}} \bar{h}_i[1-\psi(x^r(u_i^r),y^r(u_i^r))] - \sum_{i\in\mathcal{H}} \bar{h}_i[1-\psi(x(u_i),y(u_i))]\end{aligned}$$

where

$$l_i^r = \inf\{t \mid x^r(t) = \bar{q}_i\}, \quad u_i^r = \sup\{t \mid x^r(t) = \bar{q}_i\}.$$

Now observe by construction that $x^r(l_i^r) = x(l_i)$ and $x^r(u_i^r) = x(u_i)$, but $y^r(l_i^r) = y(l_i) - \epsilon$ and $y^r(u_i^r) = y(u_i) - \epsilon$. Thus

$$\begin{aligned}\psi(x(l_i),y(l_i)) - \psi(x^r(l_i^r),y^r(l_i^r)) &= \psi(x(l_i),y(l_i)) - \psi(x(l_i),y(l_i)-\epsilon) \\ &= \epsilon\psi_p(x(l_i),y(l_i)) + o(\epsilon)\end{aligned}$$

and

$$\psi(x(u_i),y(u_i)) - \psi(x^r(u_i^r),y^r(u_i^r)) = \epsilon\psi_p(x(u_i),y(u_i)) + o(\epsilon).$$

This gives

$$
\begin{aligned}
V(\mathbf{r}) - V(\mathbf{s}) &= \epsilon \int_{t_0}^{t_1} Z(x(\tau), y(\tau)) x'(\tau) d\tau \\
&+ \epsilon \sum_{i \in \mathcal{G}(t_1)} \bar{g}_i \psi_p(x(l_i), y(l_i)) + \epsilon \sum_{i \in \mathcal{H}(t_1)} \bar{h}_i \psi_p(x(u_i), y(u_i)) + o(\epsilon) \\
&= \epsilon(w_x(t_1) - w_x(t_0)) + o(\epsilon) \\
&> 0
\end{aligned}
$$

thus showing that $\mathbf{s}$ is not optimal. ∎

Lemma 4.3 *Suppose $t_1 > t_0$, $t_1 \notin \{l_i \mid i \in \mathcal{G}\} \cup \{u_i \mid i \in \mathcal{H}\}$, and $x(t_0) \notin \bar{Q}$. If $w_x(t_1) - w_x(t_0) < 0$ and $y'_+(t_1) > 0$, then $\mathbf{s}$ is not a local optimum for P.*

Proof. Let $t(\epsilon) = \inf\{t \mid y(t) = y(t_1) + \epsilon\}$. Now define

$$
\mathbf{r}(t) = \begin{cases}
(x(t), y(t)), & 0 \le t \le t_0, \\
(x(t_0), y(t_0) + t - t_0), & t_0 < t \le t_0 + \epsilon, \\
(x(t-\epsilon), y(t-\epsilon) + \epsilon)) & t_0 + \epsilon < t \le t_1 + \epsilon, \\
(x(t-\epsilon), y(t_1) + \epsilon), & t_1 + \epsilon < t \le t(\epsilon) + \epsilon, \\
(x(t-\epsilon), y(t-\epsilon)), & t(\epsilon) + \epsilon < t \le 1 + \epsilon.
\end{cases}
$$

Here $\mathbf{r}$ is the result of shifting $\mathbf{s}$ up by ϵ, so now we have

$$
\int_r R(q,p) d\psi(q,p) - \int_s R(q,p) d\psi(q,p) = -\epsilon \int_{t_0}^{t_1} Z(x(\tau), y(\tau)) x'(\tau) d\tau + o(\epsilon).
$$

This gives

$$
\begin{aligned}
V(\mathbf{r}) - V(\mathbf{s}) &= \int_r R_{gh}(q,p) d\psi(q,p) - \int_s R_{gh}(q,p) d\psi(q,p) \\
&= \int_r R(q,p) d\psi(q,p) - \int_s R(q,p) d\psi(q,p) \\
&+ \sum_{i \in \mathcal{G}} \bar{g}_i [1 - \psi(x^r(l_i^r), y^r(l_i^r))] - \sum_{i \in \mathcal{G}} \bar{g}_i [1 - \psi(x(l_i), y(l_i))] \\
&+ \sum_{i \in \mathcal{H}} \bar{h}_i [1 - \psi(x^r(u_i^r), y^r(u_i^r))] - \sum_{i \in \mathcal{H}} \bar{h}_i [1 - \psi(x(u_i), y(u_i))]
\end{aligned}
$$

$$
\begin{aligned}
&= -\epsilon \int_{t_0}^{t_1} Z(x(\tau), y(\tau))x'(\tau)d\tau \\
&+ \sum_{i\in\mathcal{G}} \bar{g}_i[\psi(x(l_i), y(l_i)) - \psi(x^r(l_i^r), y^r(l_i^r))] \\
&+ \sum_{i\in\mathcal{H}} \bar{h}_i[\psi(x(u_i), y(u_i)) - \psi(x^r(u_i^r), y^r(u_i^r))] + o(\epsilon) \\
&= -\epsilon \int_{t_0}^{t_1} Z(x(\tau), y(\tau))x'(\tau)d\tau \\
&- \epsilon \sum_{i\in\mathcal{G}(t_1)} \bar{g}_i\psi_p(x(l_i), y(l_i)) - \epsilon \sum_{i\in\mathcal{H}(t_1)} \bar{h}_i\psi_p(x(u_i), y(u_i)) + o(\epsilon) \\
&= -\epsilon(w_x(t_1) - w_x(t_0)) + o(\epsilon) \\
&> 0
\end{aligned}
$$

thus showing that **s** is not optimal. ∎

Although the above lemmas restrict the form of the optimal offer curve, they are quite general and so are not straightforward to apply in any given situation, since the exact form of optimality conditions in any circumstance will vary with the form of the level curves of Z, as well as the particular form of $g(q) + h(q)$.

Because our focus in this paper is the unit commitment problem, we shall in the remainder of this section confine attention to this special case. For simplicity we shall assume as in the previous section that all units are identical with fixed startup and shutdown costs (section 6 discusses the extension to non-identical units). Under this assumption we may show that the step function

$$G_n(q) = \max_{j\in J(q)} \{V_k(j) - U[j-n]_+ - D[n-j]_+\}$$

is unimodal with a maximum attained at any $q \in [na, nb]$. This is because the extra future benefit in having one fewer unit running is no more than D. Similarly the extra future benefit of having one more unit on is no more than U. (In effect there is no benefit to be gained from switching machines before it is necessary). These statements are made precise by the following lemmas. In the proof of these lemmas it is helpful to define

$$
\begin{aligned}
F(s,n) = &\int_s (pq + q_c(f-p) - C(q))d\psi_k(q,p) + \\
&\int_s \max_{j\in J(q)} \{V_k(j) - U[j-n]_+ - D[n-j]_+\} d\psi_k(q,p),
\end{aligned}
$$

where the Stieltjes integral of $\max_{j\in J(q)}\{V_k(j) - U[j-n]_+ - D[n-j]_+\}$ is well-defined by virtue of the continuity of ψ.

Lemma 4.4 *For all $k \leq K$, and all $n > 0$, $V_k(n-1) \leq V_k(n) + D$.*

Proof. The result is trivial for $k = K$. We fix n and prove the result for any $k < K$. For any $k \leq K$, we have for any j,

$$[j-n]_+ \leq [j-(n-1)]_+$$

and so

$$V_k(j) - U[j-n]_+ - D[n-j]_+ + D \geq V_k(j) - U[j-(n-1)]_+ - D[(n-1)-j]_+$$

Thus, for any q,

$$\begin{aligned}&\max_{j\in J(q)}\{V_k(j) - U[j-n]_+ - D[n-j]_+\} + D \\ &\geq \max_{j\in J(q)}\{V_k(j) - U[j-(n-1)]_+ - D[(n-1)-j]_+\}.\end{aligned}$$

Since, for any offer curve

$$\int_s D d\psi_k(q,p) = D,$$

we have

$$F(s, n-1) \leq F(s,n) + D$$

for any curve s. Denote by s_n, the curve that maximizes $F(s,n)$. Then

$$\begin{aligned}V_{k-1}(n-1) &= \max\{S_k - (n-1)D + V_k(0), F(s_{n-1}, n-1)\} \\ &\leq \max\{S_k - (n-1)D + V_k(0), F(s_{n-1}, n) + D\} \\ &= \max\{S_k - nD + V_k(0), F(s_{n-1}, n)\} + D \\ &\leq \max\{S_k - nD + V_k(0), F(s_n, n)\} + D \\ &= V_{k-1}(n) + D,\end{aligned}$$

which gives the result for $k < K$. ∎

Lemma 4.5 *For all $k \leq K$, and all $n > 0$, $V_k(n) - U \leq V_k(n-1)$.*

Proof. The result is trivial for $k = K$. We fix n and prove the result for any $k < K$. We have for any $k \leq K$ and for any j,

$$V_k(j) - U[j-n]_+ - D[n-j]_+ \leq V_k(j) - U[j-(n-1)]_+ - D[(n-1)-j]_+ + U.$$

Thus, for any q,

$$\begin{aligned}&\max_{j\in J(q)}\{V_k(j) - U[j-n]_+ - D[n-j]_+\} \\ &\leq \max_{j\in J(q)}\{V_k(j) - U[j-(n-1)]_+ - D[(n-1)-j]_+\} + U.\end{aligned}$$

Since, for any offer curve s

$$\int_s U d\psi_k(q,p) = U,$$

we have

$$F(s,n) \leq F(s,n-1) + U.$$

Again using s_n to denote the curve that maximizes $F(s,n)$, we have

$$\begin{aligned} V_{k-1}(n) - U &= \max\{S_k - nD + V_k(0), F(s_n,n)\} - U \\ &\leq \max\{S_k - nD + V_k(0), F(s_n,n-1) + U\} - U \\ &\leq \max\{S_k - (n-1)D + V_k(0), F(s_n,n-1)\} \\ &\leq \max\{S_k - (n-1)D + V_k(0), F(s_{n-1},n-1)\} \\ &= V_{k-1}(n-1), \end{aligned}$$

which gives the result. ∎

The lemmas above simplify the integral (2). By an abuse of notation let us write $n > J(q)$ to mean $n > j$ for every $j \in J(q)$, and $n < J(q)$ to mean $n < j$ for every $j \in J(q)$. Then it is easy to see from the lemmas above that $V_k(j) - D[n-j]_+$ is nondecreasing in j when $j < n$, and $V_k(j) - U[j-n]_+$ is nonincreasing in j when $j > n$. This gives

$$\max_{j\in J(q)}\{V_k(j) - U[j-n]_+ - D[n-j]_+\}$$

$$= \begin{cases} V_k(\max J(q)) - D(n - \max J(q)), & n > J(q), \\ V_k(n), & n \in J(q), \\ V_k(\min J(q)) - U(\min J(q) - n), & n < J(q). \end{cases}$$

In other words, if the generator has to shut down or start up any units it should switch as few as possible. This allows us to use the cost function

$$C_n(q) = \begin{cases} C(q) - V_k(\max J(q)) + D(n - \max J(q)), & n > J(q), \\ C(q) - V_k(n), & n \in J(q), \\ C(q) - V_k(\min J(q)) + U(\min J(q) - n), & n < J(q), \end{cases}$$

and since the station has identical units with a running range $q \in [a,b]$, we obtain

$$C_n(q) = \begin{cases} C(q) - V_k(\lfloor \frac{q}{a} \rfloor) + D(n - \lfloor \frac{q}{a} \rfloor), & a \leq q < na, \\ C(q) - V_k(n), & na \leq q \leq nb, \\ C(q) - V_k(\lceil \frac{q}{b} \rceil) + U(\lceil \frac{q}{b} \rceil - n), & q > nb. \end{cases}$$

In the first case, the dispatch q is so small that the generator must shut down at least one machine. By Lemma 4.4 it makes sense to shut down as few units as possible, thus leaving $\lfloor \frac{q}{a} \rfloor$ running. In the next case the generator can keep n machines running so it does, and in the final case, q is so large that it must start up at least one machine. By Lemma 4.5 it should only start as many units as required, so this gives $\lceil \frac{q}{b} \rceil$ running.

Observe that the generator may choose to either not be dispatched for the coming period, in which case it switches all machines off and does not offer at all, or to be dispatched at least a, in which case at least part of its offer curve must offer a at price zero. The remainder of the curve $\mathbf{s}$ starting from the point $(a, 0)$ maximizes its expected current revenue minus fuel costs plus expected future profit from optimally switching units on or off and offering generation in the next and future periods. In this case we set $t_0 = \inf\{t \mid x(t) > a, y(t) > 0, \psi(x(t), y(t)) > 0\}$ and do not include a in $\bar{Q}$. This allows us to apply Lemmas 4.2 and 4.3 in the case where $x(t_0) = a$.

We conclude this section by considering the special case in which $Z(x(t), y(t)) = 0$ defines a unique nondecreasing curve $\mathbf{z}$ with $Z(x(t), y(t)) > 0$ above $\mathbf{z}$ and $Z(x(t), y(t)) < 0$ below. In the absence of unit-commitment effects (and limitations on the functional form of $\mathbf{z}$), this defines an optimal offer curve. The following theorem gives optimality conditions for the unit commitment problem in this case.

Theorem 4.6 *Suppose $Z(q,p) = 0$ defines a unique nondecreasing curve $\mathbf{z} = (x_z(t), y_z(t))$. Let $Z_0 = \{(q,p) \mid Z(q,p) = 0\}$, $Z_+ = \{(q,p) \mid Z(q,p) > 0\}$, and $Z_- = \{(q,p) \mid Z(q,p) < 0\}$, and suppose that $Z(q,p) > 0$ if for any t, $q < x_z(t)$ and $p > y_z(t)$, and $Z(q,p) < 0$ if for any t, $q > x_z(t)$ and $p < y_z(t)$. Then any curve $\mathbf{s} = (x(t), y(t))$ that is a local maximum for P satisfies the following conditions.*

1. *Either $(x(t), y(t)) \in Z_0$ or $x'(t) = 0$ or $y'(t) = 0$.*
2. *$w_x(t) = w_x(t_0)$ for all t where $(x(t), y(t)) \in Z_0$, and for all t where $(x(t), y(t)) \notin Z_0$ and $y'(t) > 0$.*
3. *Suppose $(x(t), y(t)) \notin Z_0$.*
   - *(a) If $y'(t) > 0$ then $x(t) \in \bar{Q}$.*
   - *(b) If $y'(t) > 0$ and $x(t) = \bar{q}_i \leq na$, then $(x(t), y(t)) \in Z_-$.*
   - *(c) If $y'(t) > 0$ and $x(t) = \bar{q}_i \geq nb$, then $(x(t), y(t)) \in Z_+$.*
   - *(d) If $x'(t) > 0$ and $x(t) \leq na$, then $(x(t), y(t)) \in Z_-$.*
   - *(e) If $x'(t) > 0$ and $x(t) \geq nb$, then $(x(t), y(t)) \in Z_+$.*
4. *If $na < x(t) < nb$ then $(x(t), y(t)) \in Z_0$.*

Proof. It is easy to see that (1) follows directly from Lemma 4.1. Thus any curve $\mathbf{s} = (x(t), y(t))$ that is a local maximum for P will be horizontal or vertical when $(x(t), y(t)) \notin Z_0$.

To show (2) observe that $w_x(t) = w_x(t_0)$ if $\mathbf{s}(\tau) = \mathbf{z}(\tau)$, $\tau \in [t_0, t]$. For larger values of t, Lemmas 4.2 and 4.3 together give $w_x(t) = w_x(t_0)$ for t lying strictly between the endpoints of every vertical section of $\mathbf{s}$. If this section is at $\bar{q}_i$, $i \in \mathcal{G}$, and finishes on $\mathbf{z}$, then $w_x(t)$ is continuous at u_i and so $w_x(u_i) = w_x(t_0)$. Thus

$$w_x(t) = w_x(t_0) + \int_{u_i}^{t} Z(x(\tau), y(\tau))x'(\tau)d\tau \tag{3}$$

on the next section of $\mathbf{s}$. If this section matches $\mathbf{z}$, then the integrand in (3) is zero, so $w_x(t) = w_x(t_0)$ on this section. Similarly, if $\mathbf{s}$ has a vertical section at $\bar{q}_i$, $i \in \mathcal{H}$, then $w_x(\cdot)$ is continuous at l_i. If $\mathbf{s}$ matches $\mathbf{z}$ just before then, we have for $t < l_i$,

$$w_x(t_0) = w_x(l_i) = w_x(t) + \int_{t}^{l_i} Z(x(\tau), y(\tau))x'(\tau)d\tau = w_x(t).$$

Finally if $\mathbf{s}$ leaves $\{(q, p) \mid q > 0, p > 0, \psi(q, p) < 1\}$ at $(x(t_M), y(t_M))$, and matches $\mathbf{z}$ just before this, then it is easy to show using a similar argument to Lemma 4.2 that $w_x(t_M) = w_x(t_0)$. This then gives $w_x(t) = w_x(t_0)$ on the last section of $\mathbf{s}$.

To show (3a), observe that the conditions imply the existence of a vertical section of $\mathbf{s}$ between t_1 and t_2 with at least one endpoint not in Z_0 with $y'_-(t_1) = 0$ or $y'_+(t_2) = 0$. Suppose $x(t_1) = x(t_2) \notin \bar{Q}$, so small horizontal perturbations of the vertical section do not alter $(x(l_i), y(l_i))$ and$(x(u_i), y(u_i))$. Then in the case where $y'_-(t_1) = 0$, $V(\mathbf{s})$ can be improved by perturbing the section below Z_0 to the left, and if $y'_+(t_2) = 0$ then $V(\mathbf{s})$ can be improved by perturbing the section above Z_0 to the right. This gives a contradiction, and so establishes (3a).

For (3b), suppose that the vertical section of $\mathbf{s}$ between t_1 and t_2 with $x(t_1) = x(t_2) = \bar{q}_i \leq na$ meets Z_+. If $x'_-(t_1) > 0$, $x'_+(t_2) > 0$, then we have

$$\int_{t_1}^{t_2} Z(x(\tau), y(\tau))y'(\tau)d\tau \leq 0$$

otherwise $V(\mathbf{s})$ can be improved by perturbing the section to the right. This means that the vertical section crosses $\mathbf{z}$, and so $V(\mathbf{s})$ can be improved by perturbing the section above $\mathbf{z}$ to the right giving a contradiction, and thus showing that $(x(t), y(t)) \in Z_-$. A similar argument shows that any vertical piece at $\bar{q}_i \geq nb$ does not intersect Z_-, giving (3c).

To show (3d) observe that $(x(t), y(t))$ lies on a horizontal section of $\mathbf{s}$. Suppose $(x(t), y(t)) \in Z_+$. By (3b), for $x(t) \leq na$ there are no vertical sections of $\mathbf{s}$ meeting Z_+, so $\mathbf{s}$ is horizontal for all $\tau \leq t$, and so lies in Z_+ for all such τ. It follows that

$$w_x(t) = \int_{0}^{t} Z(x(\tau), y(\tau))x'(\tau)d\tau + \sum_{i \in \mathcal{G}(t)} \bar{g}_i \psi_p(x(l_i), y(l_i)) > w_x(t_0)$$

contradicting the optimality of $\mathbf{s}$ by Lemma 4.2. Similarly (3e) can be established using (2) and Lemma 4.3.

For (4) suppose $na < x(t) < nb$. By (3a), either $y'(t) = 0$ or $(x(t), y(t)) \in Z_0$, and so $\mathbf{s}$ consists of horizontal pieces alternating with sections of $\mathbf{z}$, and since $\mathbf{z}$ is nondecreasing there are at most two horizontal sections (one at either end of the interval). Consider the horizontal section at the left end of the interval. For $x(t) \le na$ there are no vertical sections of $\mathbf{s}$ meeting Z_+, so $\mathbf{s}$ is horizontal for all $x(t) \le na$. Thus for any t where $x(t) > na$ and lies in the left interval we have

$$w_x(t) = \int_0^t Z(x(\tau), y(\tau))x'(\tau)d\tau + \sum_{i \in \mathcal{G}(t)} \bar{g}_i \psi_p(x(l_i), y(l_i)) > w_x(t_0)$$

contradicting the optimality of $\mathbf{s}$ by Lemma 4.2. A similar argument yields a contradiction for the horizontal section at the right-hand end. ∎

5 Example

When the $Z(q,p) = 0$ curve is known the optimality conditions of Theorem 4.6 can be used to determine the optimal offer curve for each state n (assuming the generator wishes to be dispatched).

As an illustrative example consider a market distribution function defined by

$$\psi(q,p) = \begin{cases} \frac{q^2+p^2}{400}, & q \ge 0, \quad p > 0, \quad q^2+p^2 \le 400, \\ 1, & q \ge 0, \quad p \ge 0, \quad q^2+p^2 > 400, \\ 0, & \text{otherwise.} \end{cases}$$

and set

$$R(q,p) = qp - q$$

corresponding to a fuel cost of 1, and zero contracts. This gives

$$Z(q,p) = (p-1)\frac{1}{200}p - \frac{1}{200}q^2$$

whereby the contour $Z = 0$ is the hyperbola

$$(p-1)\,p - q^2 = 0$$

as shown in Figure **??** below. Here $Z > 0$ for all points above the hyperbola, and $Z < 0$ for all points below the hyperbola, and the curve $Z = 0$ gives the optimal offer if we ignore unit-commitment effects.

Now suppose $a = 1$, $b = 3$, and $U = 7$, and $V_k(n)$ is defined by the following table

n	0	1	2	3	4	5	6
$V_k(n)$	0	6	12	18	24	29	34

To illustrate the optimality conditions, we shall suppose that there are currently two machines running (so $n = 2$). This means that the current running range is

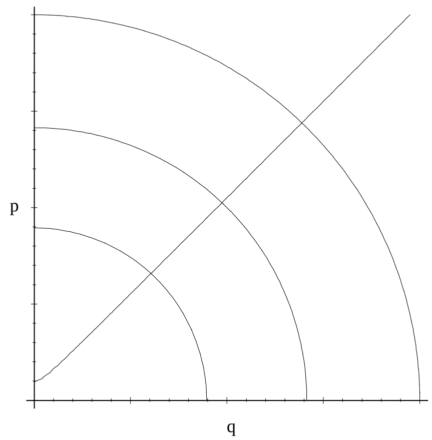

Figure 2: $Z(q,p)=0$ and $\psi(q,p)$ contours.

[2, 6]. We have

$$C_2(q) = \begin{cases} q-6, & 1 \le q < 2, \\ q-12, & 2 \le q \le 6, \\ q-11, & 6 < q \le 9, \\ q-10, & 9 < q \le 12, \\ q-8, & 12 < q \le 15, \\ q-6, & 15 < q \le 18, \end{cases}$$

as shown in Figure 3.

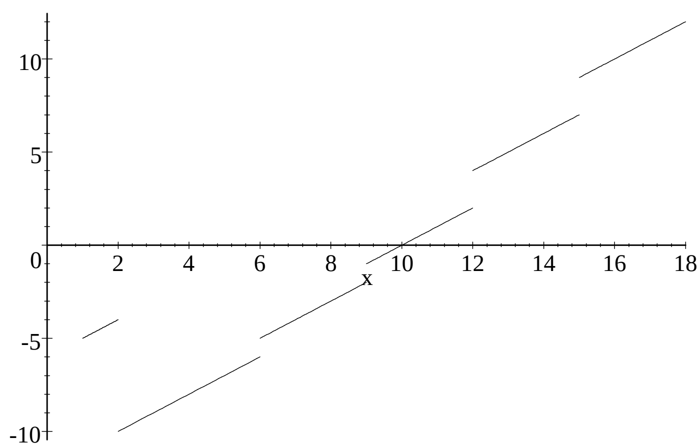

Figure 3: Plot of $C_2(q)$

The optimal stack can be written as

$$p(q) = \begin{cases} 0, & q < 1, \\ 0.430, & 1 \le q < 2, \\ \frac{1}{2} + \frac{1}{2}\sqrt{(1+4q^2)}, & 2 \le q < 6, \\ 7.581, & 6 \le q < 7.063, \\ \frac{1}{2} + \frac{1}{2}\sqrt{(1+4q^2)}, & 7.063 \le q < 9, \\ 10.556, & 9 \le q < 10.043, \\ \frac{1}{2} + \frac{1}{2}\sqrt{(1+4q^2)}, & 10.043 \le q < 12, \\ 13.977, & 12 \le q < 13.468, \\ \frac{1}{2} + \frac{1}{2}\sqrt{(1+4q^2)}, & 13.468 \le q < 15, \end{cases}$$

as shown in bold in Figure 4.

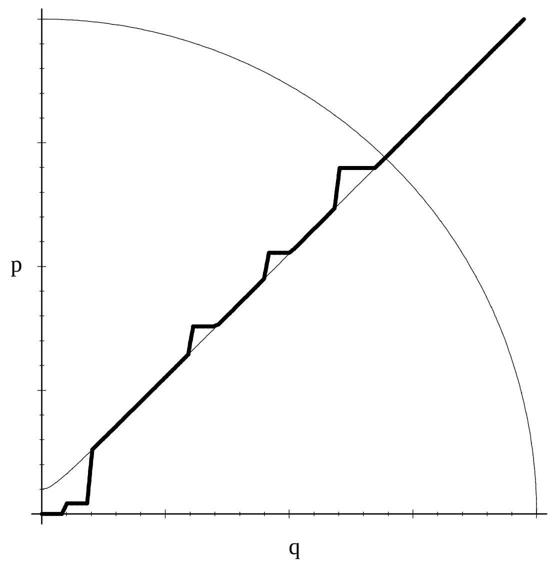

Figure 4: Optimal stack for example problem

This gives an optimal value of

$$\begin{aligned} \int_s R_{gh}(q,p)d\psi(q,p) &= \int_s [q(p-1)+6]d\psi(q,p) \\ &+ 6[1-\psi(2,0.430)] \\ &- [1-\psi(6,7.581)] \\ &- [1-\psi(9,10.556)] \\ &- 2[1-\psi(12,13.977)] \\ &= 96.565 + 4.349 \\ &= 100.914 \end{aligned}$$

and so we set $V_{k-1}(2) = 100.914$ in (1). (In the absence of a contract, this exceeds $S_k + V_k(0)$.)

The offer curve shown is easily verified to satisfy the local optimality conditions. We set $x(t_0) = 1$, $y(t_0) = 0.430$, and for example consider $x(t_1) = 2$, $y(t_1) = 2.562$, so $(x(t_1), y(t_1))$ lies in $\mathbf{z}$. Then

$$\int_{t_0}^{t_1} Z(x(\tau), y(\tau))x'(\tau)d\tau = \int_1^2 Z(q, 0.430)dq$$
$$= -0.0129$$

and

$$\sum_{i \in \mathcal{G}(t_1)} \bar{g}_i \psi_p(x(l_i), y(l_i)) = 6\frac{\partial \psi}{\partial p}(2, 0.430)$$
$$= \frac{6}{200}0.430$$
$$= 0.0129$$

and so

$$w(t_1) - w(t_0) = \int_{t_0}^{t_1} Z(x(\tau), y(\tau))x'(\tau)d\tau + \sum_{i \in \mathcal{G}(t_1)} \bar{g}_i \psi_p(x(l_i), y(l_i))$$
$$= 0$$

as required by condition (2) of Theorem 4.6.

Similarly consider $x(t_1) = 12$, $y(t_1) = 12.510$, and $x(t_2) = 13.468$, $y(t_2) = 13.977$, both lying in $\mathbf{z}$. Then we get

$$\int_{t_1}^{t_2} Z(x(\tau), y(\tau))x'(\tau)d\tau = \int_{12}^{13.468} Z(q, 13.977)dq$$
$$= 0.1398$$

and

$$\sum_{i \in \mathcal{H}(t_2)} \bar{h}_i \psi_p(x(u_i), y(u_i)) - \sum_{i \in \mathcal{H}(t_1)} \bar{h}_i \psi_p(x(u_i), y(u_i)) = -2\frac{1}{200}13.977$$
$$= -0.1398.$$

Therefore

$$w(t_2) - w(t_1) = \int_{t_1}^{t_2} Z(x(\tau), y(\tau))x'(\tau)d\tau$$
$$+ \sum_{i \in \mathcal{H}(t_2)} \bar{h}_i \psi_p(x(u_i), y(u_i)) - \sum_{i \in \mathcal{H}(t_1)} \bar{h}_i \psi_p(x(u_i), y(u_i))$$
$$= 0,$$

as required by condition (2) of Theorem 4.6.

The optimal offer curve has an appealing interpretation. Observe that at the discontinuities of C_n that arise from $\bar{g}_i$ (an expected future cost incurred by shutting down and having to restart a unit later) the generator offers at a discount to its optimal offer. (This behaviour can be seen in the figure for dispatches less than 2.) Offering at a discount will increase the likelihood of sufficient dispatch to avoid the shutdown. The discount is computed so that the marginal expected loss from discounting the offer price matches the change in expected future cost avoided.

At the discontinuities of C_n that arise from $\bar{h}_i$ (a cost incurred by starting a new unit) the generator offers at a premium to its optimal offer. This will decrease the likelihood of being dispatched an amount that is large enough to force a startup. The premium is computed so that the marginal expected loss from offering too high matches the change in expected future cost avoided (which includes the startup cost).

6 Non-identical units

The example in the previous section assumed identical units. This allowed us to define the state of a generator at stage k to be an integer n indicating how many units are running at that stage. We could then define a piecewise continuous cost function

$$C_n(q) = \begin{cases} C(q) - V(\lfloor \frac{q}{a} \rfloor, t), & q < na, \\ C(q) - V(n, t), & na \leq q \leq nb, \\ C(q) - V(\lceil \frac{q}{b} \rceil, t) + U(\lceil \frac{q}{b} \rceil - n), & q > nb, \end{cases}$$

for which an optimal offer stack can be constructed.

Suppose now that we have N non-identical units. Suppose that unit $n = 1, 2, \ldots, N$, has startup cost U_n, shutdown cost D_n, operating range $[a_n, b_n]$, and fuel cost ϕ_n. The state of the generator's plant at stage k can be encoded by the Boolean vector $(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N)$. The expected future profit is now

$$V_{k-1}(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N) = \max_s \int (pq + q_c(f - p) - C_k(\bar{z}, q)) d\psi_k(q, p)$$

where

$$\begin{aligned} C_k(\bar{z}, q) = \quad & -\max_{z_k, q_k} \quad V_k(z_1, z_2, \ldots, z_N) - \textstyle\sum_n \phi_n q_n \\ & \qquad\qquad - \textstyle\sum_n U_n [z_n - \bar{z}_n]_+ - \textstyle\sum_n D_n [\bar{z}_n - z_n]_+ \\ & \text{subject to} \quad a_n z_n \leq q_n \leq b_n z_n, \quad n = 1, 2, \ldots, N, \\ & \qquad\qquad \textstyle\sum_n q_n = q, \\ & \qquad\qquad z_n \in \{0, 1\}. \end{aligned}$$

With an appropriate definition of W, the cost function can be defined as

$$\begin{array}{lll} C_k(\bar{z}, q) = & \min_{z_n} \min_{q_n} & W(z,\bar{z},t) + \sum_n \phi_n q_n \\ & \text{subject to} & a_n z_n \le q_n \le b_n z_n, \quad n = 1,2,\ldots,N, \\ & & \sum_n q_n = q, \\ & & z_n \in \{0,1\}. \end{array}$$

The offer optimization with this cost function can be carried out as before. It is helpful here to be able to identify the values of q at which $C_k(\bar{z}, q)$ is discontinuous. Observe that for any fixed $z \in \{0,1\}^N$, the optimal value function

$$\begin{array}{lll} \varphi_z(q) = & \min_{q_n} & W(z,\bar{z},t) + \sum_n \phi_n q_n \\ & \text{subject to} & a_n z_n \le q_n \le b_n z_n, \quad n = 1,2,\ldots,N, \\ & & \sum_n q_n = q. \end{array}$$

is piecewise linear and convex on its domain of definition, and it is assumed to be $+\infty$ outside this.

Now

$$C_k(\bar{z}, q) = \min\{\varphi_z(q) \mid z \in \{0,1\}^N\},$$

whereby $C_k(\bar{z}, \cdot)$ is the pointwise minimum of extended real-valued convex functions. Discontinuities of $C_k(\bar{z}, \cdot)$ can therefore only occur at points that lie on the boundary of $dom\varphi_z$, for some z. These points have the following characterization.

Lemma 6.1 *If q belongs to the boundary of $dom\varphi_z$ then $q = \sum_n a_n z_n$ or $q = \sum_n b_n z_n$.*

Proof. The domain of definition of $\varphi_z(\cdot)$ coincides with the Minkowski sum of one-dimensional intervals

$$\sum_{n:z_n=1} [a_n, b_n] = [\sum_n a_n z_n, \sum_n b_n z_n]$$

which yields the result. ∎

By virtue of Lemma 6.1 we have that $C_k(\bar{z}, q)$ has a finite number of jump discontinuities, so the optimality conditions of section 4 are still applicable, albeit with more computational effort.

7 Conclusion

In this paper we have studied the unit commitment problem from the perspective of a generator making strategic offers to an electricity pool market. Our analysis has distinguished between two essentially different models. In the first we study a single unit that is either offered or withdrawn from the pool. This model can be extended to cover minimum up-times and down-times, and possible ramping rates. It also extends in a natural way to the case of N units that are committed prior to an offer being made.

The second model concerns N units that are collectively offered to the pool using an offer curve that ensures an optimal dispatch and unit commitment. In this model we have focussed on a generator with identical units, and derived a series of lemmas that can be used to determine an optimal stack for a single period offer when the system is in a given state (i.e. has n machines running). The (optimal) expected value of this offer is then used to compute an expected future cost that can be used in a dynamic programming recursion. The second model is difficult to apply when the units have minimum up-times, minimum down-times, or ramping rates.

In either model in order to use our dynamic programming recursion, we need to assume perfect knowledge of the market distribution function ψ_k for $k = 1, 2, \ldots, K$. Even if these functions were straightforward to estimate, it is probable that they will change over time as other agents in the market respond to the optimal decisions of the generator we are studying. Our models assume that the other agents are not behaving strategically in this way.

One extension to our model to accommodate the reactions of other participants might represent ψ_k as a Markov process that depends on some "market" state variable. In this framework a random transition from market state μ in period k to market state ν in period $k+1$ would occur as a result of observed market outcomes in period k. Since these outcomes are to some extent influenced by the generator constructing optimal offers, the optimization problem to determine $V_{k-1}(n, \mu)$ at the start of period k in market state μ will use $\psi_k(q, p, \mu)$ and replace G_n by

$$G_n(p, q, \mu) = \max_{j \in J(q)} \{ \sum_{\nu} P_k(\mu, \nu, q, p) V_k(j, \nu) - U[j-n]_+ - D[n-j]_+ \},$$

where $P_k(\mu, \nu, q, p)$ is the probability of moving from market state μ in period k to market state ν in period $k+1$, when the generator is dispatched q at price p in period k. Clearly estimating the state structure and the form of $P_k(\mu, \nu, q, p)$ represents a major modelling hurdle to this approach.

A further extension might seek a sub-game perfect Nash equilibrium in the dynamic game played by a number of generating companies. We conjecture that a feature of such an equilibrium would be sub-optimal offers by some thermal generators (as played in the single-period model) that force shutdowns in competitors in order to secure profits in later periods. Computing such an equilibrium in supply curves would seem also to be a very challenging project.

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