Welfare-maximizing transmission capacity expansion under uncertainty

S. Wogrin [*], D. Tejada-Arango [†], A. Downward [‡], A.B. Philpott[§]

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Abstract

We apply the JuDGE optimization package to a multistage stochastic leader-follower model that determines a transmission capacity expansion plan to maximize expected social welfare of consumers and producers who act as Cournot oligopolists in each time period. The problem is formulated as a large-scale mixed integer program and applied to a 5-bus instance over scenario trees of varying size. The computational effort required by JuDGE is compared with solving the deterministic equivalent mixed integer program using a state-of-the-art integer programming package.

1 Introduction

Capacity expansion modeling in the electricity industry has a long history dating back to [19] for social planning models and [21] for investment in a competitive setting. The liberalization of the electricity sector and the introduction of electricity markets, which first emerged in the 1980s in countries such as Chile, the United Kingdom, and New Zealand, has shifted many capacity expansion responsibilities away from a centralized entity and towards private companies that can act strategically. In most electricity markets of

[*]Institute for Research in Technology (IIT), School of Engineering (ICAI), Comillas Pontifical University & Institute of Electricity Econonomics and Energy Innovation, Graz University of Technology

[†]ICADE Business School, Comillas Pontifical University & Global Trading - Fundamentales y Competencia, Endesa SA

[‡]Electric Power Optimization Centre, University of Auckland

[§]Electric Power Optimization Centre, University of Auckland

developed countries transmission capacity and generation capacity are decided by different entities, which mathematically speaking can be described by multi-level optimization (and equilibrium) problems à la Stackelberg [30]. Since electricity generation plants and transmission lines are generally large capital items with long lifetimes, their sizing and location must be chosen carefully to ensure that they can accommodate future uncertainty. If they are chosen suboptimally then they may either be insufficient to meet demand (incurring losses from unmet load) or overbuilt (becoming stranded assets).

This paper describes a multistage stochastic programming model for planning the capacity expansion of a transmission network to maximize the increase in expected social welfare produced by this investment. We treat future uncertainty in the operating conditions of the electricity system as a scenario tree that spans a long planning horizon (of say 30 years). At each node of the scenario tree transmission investment choices are made based on the history of the system up to that point in time. Between these points in time, the electricity system is operated with the provided transmission capacity to meet demand.

In our model the electricity supplied by generators is dispatched by an independent system operator (ISO) and travels through the transmission grid to satisfy demand at different locations. Generators supply energy under varying conjectures on the effect that this supply will have on energy prices. In the simplest setting generators are perfectly competitive and offer all their available capacity at their (assumed constant) marginal costs. In a risk-neutral setting this assumption gives a transmission planning model that minimizes the expected social cost of the expansion using a stochastic program.

An alternative model treats generators as Cournot players who anticipate the effect of their generation on the clearing price assuming all other agents fix their actions. In this model, generators compete strategically in the times between each transmission investment, so the optimal transmission expansion plan is not a straightforward system optimization. Here the transmission investments are chosen to maximize the social welfare that would result from generators acting strategically in competition using the transmission assets available. This yields a multistage optimization problem in which the outcome in each stage is the solution to an equilibrium problem, rather than an optimization problem as in risk-neutral perfect competition.

To summarize, the model we have in mind has the following structure. A transmission planner is to determine a plan of transmission investments over a long time horizon that adapts to changes in circumstances as exogenous market conditions (e.g. demand, fuel prices etc.) become revealed over time. The plan accounts for the fact that in the time interval between each transmission expansion[1], each generator in the network will choose a capacity level, and generation amounts for each period in this interval that will maximize its profits over this time interval, accounting for the actions of competing generators in a Cournot oligopoly. The transmission investments of the planner are chosen to maximize the total expected welfare of generators and consumers less transmission capital costs over the planning horizon.

The approach we take to solving such a problem is to formulate a multistage stochastic mixed integer program, where binary variables are used to represent the complementarity conditions that model the Cournot equilibrium. In practical instances the deterministic equivalent version of this problem is intractable, so we solve it using Dantzig-Wolfe decomposition as implemented in the JuDGE package [9] written in the Julia language. JuDGE enables the solution of problems of unprecedented scale using modest computing resources.

The contributions of the paper can be summarized as follows:

1. We show how to formulate stochastic leader-follower games in the JuDGE system to yield computationally tractable models;
2. We solve large-scale[2] instances of these models and compare their solution times with those of deterministic equivalent Mixed Integer Problems (MIPs);
3. We compare optimal transmission capacity investment plans for problem instances under perfect and imperfect competition.

Multilevel or hierarchical optimization and equilibrium models have been used by a number of authors (e.g. [3, 8, 17, 25, 31]) to represent sequential decision making à la Stackelberg within wholesale electricity markets. When it comes to electricity transmission expansion planning (TEP), modelers are faced with a dilemma stemming from the corresponding timing of generation expansion planning (GEP). In *proactive* TEP the transmission system is decided first and generators react by locating and sizing their generation investments to take advantage of the grid. In *reactive* TEP the transmission system is designed and built in response to growth in demand and generation expansion decisions. The choice of either a proactive model or a reactive

[1] There may be multiple transmission expansions or none at all before each uncertainty node.

[2] In this context *large-scale* refers to problem instances of unprecedented scale for a problem of this structure. Moreover, we want to stress that *large-scale* is used to describe the size of the resulting overall MIP, and not the size of the power system instance that we use for illustration.

model identifies who are the leaders and who are the followers in the multi-level game.

The model considered in this paper adopts a proactive transmission expansion approach. In the remainder of this section we discuss the literature most relevant for the TEP problem tackled here. This literature review is by no means exhaustive. For a more detailed literature review about transmission expansion planning, the reader is referred to [15, 34].

One of the first works on multi-level proactive TEP is presented by Sauma and Oren [28], which is an extension of [27], where the authors explore the impact of different objectives on TEP, considering policy implications and anticipating responses of strategic GEP players with ownership structures as proposed in [35]. The generation expansions are modeled as actions that decrease the marginal cost of generation at every level. The model itself has three levels: TEP, GEP and the market. In order to solve this very complex type of nonconvex problem the authors formulate a Mathematical Problem with Equilibrium Constraints (MPEC) representing the two lower levels. This MPEC is solved iteratively, eliminating dominated GEP strategies. The third layer is solved by an enumeration of TEP strategies and repeating the iterative process. While [28] takes the first ambitious step into modeling proactive multi-level TEP, it does not guarantee global optimality.

In [22], Pozo et al. propose the first complete formulation of a three-level TEP problem. In the second level (the GEP stage), each strategic firm's investment strategies are enumerated and expressed using a binary expansion. Since generation capacity actions are discrete, conditions for a pure-strategy Nash equilibrium can be expressed as a finite set of inequalities. There is no guarantee that these have a solution. Nevertheless, this approach allows the authors to formulate the full problem as an MPEC, which is transformed into a Mixed Integer Linear Problem (MIP) by linearizing complementarity conditions using binary variables as described in [14]. The optimal solution of a MIP model gives a guarantee of global optimality for this nonconvex problem. If the MIP is provably infeasible then one might suppose that no pure-strategy Nash equilibrium exists, at least for the discrete approximation of investment strategies. Demonstrating this computationally is challenging for large problems that are difficult to solve at scale (e.g. if demands are stochastic) because of the many "big-M" constraints.

To tackle large-scale problems, [24] develops a column-and-row decomposition technique for solving the arising GEP and market-clearing equilibrium problem, and applies this to a realistic power system in Chile with uncertain demand. This technique ultimately yields the globally optimal solution and greatly increases computational efficiency with respect to previous works, while not compromising the rigor of the mathematical formulation.

Alternatives to MIP formulations have also received some attention, although these provide no guarantee that the solution found is a global optimum. A hybrid approach to solving a three-level TEP problem is presented in [18], which applies a diagonalization method to a complementarity formulation to yield a convergent algorithm. In [1] the authors propose a stochastic adaptive robust optimization model, which is solved by iterating between a master and subproblems.

However, to ensure global optimality, the majority of multi-level TEP models described in the literature (both deterministic and stochastic) are solved as follows: the model is first formulated with complementarity constraints, which are then transformed into linear constraints using binary variables. The resulting MIP is then solved using commercial solvers. Examples of such works are [2, 16, 20, 32]. As remarked above, this approach does not scale well because of the "big-M" constraints. Since these constraints are replicated for every state of the world represented in a stochastic model, the approach is unsuitable for TEP problems with many scenarios, unless some form of decomposition is applied.

In the literature, there have been works that applied decomposition techniques to transmission expansion models. For example, [26] applied Benders decomposition; however, their model considers a more centralized type of planning where both generation and transmission expansion planning is taken in the upper level, and market clearing happens in the lower level. This does however not account for strategic market feedback by generation companies that can behave a la Cournot as in our model. Moreover, [24] resort to the computationally more efficient column-generation technique; however, their model substantially differs from ours in two aspects: they consider only a finite number of generation expansion options and they only account for a perfectly competitive market. Finally, [12] also uses column generation but the first mover in their model is a merchant storage investor not a TSO, and their market is also perfectly competitive. Hence, to the best of our knowledge, this is the first example of a stochastic bilevel model that considers strategic feedback from both generation expansion and operation from a market that can be both perfectly competitive or a Cournot oligopoly.

The rest of the paper is laid out as follows. In section 2 we formulate an equilibrium-constrained model that determines optimal transmission capacity investments when electricity producers behave as Cournot agents. Section 3 expands the model of the previous section to a multistage setting, where it is formulated in a scenario tree. Section 4 then describes the JuDGE package which applies Dantzig-Wolfe decomposition to investment planning problems. Section 5 presents a case study that illustrates the model, and demonstrates its computational performance as the number of scenarios increases. Section

6 closes the paper with some concluding remarks.

2 Models with equilibrium constraints

The model we consider has a set of agents that compete to supply electricity to consumers located at nodes of a transmission network in a set of trading periods $t \in \mathcal{T}$. Our model closely follows that of [27], which we summarize here for completeness. To reduce notation, we only label the equations that we refer to in the remainder of the paper. A complete nomenclature of all parameters and variables is provided in the appendix of the paper. We assume that consumption of electricity d_{kt} at each node k of the network in period t is modeled by a representative consumer with a quadratic utility function $a_k d_{kt} - \frac{1}{2} b_k d_{kt}^2$. Given a price p_{kt}, the consumer at node k maximizes consumer surplus

$$a_k d_{kt} - \frac{1}{2} b_k d_{kt}^2 - p_{kt} d_{kt}$$

subject to $d_{kt} \geq 0$, so they solve

$$\min_{d_{kt} \geq 0} \; p_{kt} d_{kt} - a_k d_{kt} + \frac{1}{2} b_k d_{kt}^2.$$

This convex optimization problem has Karush-Kuhn-Tucker (KKT) conditions

$$0 \leq p_{kt} - (a_k - b_k d_{kt}) \perp d_{kt} \geq 0.$$

We denote each electricity plant by an index i, and use the notation $k(i)$ and $i \in k$ to denote the location of each plant in the network, where we assume that each generator operates exactly one plant. In our model producer i simultaneously chooses a production capacity u_i (costing $K_i u_i$) and an amount of energy x_{it} to supply to the market in every trading period $t \in \mathcal{T}$ to maximize their total profit at price $p_{k(i)t}$ given the cost of capacity and a marginal production cost c_{it}.

It is useful to outline here how our approach differs from previous authors ([27], [28],[23], [22], [24]). They model capacity expansion by generators as a second-stage decision that alters the marginal cost of producer i. Dispatch decisions are made in a third stage of the game given the capacity decisions of the generators. Optimal capacity decisions then give an equilibrium problem with equilibrium constraints (EPEC). In our model, generators choose generation capacity at the same time as they choose operating quantities for each period that uses this capacity. As outlined below, the system operator chooses transmission quantities for each period at the same time as generator decisions.

Under Cournot conjectures, each generator and the system operator assumes when optimizing that other generators' capacity decisions and generation quantities in every time period are fixed. This is a form of *open-loop* equilibrium which we can guarantee exists as the solution of simultaneous KKT systems. In a more realistic *subgame-perfect* model agents would optimize assuming others actions are fixed only in the same stage of the game, and accounting for the payoffs that result from future (subgame) equilibria.

The optimization problem faced by producer i under these assumptions is

$$\begin{array}{llll} \mathrm{P}(i)\colon & \min & \sum_{t\in\mathcal{T}} x_{it}(c_{it}-p_{k(i)t})+K_i u_i & \\ & \text{s.t.} & x_{it}-u_i\leq 0, & t\in\mathcal{T}, \\ & & x_{it},u_i\geq 0, & t\in\mathcal{T}. \end{array}$$

The KKT conditions for $\mathrm{P}(i)$ are

$$\begin{array}{lllll} 0 & \leq & c_{it}-p_{k(i)t}-\frac{dp_{k(i)t}}{dx_{it}}x_{it}+\lambda_{it} & \perp x_{it} & \geq 0, \quad t\in\mathcal{T}, \\ 0 & \leq & K_i-\sum_{t\in\mathcal{T}}\lambda_{it} & \perp u_i & \geq 0, \\ 0 & \leq & u_i-x_{it} & \perp \lambda_{it} & \geq 0, \quad t\in\mathcal{T}. \end{array}$$

Here λ_{it} is the Lagrange multiplier on the generation capacity constraint for generator i which provides a capacity rent every time this constraint is binding. If we assume $\frac{dp_{k(i)}}{dx_i}=0$ then these conditions represent perfectly competitive producer behaviour. If we set $\frac{dp_{k(i)}}{dx_i}=-b_{k(i)}$ then producer i is behaving as a Cournot agent. We distinguish between these two cases by using a parameter φ_i that is set to $b_{k(i)}$ for Cournot producers and 0 for perfectly competitive producers. (It is possible to study various levels of imperfect competition by choosing $\varphi_i\in[0,b_{k(i)}]$ but we will confine ourselves to the extreme cases in this paper.) If the transmission network has a single node k then, with appropriate choices of φ_i, the market equilibrium is defined by the complementarity problem

$$\begin{array}{lrcll} 0\leq & p_k-(a_k-b_k d_{kt}) & \perp & d_{kt} & \geq 0, \quad t\in\mathcal{T}, \\ 0\leq & c_{it}-p_{kt}+\varphi_i x_{it}+\lambda_{it} & \perp & x_{it} & \geq 0, \quad i\in k, t\in\mathcal{T}, \\ 0\leq & K_i-\sum_{t\in\mathcal{T}}\lambda_{it} & \perp & u_i & \geq 0, \quad i\in k, \\ 0\leq & u_i-x_{it} & \perp & \lambda_{it} & \geq 0, \quad i\in k, t\in\mathcal{T}, \\ 0\leq & \sum_{i\in k}x_{it}-d_{kt} & \perp & p_{kt} & \geq 0, \quad t\in\mathcal{T}. \end{array}$$

A transmission system with multiple nodes complicates this model when competition is imperfect. It is well known (see e.g. [5, 10]) that there may not exist a pure-strategy Cournot equilibrium, even in radial networks. To overcome this, Yao et al. [35] present two models with different conjectures on the bounded rationality of electricity producers. In the first of these models, agents assume that transmission flows are fixed and do not change in response to their production choices. In the second model agents anticipate changes in transmission flows from changes in production, but assume that price differences between nodes do not vary as production changes. We adopt the first of these models, so agents assume that transmission flows are unaffected by their production choices.

The demand and suppliers have the same KKT conditions as before

$$0 \leq \quad p_{kt} - (a_k - b_k d_{kt}) \quad \perp \quad d_{kt} \quad \geq 0, \tag{1}$$

$$\begin{array}{lrcll} 0 \leq & c_{it} - p_{k(i)t} + \varphi_i x_{it} + \lambda_{it} & \perp & x_{it} & \geq 0, \\ 0 \leq & K_i - \sum_{t \in \mathcal{T}} \lambda_{it} & \perp & u_i & \geq 0, \\ 0 \leq & u_i - x_{it} & \perp & \lambda_{it} & \geq 0. \end{array} \tag{2}$$

In time period $t \in \mathcal{T}$ the system operator given a transmission network defined by lines $(k,l) \in \mathcal{A}$, chooses line flows f_{klt}, $(k,l) \in \mathcal{A}$ to solve

$$\begin{array}{rll} \text{SO: min} & \sum_{(k,l) \in \mathcal{A}} (p_{kt} - p_{lt}) f_{klt} & \\ \text{s.t.} & f_{klt} \leq \tau_{kl}, & (k,l) \in \mathcal{A}, \\ & f_{klt} \geq -\tau_{kl}, & (k,l) \in \mathcal{A}, \\ & X_{kl} f_{klt} = \theta_{kt} - \theta_{lt}, & (k,l) \in \mathcal{A}. \end{array}$$

Here we adopt the convention that the flow f_{klt} in transmission line (k,l) is directed from k to l where $k < l$, and a negative value indicates a flow from l to k. This means that $\mathcal{A}$ contains only ordered pairs (k,l) with $k < l$. Transmission flows f_{klt} must satisfy thermal capacity limits τ_{kl}, $(k,l) \in \mathcal{A}$. The equality constraints (Kirchhoff's Laws) are required to represent transmission flows using a DC model, where θ_{kt} denotes the voltage phase angle at node k, and X_{kl} is the reactance of line $(k,l) \in \mathcal{A}$. For each time period $t \in \mathcal{T}$ the problem SO has KKT conditions:

$$\begin{array}{lrclll} 0 = & p_{kt} - p_{lt} + \rho_{klt} - \sigma_{klt} + \mu_{klt} X_{kl} & \perp & f_{klt}, & & (k,l) \in \mathcal{A}, \\ 0 = & \sum_{l:(l,k) \in \mathcal{A}} \mu_{lkt} - \sum_{l:(k,l) \in \mathcal{A}} \mu_{klt} & \perp & \theta_{kt}, & & k \in \mathcal{K}, \\ 0 = & X_{kl} f_{klt} - \theta_{kt} + \theta_{lt} & \perp & \mu_{klt}, & & (k,l) \in \mathcal{A}, \\ 0 \leq & \tau_{kl} - f_{klt} & \perp & \rho_{klt} & \geq 0, & (k,l) \in \mathcal{A}, \\ 0 \leq & \tau_{kl} + f_{klt} & \perp & \sigma_{klt} & \geq 0, & (k,l) \in \mathcal{A}. \end{array} \tag{3}$$

The market clearing condition at each node $k \in \mathcal{K}$ in time period $t \in \mathcal{T}$ is

$$0 \leq \sum_{i \in k} x_{it} - \sum_{l:(k,l) \in \mathcal{A}} f_{klt} + \sum_{l:(l,k) \in \mathcal{A}} f_{lkt} - d_{kt} \perp p_{kt} \geq 0. \tag{4}$$

Collecting the complementarity conditions gives

$$
\begin{aligned}
\text{MCP:}\quad & (1), \quad k \in \mathcal{K}, t \in \mathcal{T}, \\
& (2), \quad i \in k, k \in \mathcal{K}, t \in \mathcal{T}, \\
& (3), \quad (k,l) \in \mathcal{A}, t \in \mathcal{T}, \\
& (4), \quad k \in \mathcal{K}, t \in \mathcal{T},
\end{aligned}
$$

a mixed complementarity system that represents a competitive network equilibrium. The social welfare W that results from this equilibrium is

$$
W = \sum_{k \in \mathcal{K}} \left(\sum_{t \in \mathcal{T}} (a_k d_{kt} - \frac{1}{2} b_k d_{kt}^2 - \sum_{i \in k} c_{it} x_{it}) - \sum_{i \in k} K_i u_i \right). \tag{5}
$$

The problem of choosing transmission capacities to make the outcome of the competitive equilibrium socially optimal is a mathematical program with equilibrium constraints (MPEC). This transmission expansion planning problem can now be formulated as follows.

$$
\begin{aligned}
\text{TEP: min} \quad & \textstyle\sum_{(k,l) \in \mathcal{A}} C_{kl}(\tau_{kl}) - W \\
\text{s.t.} \quad W = \textstyle\sum_{k \in \mathcal{K}} \big(\sum_{t \in \mathcal{T}} (a_k d_{kt} - \frac{1}{2} b_k d_{kt}^2 - & \textstyle\sum_{i \in k} c_{it} x_{it}) - \sum_{i \in k} K_i u_i, \\
0 \leq p_{kt} - (a_k - b_k d_{kt}) & \perp d_{kt} \geq 0, \quad k \in \mathcal{K}, \\
0 \leq c_{it} - p_{k(i)t} + \varphi_i x_{it} + \lambda_{it} & \perp x_{it} \geq 0, \quad i \in k, k \in \mathcal{K}, \\
0 \leq K_i - \textstyle\sum_{t \in \mathcal{T}} \lambda_{it} & \perp u_i \geq 0, \quad i \in k, k \in \mathcal{K}, \\
0 \leq u_i - x_{it} & \perp \lambda_{it} \geq 0, \quad i \in k, k \in \mathcal{K}, \\
0 = p_{kt} - p_{lt} + \rho_{klt} - \sigma_{klt} + \mu_{klt} X_{kl} & \perp f_{klt}, \quad (k,l) \in \mathcal{A}, \\
0 = \textstyle\sum_{l:(l,k) \in \mathcal{A}} \mu_{lkt} - \sum_{l:(k,l) \in \mathcal{A}} \mu_{klt} & \perp \theta_{kt}, \quad k \in \mathcal{K}, \\
0 = X_{kl} f_{klt} - \theta_{kt} + \theta_{lt} & \perp \mu_{klt}, \quad (k,l) \in \mathcal{A}, \\
0 \leq \tau_{kl} - f_{klt} & \perp \rho_{klt} \geq 0, \quad (k,l) \in \mathcal{A}, \\
0 \leq \tau_{kl} + f_{klt} & \perp \sigma_{klt} \geq 0, \quad (k,l) \in \mathcal{A}, \\
0 \leq \textstyle\sum_{i \in k} x_{it} - \sum_{l:(k,l) \in \mathcal{A}} f_{klt} & \\
+ \textstyle\sum_{l:(l,k) \in \mathcal{A}} f_{lkt} - d_{kt} & \perp p_{kt} \geq 0, \quad k \in \mathcal{K}.
\end{aligned}
$$

Here all constraints with index t are assumed to hold for all $t \in \mathcal{T}$. The problem TEP can be solved using standard nonlinear programming solvers that exploit the complementarity structure of the lower level. However, TEP is a non-convex optimization problem due to the complementarity conditions in its constraints. Therefore, standard solvers will at best yield a local optimum.

In order to achieve global optimality we convert TEP into a mixed integer program. Complementarities are linearized introducing additional binary variables using the approach of Fortuny-Amat and McCarl [14]. This replaces a complementarity condition such as

$$0 \le \quad F(x,y) \quad \perp \quad x \quad \ge 0$$

by

$$\begin{aligned} 0 &\le x \le Mz, \\ 0 &\le F(x,y) \le M(1-z), \\ z &\in \{0,1\}, \end{aligned}$$

where M is chosen large enough to bound both x and $F(x,y)$. Observe that the complementarity conditions (3) require this construction for ρ and σ only as the other conditions can be imposed in TEP as equality constraints. Similarly we can simplify (4) by assuming that all nodal prices are strictly positive and requiring

$$\sum_{i \in k} x_{it} - \sum_{l:(k,l) \in \mathcal{A}} f_{klt} + \sum_{l:(l,k) \in \mathcal{A}} f_{lkt} - d_{kt} = 0. \qquad (6)$$

The complete mixed integer programming formulation (MIQP) of TEP can now be written out as follows. Please note that the constraints below that correspond to the linearization of the complementarity constraints have been tagged on the left-hand side by the original complementarity constraint. For example, $(2b)$ corresponds to the second complementarity contraint in (2). Also, we suppress the dependence of constraints on $t \in \mathcal{T}$ and prefix each big M constraint with the equation number of its corresponding complementarity condition. This gives

$$
\begin{array}{lrll}
\text{MIQP: min} & \multicolumn{3}{l}{\sum_{(k,l)\in\mathcal{A}} C_{kl}(\tau_{kl}) + \sum_{k\in\mathcal{K}}\sum_{i\in k} K_i u_i} \\
& \multicolumn{3}{l}{\quad + \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}}(\tfrac{1}{2}b_k d_{kt}^2 - a_k d_{kt} \quad + \sum_{i\in k} c_{it}x_{it})} \\
\text{s.t.} & p_{kt} - p_{lt} + \rho_{kl} - \sigma_{klt} + \mu_{klt}X_{kl} & = 0, & (k,l)\in\mathcal{A}, \\
& \sum_{l:(l,k)\in\mathcal{A}} \mu_{lkt} - \sum_{l:(k,l)\in\mathcal{A}} \mu_{klt} & = 0, & k\in\mathcal{K}, \\
& X_{kl}f_{klt} - \theta_{kt} + \theta_{lt} & = 0, & (k,l)\in\mathcal{A}, \\
& \sum_{i\in k} x_{it} - \sum_{l:(k,l)\in\mathcal{A}} f_{klt} + \sum_{l:(l,k)\in\mathcal{A}} f_{lkt} & = d_{kt}, & k\in\mathcal{K}, \\
(1) & p_{kt} - (a_k - b_k d_{kt}) & \le M z_{kt}, & k\in\mathcal{K}, \\
& d_{kt} & \le M(1-z_{kt}), & k\in\mathcal{K}, \\
(2a) & c_i - p_{k(i)t} + \varphi_i x_{it} + \lambda_{it} & \le M w_{it}, & i\in k, k\in\mathcal{K}, \\
& x_{it} & \le M(1-w_{it}), & i\in k, k\in\mathcal{K}, \\
(2b) & K_i - \sum_{t\in\mathcal{T}} \lambda_{it} & \le M v_i, & i\in k, k\in\mathcal{K}, \\
& u_i & \le M(1-v_i), & i\in k, k\in\mathcal{K}, \\
(2c) & u_i - x_{it} & \le M y_{it}, & i\in k, k\in\mathcal{K}, \\
& \lambda_{it} & \le M(1-y_{it}), & i\in k, k\in\mathcal{K}, \\
(3a) & \tau_{kl} - f_{klt} & \le M r_{klt}, & (k,l)\in\mathcal{A}, \\
& \rho_{klt} & \le M(1-r_{klt}), & (k,l)\in\mathcal{A}, \\
(3b) & \tau_{kl} + f_{klt} & \le M s_{klt}, & (k,l)\in\mathcal{A}, \\
& \sigma_{klt} & \le M(1-s_{klt}), & (k,l)\in\mathcal{A}, \\
& \rho_{klt}, \sigma_{klt}, \tau_{kl}, \tau_{kl}+f_{klt}, \tau_{kl}-f_{klt} & \ge 0, & (k,l)\in\mathcal{A}, \\
& p_{kt} - (a_k - b_k d_{kt}), d_{kt} & \ge 0, & k\in\mathcal{K}, \\
& c_i - p_{k(i)t} + \varphi_i x_{it} + \lambda_{it}, x_{it} & \ge 0, & i\in k, k\in\mathcal{K}, \\
& u_i, x_{it}, u_i - x_{it}, K_i - \sum_{t\in\mathcal{T}} \lambda_{it} & \ge 0, & i\in k, k\in\mathcal{K}, \\
& w_{it}, v_i, y_{it} & \in \{0,1\}, & i\in k, k\in\mathcal{K}, \\
& z_{kt}, r_{klt}, s_{klt} & \in \{0,1\}, & (k,l)\in\mathcal{A}.
\end{array}
$$

Observe that MIQP is a mixed integer program with convex quadratic constraints. It is now standard for commercial MIP solvers (such as Gurobi) to handle convex quadratic terms. It is also possible to approximate the quadratic function $\frac{1}{2}b_k d_{kt}^2$ by a piecewise linear convex function so that MIQP becomes a mixed integer linear program. Observe that care must be taken if working with a coarse approximation of $\frac{1}{2}b_k d_{kt}^2$ as the KKT conditions that appear in the constraints are based on $\frac{1}{2}b_k d_{kt}^2$ rather than its approximation.

The constraints of MIQP require some special attention in the case where a new line is built joining k and l by choosing $\tau_{kl} > 0$. In this case the set of lines $\mathcal{A}$ is enlarged, which adds new constraints to MIQP. The capacity of the new line is expressed in terms of expansion increments $T_q, q\in\mathcal{Q}$ and binary variables κ_{klq}, so

$$\tau_{kl} = \sum_{q\in\mathcal{Q}} \kappa_{klq} T_q.$$

It is well known that adding new transmission lines (even at zero cost) to a DC-load flow model can lead to a loss of welfare akin to the Braess paradox of traffic engineering [6]. In our model, welfare might be lost in some periods and gained in others depending on the demand. To ensure that line investments have nonnegative benefits, we assume that the system operator can switch out a line in periods when it detracts from welfare so that adding a zero cost line can never decrease welfare.

This feature is modeled by a binary variable η_{klt} that indicates if the line (k,l) is being used. We enlarge $\mathcal{A}$ in the formulation MIQP to include all potential arcs, so

$$\mathcal{A} = \{(k,l) : k < l, k, l \in \mathcal{K}\},$$

and replace

$$X_{kl} f_{klt} - \theta_{kt} + \theta_{lt} = 0, \quad (k,l) \in \mathcal{A}$$

by constraints

$$\begin{array}{rcl} X_{kl} f_{klt} - \theta_{kt} + \theta_{lt} & \leq & (1 - \eta_{klt})M, \quad (k,l) \in \mathcal{A}, \\ X_{kl} f_{klt} - \theta_{kt} + \theta_{lt} & \geq & -(1 - \eta_{klt})M, \quad (k,l) \in \mathcal{A}, \\ f_{klt} & \leq & \eta_{klt} M, \quad (k,l) \in \mathcal{A}, \\ f_{klt} & \geq & -\eta_{klt} M, \quad (k,l) \in \mathcal{A}, \\ \mu_{lkt} & \leq & \eta_{klt} M, \quad (k,l) \in \mathcal{A}, \\ \mu_{lkt} & \geq & -\eta_{klt} M, \quad (k,l) \in \mathcal{A} \\ \eta_{klt} & \leq & \sum_{q \in \mathcal{Q}} \kappa_{klq}, \quad (k,l) \in \mathcal{A}. \end{array}$$

If $\eta_{klt} = 0$ for some $t \in \mathcal{T}$ then the new constraints set f_{klt} and μ_{lkt} to zero, and there is no constraint on $-\theta_{kt} + \theta_{lt}$. This has the same effect on MIQP as removing the arc (k,l) from the index set $\mathcal{A}$ wherever it appears in the equality constraints applying at t, without removing arc (k,l) from the index set $\mathcal{A}$ in constraints (3) of MIQP. So capacity expansion $\tau_{kl} > 0$ is possible even though the line (k,l) is not used in period t. On the other hand, if $\tau_{kl} = 0$ then (2) implies $\eta_{klt} = 0$ for all $t \in \mathcal{T}$, and so the constraint on $-\theta_{kt} + \theta_{lt}$ is omitted.

3 Multistage transmission expansion

The transmission expansion problem MIQP provides a single opportunity to invest in extra transmission capacity. We now explore how to extend MIQP to a multistage problem in which transmission expansion decisions are planned to be implemented over a long time horizon of twenty or thirty years. In the multistage problem there are opportunities to be flexible in choosing investments. For example, the planner may wish to delay investment until there is more certainty about future demand, or make capacity decisions now that provide some flexibility for future augmentation.

Flexibility is important since many of the parameters in MIQP (denoted by the vector ξ) will be realized some time in the future, and so they will be subject to considerable uncertainty. The possible values that they take can be represented using a *scenario tree* with nodes $n \in \mathcal{N}$ and leaves in $\mathcal{L}$. At each node in this tree we acquire some new information and make a transmission investment decision based on the information we have accrued up that point. The time intervals between these decision points depend on the particular setting, but we imagine they are measured in years rather than hours (like $t \in \mathcal{T}$) so we index the decision points by $y \in \mathcal{Y}$. A pictorial representation of a scenario tree with four time stages is given in Figure 1.

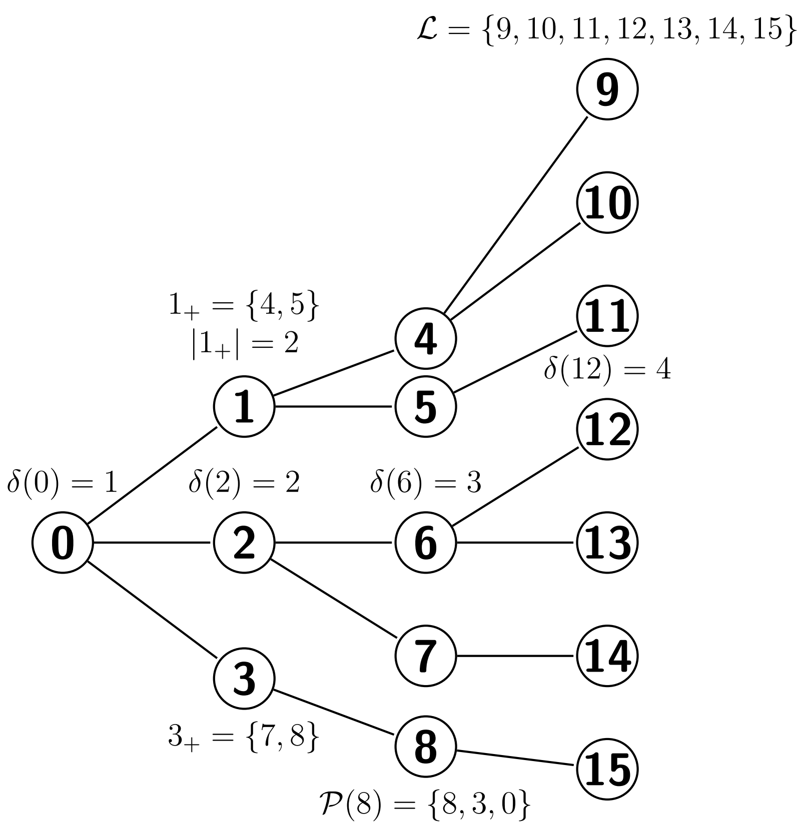


Figure 1: A scenario tree with nodes $\mathcal{N} = \{0, 1, \ldots, 15\}$, and $\mathcal{Y} = 1, 2, 3, 4$.

The probability of the event represented by node n is denoted $\phi(n)$. By convention we number the root node $n = 0$. The unique predecessor of node $n \neq 0$ is denoted by n_-. We denote the set of children of node $n \in \mathcal{N} \setminus \mathcal{L}$ by

n_+, and denote its cardinality by $|n_+|$. The set of predecessors of node n on the path from n to node 0 is denoted $\mathcal{P}(n)$ (so $\mathcal{P}(n) = \{n, n_-, n_{--}, \ldots, 0\}$), where we use the natural definitions for n_{--}. The depth $\delta(n)$ of node n is the number of nodes on the path to node 0, so $\delta(0) = 1$ and we assume that every leaf node has the same depth, say $\delta_{\mathcal{L}}$. The depth of a node $\delta(n)$ can be interpreted as its time index $y \in \mathcal{Y}$. At node n of the scenario tree the parameters of MIP are assumed to take values $\xi(n)$. We use the notation $(\#(n))$ to denote the set of constraints $(\#)$ for MIP applied at at node n by substituting the parameters $\xi(n)$ for ξ, and making all variables assume the extra index n. Thus for example (4) becomes

$$0 \leq \sum_{i \in k} x_{it}(n) - \sum_{l} f_{klt}(n) + \sum_{l} f_{lkt}(n) - d_{kt}(n) \perp p_{kt}(n) \geq 0.$$

Given realizations for the random parameters in each state of the world, we now consider a multistage stochastic transmission expansion model. Let T^0_{kl} be the initial value of transmission capacity between bus k and l, where $k < l$. For each such pair (k, l) and node n recall the binary variable $\kappa_{klq}(n)$ that denotes an expansion in node n of line capacity of type $q \in \mathcal{Q}$ (adding an increment $T_q(n)$ and costing an extra amount c_{klq}). The total capacity of the line (k, l) in node n is then

$$\tau_{kl}(n) = T^0_{kl} + \sum_{p \in \mathcal{P}(n)} \sum_{q \in \mathcal{Q}} \kappa_{klq}(p) T_q(p). \tag{7}$$

The objective function that the system planner seeks to minimize is the expected cost of the transmission expansion minus the expected social benefit that it creates, giving

$$\sum_{n} \phi(n) (\sum_{k,l,q} c_{klq} \kappa_{klq}(n) - W(n)). \tag{8}$$

There will be a set of constraints in each node n of the scenario tree that define the operations that will be in equilibrium in that state of the world. These constraints are exactly those of MIQP, reproduced so each variable and parameter assumes an extra index "(n)". The multistage transmission capacity expansion problem then takes the form of a multistage stochastic mixed integer program. Since the number of nodes in the scenario tree grows exponentially with the number of stages, the number of binary variables in the deterministic equivalent version of the multistage transmission capacity expansion problem grows rapidly, and the large-scale problem becomes impossible to solve. The JuDGE package enables us to attack this problem using decomposition (which splits the deterministic equivalent problem into many smaller MIPs).

4 JuDGE

JuDGE (which stands for Julia Decomposition for Generalized Expansion) [9] is an open-source Julia [4] package for solving multistage stochastic capacity expansion problems, and is based on the Dantzig-Wolfe decomposition algorithm specialized to capacity expansion by Singh et al. in [29]. Specifically, JuDGE implements a variant of their split-variable formulation, called the SV1 model, where each investment decision is binary, and so can be built at most once for each scenario. Fortunately, however, this is not particularly restrictive, since we are able to define separate investments that are additive, with the effect of allowing multiple upgrades (with additive costs).

The JuDGE package provides a core modelling framework for defining a scenario tree, with corresponding nodal subproblems, utilizing the JuMP package [11] in order to define both the back-end master problem, and the front-end, user-customizable subproblems and investment variables. JuDGE automates the Dantzig-Wolfe column-generation procedure and computes upper and lower bounds as the problem is solved.

We have implemented the multistage transmission expansion problem described in section 3 using JuDGE. This model is specified in terms of a scenario tree (with corresponding probabilities), the investment variables, and the subproblems, which in our case is the MIP model (with appropriate parameters for each node of the scenario tree), as described in section 2, defined as JuMP models. JuDGE also provides functionality to automatically generate a deterministic-equivalent formulation of the JuDGE model; we have utilized this for our computational results in section 5.

The core decomposition method of JuDGE, involves constructing a restricted master problem (SV1-RMP, below, which has been reproduced from [29], with modified notation) which defines the non-anticipativity constraints for the investments. The master problem does not strictly enforce integrality.

$$\begin{aligned}
\text{SV1-RMP:} \qquad \min \quad & \sum_{n\in\mathcal{N}} \phi_n c_n^\top \kappa'_n + \sum_{n\in\mathcal{N}} \sum_{j\in J_n} \phi_n \psi_n^j \omega_n^j \\
\text{s.t.} \quad & \sum_{j\in\mathcal{J}_n} \hat{\kappa}_n^j \omega_n^j \leq \sum_{h\in\mathcal{P}(n)} \kappa'_h, \quad n\in\mathcal{N}, \qquad [\pi_n] \qquad (9) \\
& \sum_{j\in\mathcal{J}_n} \omega_n^j = 1, \quad n\in\mathcal{N}, \qquad [\nu_n] \qquad (10) \\
& \omega_n^j \geq 0, \quad n\in\mathcal{N}, j\in\mathcal{J}_n, \\
& \kappa'_n \geq 0, \quad n\in\mathcal{N}.
\end{aligned}$$

Here κ'_n is the vector of (binary) investments chosen to be made for node

n; naturally, these investments will also be available for all descendants of node n. ω_n^j are weightings that are used to form convex combinations of the columns, $j \in \mathcal{J}_n$ corresponding to each node n. For node n, the j^{th} column has investments $\hat{\kappa}_n^j$, and has a corresponding cost of operations ψ_n^j. The dual vectors π_n associated with (9) can be thought of as the marginal costs of utilizing investments in node n.

When the restricted master problem is solved, it will seek some convex combinations of columns, $j \in \mathcal{J}_n$ for each node n, as enforced by constraint (10). Together, the corresponding investments must satisfy the non-anticipativity constraints (9), and minimize the overall investment and operational costs.

The details of the algorithm are provided in [29, 9], but the main loop of JuDGE is a column generation procedure, which we will briefly outline. Suppose we solve SV1-RMP, given some sets of columns for each node, and compute the optimal objective function value as z for this restricted master problem. For each node, the method seeks to find the column with the most negative reduced cost for SV1-RMP; this minimum reduced cost for node n is $RC(n) = \psi_n^j - \pi_n^\top \hat{\kappa}_n^j - \nu_n$. If this is negative, we add this column to SV1-RMP for the next iteration. This column generation will continually improve the objective of the restricted master problem, thereby reducing the upper bound. This also enables us to compute a valid lower bound $\underline{z}$ for the optimal objective function value (z^*) of the full master problem.

$$z^* \geq \underline{z} = z - \sum_{n \in \mathcal{N}} RC(n).$$

Moreover, this lower bound is tight, since at the optimal solution the smallest reduced cost for every node will be 0, providing a certificate of optimality for the relaxed master problem.

This procedure can often result in a naturally integer optimal solution; however, in some cases this solution can be fractional. For such instances JuDGE provides an implementation of branch-and-price, described in [9]. This will branch on fractional investments, making use of the column generation procedure and lower bounds, as outlined above, in order to find provably-optimal integer solutions.

5 Results

In this section we present a case study illustrating the stochastic MPEC solved using JuDGE, and the computational results of JuDGE applied to some large problem instances.

5.1 Illustrative Case Study

To illustrate the model, we solve an instance of the multistage transmission expansion problem for a 5-bus transmission network over a scenario tree with 7 nodes. Please note that for clarity we will say *bus* (and use notation k and l in referring to buses) when referring to the transmission power network, and *node* (and use nomenclature n) when we refer to the stochastic tree.

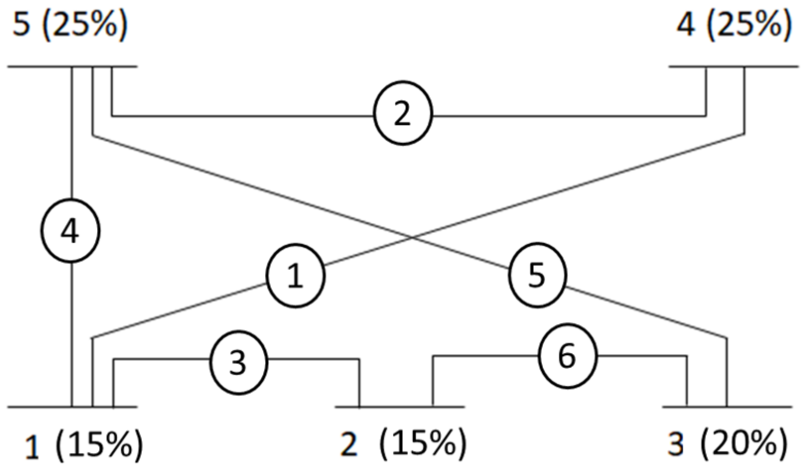


Figure 2: 5-bus transmission network with possible candidate lines indexed 1 to 6, and percentage of demand intercept in parentheses.

Figure 2 shows the 5-bus power network, where numbers in parenthesis correspond to the percentage of the total demand observed at each bus k when all prices are zero. The 6 lines drawn between the buses define the candidate lines that the model can build.

The constraint (7) defines the expansion of line (k, l) in node n by binary decision variables $\kappa_{klq}(n)$ that add fixed increments $T_q(n)$ to the transmission line. In our case study we assume $T^0_{kl} = 0$ and possible expansion increments are the same for all n, as defined by Table 1. The cost of the transmission expansion decision for line (k, l) in node n is then $\sum_{q=1}^{5} c_{klq}\kappa_{klq}(n)$, and the amount of transmission capacity this yields in node n is $\sum_{p\in\mathcal{P}(n)} \sum_{q=1}^{5} \kappa_{klq}(p)T_q$. All lines built are assumed to yield equal values of reactance between their endpoints, and line expansion costs c_{klq} are chosen to be proportional to T_q.

The consumer utility at bus k gives a linear demand function $\frac{a_k}{b_k} - \frac{1}{b_k}p$ for demand at price p. We set b_k =10 M€/(GWh)^2 at each node k, and let a_k depend on the bus and the nodes of the scenario tree. In the root node of the tree a_k takes the values shown in Table 2.

q	T_q(MW)
1	40
2	100
3	160
4	250
5	460

Table 1: Values of expansion increments T_q.

Bus k	a_k(M€)
1	2.421
2	2.421
3	3.228
4	4.035
5	4.035

Table 2: Values of a_k at each bus k. Total demand at zero price is 1614 MW shared amongst buses according to the percentages shown in Figure 2.

Table 3 and Table 4 contain generator and line data respectively. Note that the investment costs K_i and c_{klq}/T_q represent annual investment costs per MW of generation capacity and transmission capacity respectively. Since in our case study we only solve for one representative hour, these investment costs are deflated by 8760 in the model to represent the hourly investment costs. We also discount all costs and welfare in nodes 1 and 2 by the factor 0.9, and by 0.81 in nodes 3,4,5,6. Base power is 0.1 GW, and the cost of CO_2 emissions is set at 18 €/tonne.

The scenario tree has depth 3 and degree 2, leading to a total of 7 nodes and 4 scenarios as depicted by Figure 3. At each node n of the stochastic tree we scale a_k by the factor shown to give $a_k(n)$. Here scenarios 3 and 4 have no growth in demand in the southern buses 1, 2 and 3, but growth in demand in buses 4 and 5. In contrast, scenarios 5 and 6 experience growth in demand in the southern buses 1, 2 and 3, but none in the north. All scenarios are assumed to have equal probability.

We now present the results of applying JuDGE to three instances of this stochastic problem. The first instance (Competitive) assumes all agents are perfectly competitive so we set $\varphi_i = 0$ for every i. In the second instance (Coal) we set we set $\varphi_1 = 10$ (Cournot) for the coal-fired generator ($i = 1$)

Generator i	Bus $k(i)$	Type	c_i (€/MWh)	K_i (k€/MWy)	Emission Rate tCO_2/MWh
1	5	Coal	28	19.0	0.9
2	1	CCGT	41	8.0	0.3
3	1	CCGT	41	8.0	0.3
4	2	CCGT	41	8.0	0.3

Table 3: Generator data.

From k	To l	Line reactance X_{kl} (p.u.)	Investment cost c_{klq}/T_q (k€/MWy)
1	4	0.030	1860
4	5	0.030	1800
1	2	0.030	1900
1	5	0.030	1810
3	5	0.030	1820
2	3	0.030	1830

Table 4: Line data.

and assume all other generators are perfectly competitive. Finally in the third instance we set $\varphi_i = 10$ for every i, which corresponds to all generators acting as Cournot oligopolists. All results are reported in terms of total system welfare, which is maximized by JuDGE.

5.1.1 Competitive generators

The first experiment assumes that all generators are perfectly competitive so $\varphi_i = 0$ for every i. JuDGE gives the line investments shown in Table 5.

Observe that in node 0 of the scenario tree the optimal solution builds lines 2, 5 and 6. Line 2 connects the demand at bus 4 to the coal generator at bus 5, lines 5 and 6 connect the demand at buses 2 and 3 to the coal generator at bus 5. All of the demand in buses 1, 2 and 5 is met by local generators, bus 4 imports all power to meet demand from the coal generator at bus 5, finally bus 3's demand is met by a combination of the coal generator from bus 5 and the CCGT generator at node 2.

In nodes 2, 5 and 6 of the scenario tree, where the demand grows in the south (buses 1, 2, 3, as shown Figure 3), we observe further expansions in line capacity in line 5 (between buses 5 and 3) and line 6 (between buses 2

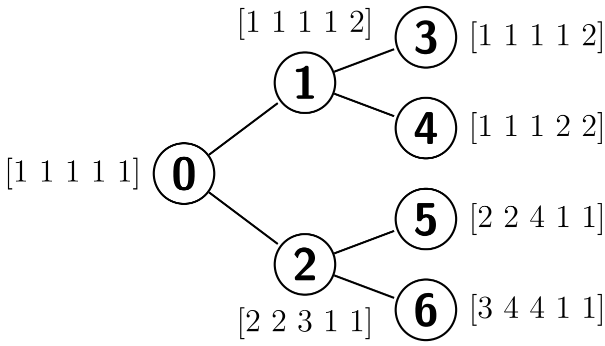


Figure 3: A scenario tree with 7 nodes $\mathcal{N} = \{0, 1, \ldots, 6\}$, and $\mathcal{Y} = 1, 2, 3$. The vector $s(n)$ shown at each node n scales a_k by $s_k(n)$ at bus k.

Line	1	2	3	4	5	6
n=0	0	390	0	0	160	160
n=1	0	390	0	0	160	160
n=2	0	390	0	0	720	260
n=3	0	390	0	0	160	160
n=4	390	390	0	0	160	160
n=5	0	390	0	0	1010	260
n=6	0	390	0	0	1010	260

Table 5: Optimal line capacities (MW) by scenario node for perfect competition. Total expected discounted welfare = 17.0151 M Euro.

and 3). However, in node 4 of the scenario tree we have increasing demand in bus 4 in scenario node $n = 4$ (shown by $s_4(4) = 2$ in Figure 3); here we observe that line 1's capacity is expanded to enable the CCGT at bus 1 to supply the demand at bus 4.

We can compare the solution in Table 5 with solutions obtained in each of the four scenarios as shown in Table 6. The expected welfare in Table 6 is slightly higher, since the transmission investment solution can adapt to each scenario, for example by making expansion decisions for line 5 in stage 1 of 0MW in scenarios 1 and 2, but 160MW in scenarios 3 and 4 (to be expanded to 1010 MW by stage 3).

Finally the solution in Table 5 can be compared with the solution obtained by averaging demand outcomes over the four scenarios in each time period. This gives the solution shown in Table 7. Expected welfare from this solution is lower than that in the optimal solution to the stochastic problem because

scenario	stage	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$
1	1	0	390	0	0	0	300
1	2	0	390	0	0	0	300
1	3	0	390	0	0	0	300
2	1	0	410	0	0	0	300
2	2	0	410	0	0	0	300
2	3	260	510	0	0	0	300
3	1	0	390	0	0	160	160
3	2	0	390	0	0	720	260
3	3	0	390	0	0	1010	260
4	1	0	390	0	0	160	160
4	2	0	390	0	0	720	260
4	3	0	390	0	0	1010	260

Table 6: Total invested line capacities (MW) optimized for each scenario under perfect competition. Total expected discounted welfare = 17.0153 M Euro.

of reduced investment in the high demand scenarios. Thus the value of a stochastic solution is high in this case, as the flexibility it affords nearly captures all the possible value that would accrue from solving the problem with perfect foresight.

Line	1	2	3	4	5	6
y=1	0	410	0	0	350	0
y=2	0	410	0	0	390	250
y=3	40	450	0	0	390	390

Table 7: Total invested line capacities (MW) by stage y for perfect competition. Total expected discounted welfare = 14.4746 M Euro.

5.1.2 Coal monopolist

The first experiment assumed that all generators are perfectly competitive. We relax this assumption by setting $\varphi_1 = 10$ for the coal generator at bus 5. Under the conjectural assumptions made here (as in [35]), the coal generator behaves as a local monopolist at bus 5, to meet demand in this node without accounting for the effect of their actions on transmission flows. JuDGE gives the line investments shown in Table 8.

Line	1	2	3	4	5	6
$n = 0$	390	0	0	460	0	300
$n = 1$	390	0	0	760	0	300
$n = 2$	390	0	0	460	0	1010
$n = 3$	390	0	0	760	0	300
$n = 4$	850	0	0	760	0	300
$n = 5$	390	0	0	560	250	1010
$n = 6$	390	0	0	560	250	1010

Table 8: Optimal line capacities (MW) by scenario node for perfect competition with monopolist coal generator at bus 5. Total expected discounted welfare = 16.8674 M Euro.

In this example it is welfare enhancing to expand line 4 that connects buses 1 and 5. This enables the price-taking CCGT generator at bus 1 to send power to consumers in bus 5 to alleviate the price-setting behaviour of the coal generator at this location. In scenario-tree nodes 5 and 6 when demand is high in bus 3, line 5 is expanded to carry some the power from bus 1 to bus 3, via bus 5.

5.1.3 Cournot generators

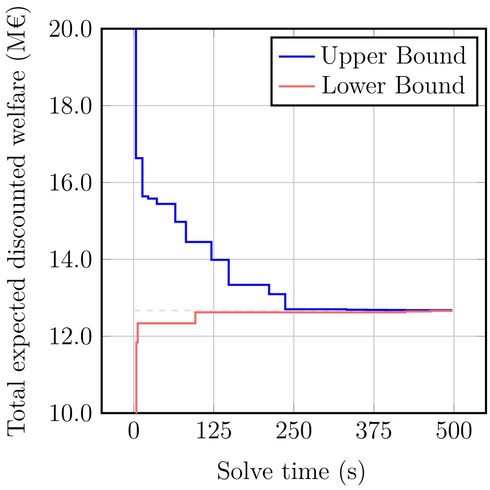
We now assume that all generators are strategic, and set $\varphi_i = 10$ for all i. JuDGE gives the line investments shown in Table 9.

Line	1	2	3	4	5	6
$n = 0$	250	0	40	0	0	160
$n = 1$	250	40	40	260	0	160
$n = 2$	250	40	40	250	500	160
$n = 3$	250	40	40	260	0	160
$n = 4$	390	140	0	260	40	160
$n = 5$	250	40	40	260	160	260
$n = 6$	250	40	200	260	260	160

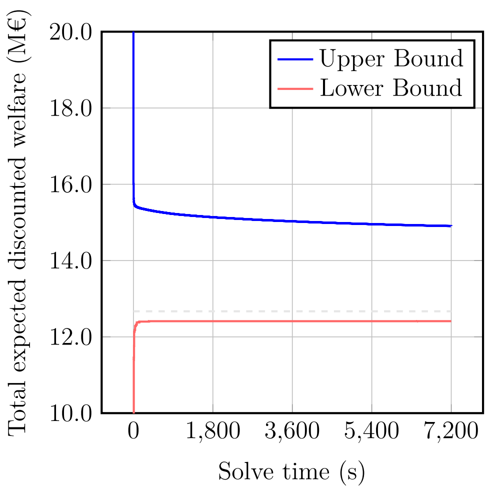
Table 9: Optimal line capacities (MW) by scenario node for Cournot competition. Total expected discounted welfare = 12.6764 M Euro.

This solution builds less capacity in line 4, than for the previous example (Table 8), since the generators at bus 1 are no longer behaving competitively, so there is less value in transporting power from the CCGTs at bus 1 to the demand at bus 5.

The Cournot example also gives some indication of the efficiency gains from JuDGE that we explore more fully in the next section. We formulated this problem both as a JuDGE model and as a deterministic-equivalent MIP using Gurobi 9.02 as the solver. Figure 4 shows that the JuDGE decomposition solves to a bound gap of 0.1% within 500s, whereas the deterministic equivalent still has a bound gap exceeding 20% after two hours of CPU time and the best integer solution it has found by this point is still some distance from optimality (indicated by the grey dashed line).



(a) JuDGE decomposition.



(b) Deterministic equivalent.

Figure 4: Comparison of convergence for Cournot agent model.

5.2 Computational Efficiency

In this section we present the results of some experiments that explore the computational efficiency of the JuDGE implementation when applied to the 5-bus network data of section 5.1. As shown in Figure 4 the deterministic equivalent MIP failed to solve this problem with Cournot agents. Here we investigate the effect of both increasing the size of the scenario tree and increasing the number of time periods ($t \in \mathcal{T}$) on JuDGE computation time in comparison with the deterministic equivalent MIP.

5.2.1 Scenario tree size

To do this, we construct scenario trees of varying maximum depth and degree with randomly generated demand growth data, with Cournot agents. Here we use the notation (d, m) to denote a tree with degree d and maximum depth of m, giving $1 + d + d^2 + \ldots d^{m-1}$ nodes. The computational results of applying JuDGE to the 5-bus problem with these trees are shown in Table 10.

All computations are carried out on a virtual machine with an Intel Xeon E5-2690 with 16 cores @1.90GHz and 128GB RAM running under Windows Server 2013. The solver we have used in JuDGE is Gurobi 9.02. The convergence criterion for the stochastic MPEC using JuDGE is set to a 1% relative gap. In other words JuDGE terminates when the difference between the upper and lower bound on the optimal objective value is less than 1% of the upper bound. The objective referred to here is expected total discounted welfare. JuDGE returns a candidate integer solution (Incumbent) and an upper bound on its value (BestBd). We report the actual relative gap using the best integer solution found, where the relative gap is defined to be BestBd-V(Incumbent) divided by V(incumbent). This is occasionally significantly smaller than the termination tolerance, due to the nature of the algorithm.

Tree	Nodes	V(Incumbent)	BestBd	Gap (%)	Time (s)
$(3,3)$	13	8.2417	8.3238	0.99	121.3
$(3,4)$	40	12.5860	12.7126	1.00	131.1
$(3,5)$	121	19.5999	19.7772	0.90	224.6
$(3,6)$	364	37.6992	37.9116	0.56	881.7
$(5,3)$	31	9.1504	9.2393	0.97	217.7
$(5,4)$	156	16.1912	16.3403	0.92	484.8
$(5,5)$	781	49.4345	49.6743	0.48	1683.1
$(9,3)$	91	10.8957	10.9963	0.92	395.2
$(9,4)$	820	32.1330	32.3357	0.63	1460.2

Table 10: JuDGE CPU times for solving 5-bus Cournot model with different size scenario trees.

The deterministic equivalent problems all failed to solve within two hours on these problems. In many cases they failed to find any integer solutions.

5.2.2 Varying the number of time periods

The second set of experiments that we have carried out examines how the solution time scales as we increase the number of time periods in each node of the scenario tree. We will extend the coal monopolist example from section 5.1.2, for a scenario tree of depth 3 and degree 3 (13 nodes), with the number of time periods per node ($|\mathcal{T}|$) increasing from 1 to 12. For each problem instance, we solve the model using JuDGE, and compare this with the deterministic equivalent MIP, both stopping when they reach a 1% bound-gap. The results are plotted on a log-scale in Figure 5 below. From these results we can see that for small problems, with fewer than 6 time periods per node, the deterministic equivalent formulation outperforms JuDGE. However, for larger problems where there are 7-12 time periods per node, the decomposition method solves 2-3 times faster. As an example, the model with 9 time periods takes more than 30 minutes to reach a 1% bound-gap for the deterministic equivalent, whereas JuDGE solves the same problem in under 10 minutes.

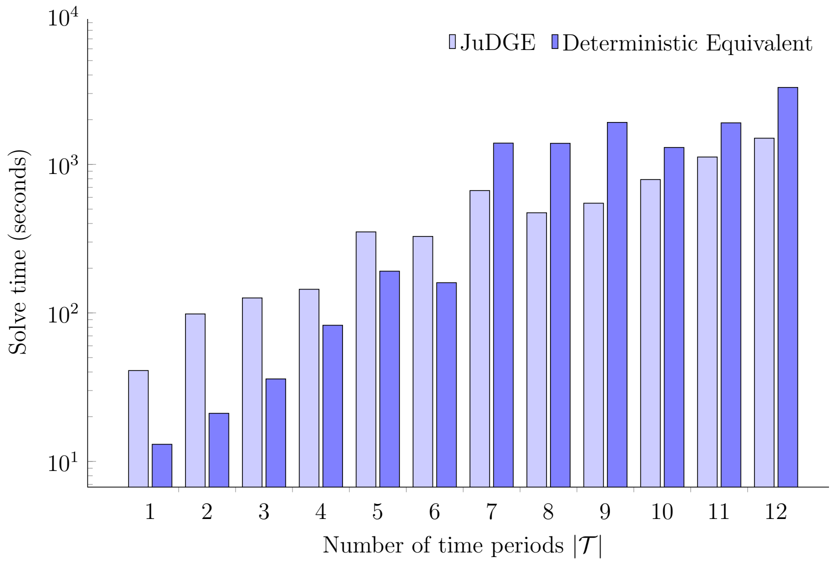


Figure 5: Comparison of solve times for the JuDGE decomposition and the deterministic equivalent model with varying numbers of time periods per node.

6 Conclusion

We have presented a model for socially optimal transmission capacity expansion that can be applied to settings with electricity generators behaving either as perfectly competitive or Cournot players. Note (see [33]) that it is possible to choose the parameter $\varphi_i \in (0, b_{k(i)})$ to model the behaviour of generator i as falling between these extremes. The structure of our model enables one to study changes in agent behaviour that evolve randomly over time (perhaps in response to evolving regulatory intervention).

The representation of generator capacity expansion in our model could be made more realistic. We assume that generator capacity decisions are made in each node of the scenario tree, independently of previous generation capacity decisions, and are made following the transmission capacity decisions that are made in that node. However generation capacity has a long life, and so a generator would need to account not only for the history of her actions when making a capacity decision in a given state of the world, but also the possible future states of the world in which this capacity will be used. An arguably more realistic model here would be an optimal transmission plan formulated in a scenario tree that accounts for a dynamic equilibrium for generators (formulated in the same tree) that uses the planned transmission. Unfortunately this model does not admit a decomposition by node that is required to apply JuDGE.

The deterministic MIPs that result from our model can be at very large scale, and have many "big-M" logical constraints, so they are computationally challenging for current MIP solvers. By decomposing into many smaller MIPs, the JuDGE platform enables a multistage stochastic model with many scenarios to be formulated and solved to a high degree of accuracy.

The examples we have presented show that JuDGE can (approximately) solve realistic instances of these models with modest resources. In practice the instances to be solved will have many more electrical buses and might involve short-term variations in intermittent renewable energy supply and random plant outages. This makes each subproblem a *stochastic* MPEC which will yield a very large deterministic equivalent problem that will not be solvable using current MIP solvers. Decomposition techniques like JuDGE provide some hope of discovering optimal transmission plans for such problems.

All our models maximize expected discounted welfare. The discount factor can be chosen to represent a risk-adjusted cost of capital as defined by a CAPM methodology [7]. Alternatively the model can be modified to incorporate a dynamic risk measure for the transmission planner. Incorporating a (possibly different) risk measure for each generator then gives a JuDGE optimization subproblem with risked-equilibrium constraints as studied by [13].

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Nomenclature

Parameters

$\mathcal{T}$	=	set of time periods $t \in \mathcal{T}$
$\mathcal{K}$	=	set of buses k (or l) $\in \mathcal{K}$
$\mathcal{A}$	=	set of transmission lines $(k, l) \in \mathcal{A}$
$\mathcal{Q}$	=	set of transmission capacity increments $q \in \mathcal{Q}$
$\mathcal{N}$	=	set of nodes $n \in \mathcal{N}$ in scenario tree
$\mathcal{L}$	=	set of leaves in scenario tree
$\mathcal{Y}$	=	set of decision points $y \in \mathcal{Y}$ in scenario tree
$\mathcal{P}(n)$	=	set of predecessors of node n
$i \in k$	=	generator i at bus k
$k(i)$	=	bus location for generator i, i.e. k such that $i \in k$
a_k	=	inverse demand curve intercept at bus k
b_k	=	inverse demand curve slope at bus k
c_{it}	=	marginal cost of generator i in period t
X_{kl}	=	transmission line reactance for line from k to l
K_i	=	cost per MW of expanding capacity of generator i
φ_i	=	competition parameter for generator i
C_{kl}	=	cost of expanding transmission capacity for line from k to l
M	=	large scalar used for linearisation
$\xi(n)$	=	uncertain parameters at node n
T_q	=	transmission increment q
T^0_{kt}	=	initial value of transmission capacity between bus k and l
$\delta(n)$	=	depth of node n in scenario tree
$\phi(n)$	=	probability of the event represented by node n

Variables

τ_{kl}	=	transmission capacity choice for line from k to l
u_i	=	capacity choice of generator i
x_{it}	=	output of generator i in period t
f_{klt}	=	transmission flow in line from k to l in period t
d_{kt}	=	demand at bus k in period t
p_{kt}	=	price at bus k in period t
θ_{kt}	=	voltage phase angle at bus k in period t
λ_{it}	=	dual variable on generator i capacity constraint in period t
μ_{klt}	=	dual variable on voltage law constraint for line from k to l in period t
ρ_{klt}	=	dual variable on capacity constraint for line from k to l in period t
σ_{klt}	=	dual variable on (reverse) capacity constraint for line from k to l in period t
κ_{klq}	=	binary variable to decide transmission capacity increment q on line from k to l
z_{kt}	=	binary variable used in linearisation of complemenarity constraints
w_{it}	=	binary variable used in linearisation of complemenarity constraints
v_i	=	binary variable used in linearisation of complemenarity constraints
y_{it}	=	binary variable used in linearisation of complemenarity constraints
r_{klt}	=	binary variable used in linearisation of complemenarity constraints
s_{klt}	=	binary variable used in linearisation of complemenarity constraints
W	=	social welfare

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