Mixed strategies in discriminatory divisible-good auctions*

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Abstract

We introduce the concept of an offer distribution function to analyze randomized offer curves in multi-unit procurement auctions. We characterize mixed-strategy Nash equilibria for pay-as-bid auctions where demand is uncertain and costs are common knowledge; a setting for which pure-strategy supply function equilibria typically do not exist. We generalize previous results on mixtures over horizontal offers as in Bertrand-Edgeworth games, and we also characterize novel mixtures over partly increasing supply functions. We show that the randomization can cause considerable production inefficiencies.

Key words: Pay-as-bid auction, divisible-good auction, mixed strategy equilibria, wholesale electricity markets

JEL Classification D43, D44, C72

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1 Introduction

Modelling auctions has been one of the major successes in the application of game theory, since in this setting the rules controlling the interactions between agents are particularly well defined. With auction theory it has been possible to predict bidding behaviour under different auction formats, and this has helped auction designers to choose efficient formats and to avoid disastrous ones. In this paper we analyze multi-unit auctions where bidders are free to choose a separate price for each object. We assume that the number of traded objects is large and that each bid consists of a complete curve of price-quantity pairs. Such auctions are called divisible-goods auctions (Back and Zender, 1993; Wang and Zender, 2002), auctions of shares (Wilson, 1979) or supply function auctions (Green and Newbery, 1992; Klemperer and Meyer, 1989). Important markets with this character are treasury auctions, electricity auctions and auctions of emission permits. We focus on procurement auctions (reverse auctions), such as electricity markets, where producers compete to sell their goods. But results are analogous for sales auctions. In particular we are interested in circumstances when pure-strategy Nash equilibria (NE) do not exist and in these settings we calculate equilibria with randomized offer curves and give estimates of the production inefficiencies that they cause.

Divisible-good auctions have two typical mechanisms. In a uniform-price procurement auction all sellers are paid the clearing price (the highest accepted offer) for all of their accepted supply. The alternative, which we analyze in this paper, is a pay-as-bid (or discriminatory) procurement auction, where the auctioneer pays each accepted offer according to its offer price. A survey by Bartolini and Cottarelli (1997) has found that 39 out of 42 countries used the discriminatory format in their treasury auctions. On the other hand, most electricity markets use the uniform-price format. But there are exceptions. The electricity market in Britain switched to a pay-as-bid format in 2001 and Italy has recently decided to follow suit. A similar move has been considered in California (Kahn et al., 2001). The electricity market in Iran started to operate in 2003 with a pay-as-bid design. Moreover, the electricity markets in most European countries, Australia and some other countries/states are zonal markets with a blend of the uniform-price and pay-as-bid format.[1] Some of the power system reserves are also procured by the system operator using a discriminatory mechanism, e.g. in Germany (Swider and Weber, 2007).

Most studies of strategic bidding behaviour in divisible-good auctions are limited to characterizations of pure-strategy equilibria. But such equilibria do not always exist, especially in models of pay-as-bid electricity markets with capacity constraints and costs that are common knowledge (Fabra et al., 2006; Genc, 2009; Holmberg, 2009).[2] Thus when using a game theory approach to analyse

[1] Zonal electricity markets have the same price within a region, often a country or a state. This zonal price is normally determined in a uniform-price auction. However, post-clearing adjustments in the production (redispatch) is often needed to relax local transmission/transport constraints within regions. Normally, producers making such adjustments are paid as bid.

[2] But note that without capacity constraints, pure-strategy Bertrand Nash equilibria can be

real divisible-good auctions, it is often necessary to consider mixed strategies. For restrictive market assumptions, Anwar (2006) shows that there will be a type of mixed-strategy equilibrium in a uniform-price multi-unit auction with independent increasing offers.[3] With this exception, previous models of mixed strategy equilibria in discriminatory auctions are limited to mixtures where each producer offers its entire capacity at one price. They occur when producers are pivotal, i.e. competitors do not have enough capacity to meet maximum demand, and they are essentially Bertrand-Edgeworth Nash equilibria (Allen and Hellwig, 1986; Beckmann, 1967; Levitan and Shubik, 1972; Maskin, 1986; Vives, 1986) with the added complexity that either the auctioneer's demand (Anwar, 2006; Fabra et al., 2006; Genc, 2009; Son et al., 2004) or the bidders' costs/valuations are uncertain as in Back and Zender (1993).

Note that Reny (1999) is consistent with non-existing pure-strategy NE in our setting with complete information. He discusses pay-as-bid auctions with incomplete information (e.g. due to private costs) and shows that there will always exist a pure-strategy Bayesian Nash equilibrium. In this case privately observed random variations in an individual firm's payoff function (e.g. due to private costs) can effectively serve as a randomizing device (Wilson, 1969). Thus as shown by Harsanyi (1973), a pure-strategy Bayesian Nash equilibrium can sometimes correspond to a mixed-strategy NE in a game with complete information.[4]

In this paper we generalize previous equilibrium studies of discriminatory auctions by considering general cost functions and general probability distributions for the auctioneer's demand. We are also the first to characterize equilibria with mixtures over increasing offer curves in such auctions. We start the analysis by deriving optimality conditions for producers, who know their own production costs but face an uncertain residual demand. A first-order condition for strictly increasing offers has previously been derived by Hortacsu (2002). We give a fuller treatment including second-order conditions and optimality conditions when an offer's monotonicity constraint is binding; most divisible-good auctions only allow monotonic offer curves. The latter condition also applies to situations where offers are constrained to be horizontal by restrictions on the number of allowed steps in an offer, as in models by Fabra et al. (2006) and Bertrand games.

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found in the special case of constant marginal costs (Fabra et al., 2006; Wang and Zender, 2002). With capacity constraints, Holmberg (2009) finds pure-strategy Nash equilibria with increasing offer curves when the hazard rate of the demand shock is decreasing. Corresponding results for treasury auctions are found by Rostek et al. (2010).

[3] Anwar (2006) analyses a uniform-price auction where an offer is submitted for each discrete production unit. For constant marginal costs and uniformly distributed demand, he shows that there exists a mixed strategy Nash equilibrium with increasing offers that are independently chosen for each production unit.

[4] As in Harsanyi (1973), one can see mixed-strategy equilibria in a game with complete information, i.e. costs are common knowledge, as the limit of a sequence of Nash (Bayesian) equilibria for games with incomplete information as the variation in the individual payoff parameter becomes smaller and smaller. In this limit very small variations in the privately observed cost have a significant influence on the bidding behaviour. It is often much easier to calculate Nash equilibria in games with complete information than Bayesian equilibria in games with incomplete information, and the former can be used as an approximation of equilibrium outcomes in the latter if the individual payoff parameter variation is small relative to the demand uncertainty.

Using the optimality conditions we verify that pure-strategy equilibria are unlikely to exist in discriminatory auctions with capacity constraints when production costs are common-knowledge and demand has an additive demand shock. We analyze duopoly markets where pure-strategy equilibria do not exist and derive symmetric mixed-strategy equilibrium offers in such auctions. We consider three types of mixed-strategy equilibria: 1) Mixtures over increasing supply curves, where slope-constraints are not binding. 2) Mixtures over horizontal offers, where the whole output is offered at the same price, as in Bertrand-Edgeworth Nash equilibria analyzed in previous studies. 3) Mixtures with hockey-stick offers, which are a combination of the other two mixtures; offers are slope-constrained for low outputs and some offers are strictly increasing for high outputs.

The type of equilibria that arise in a duopoly depend on whether producers are pivotal or not. A *pivotal producer* is one for which the competitor's capacity is not sufficient to meet maximum demand. Thus the removal of a pivotal producer from the market would create a supply shortage with positive probability. We show that symmetric duopoly mixtures over strictly increasing offer curves only occur in discriminatory auctions with inelastic demand and non-pivotal producers. In this case there is a continuum of such equilibria.

The slope-constrained mixtures only occur when producers are pivotal, and then the price cap determines a unique equilibrium among the mixtures that we consider. The horizontal mixture exists when the price cap is sufficiently high relative to the curvature of the cost curve. When the price cap becomes sufficiently low, this equilibrium is continuously transformed into a hockey-stick mixture, so that the top of the mixture has horizontal offers and the bottom of the mixture has a hockey-stick shaped mixture.

The slope-constrained equilibria can be intuitively explained as follows. Ex-post, after the demand shock has been realized, it is always optimal to offer all accepted offers horizontally in a pay-as-bid auction, so that the maximum price is obtained for all the quantity supplied. Hence, unless the demand density is sufficiently decreasing or marginal costs are sufficiently steep relative to mark-ups (in which case pure-strategy Nash equilibria can be found), producers have incentives to offer the very first unit at the same price as some of the units with a higher marginal cost. Hence, the lowest part of the offer curve becomes horizontal and producers have incentives to slightly undercut each other's lowest offers down to the marginal cost, as in a Bertrand game. With constant marginal costs and non-pivotal producers there is a pure-strategy Bertrand-Nash equilibrium (Wang and Zender, 2002; Fabra et al., 2006). But similar to a Bertrand-Edgeworth game there will be profitable deviations from such an outcome if producers are pivotal (Genc, 2009; Holmberg, 2009), and then the equilibrium must be a mixed one. Increasing marginal costs may become steep relative to mark-ups for higher outputs so that the producer will have incentives to increase the offers of more expensive units. In this case the offer gets a hockey-stick shape.

As firms are symmetric in our model, it is always efficient to clear the market such that each firm has the same output, as in symmetric pure-strategy NE of uniform-price auctions (Klemperer and Meyer, 1989). But this rarely happens when firms randomize their supply curves. In our examples where the two symmetric firms have quadratic costs, the mixed-strategies lead to significant production inefficiencies: between 25% and 100% of the optimal production cost. For pivotal firms with fixed capacities, the welfare loss becomes higher for higher price caps. The highest relative welfare loss occurs for horizontal mixtures when firms' production capacity is near the maximum demand, so that one of the firms produces the whole demand for nearly all demand outcomes. This inefficiency is a significant drawback for the discriminatory format.

The setting that we consider is particularly useful when modelling strategic bidding in a wholesale electricity market operating as a discriminatory divisible-good auction. Electricity markets are special in that there are very limited storage possibilities, so supply must equal consumption at every instant. The system operator runs a real-time auction to match demand and supply in every period (normally hour or half-hour). Producer's offer curves to this auction are submitted before the delivery period starts. The demand is uncertain at this point, mainly because of unexpected changes in wind generation[5], air temperature and unexpected transmission-line failures. Production costs in wholesale electricity markets are primarily determined by fuel costs and plants' efficiency, which are well-known and common knowledge. Thus a standard assumption of electricity markets is that supplier costs are common knowledge and that the exogenous non-strategic demand is uncertain (Green and Newbery, 1992).[6] To simplify the equilibrium calculations, the demand shock is normally assumed to be additive (Klemperer and Meyer, 1989). Moreover, production capacity constraints are important in electricity markets. They are less important in treasury auctions, but otherwise corresponding assumptions have been used to analyze strategic bidding in treasury auctions. Wang and Zender (2002) and Rostek et al. (2010) assume that the values of the treasury securities are common knowledge and that there is an uncertain amount of non-competitive bids from many small non-strategic investors, which means that the remaining amount of treasury securities available to the large strategic investors is uncertain.

In a uniform-price auction, pure-strategy Nash equilibria, called Supply Function Equilibria (SFE), can often be found under the assumptions above (Green and Newbery, 1992; Klemperer and Meyer, 1989). They are ex-post optimal, and accordingly independent of the probability distribution of the demand shock. However, pure-strategy equilibria in pay-as-bid auctions are harder to find and they are not ex-post optimal; these equilibria and their existence depend very much on the probability distribution of the demand shock. Holmberg (2009) shows that pure-strategy SFE in pay-as-bid auctions with positive mark-ups do not exist if there is any output level for which both marginal costs are constant and the hazard rate of the demand shock is increasing. In this paper, we generalize this condition and make it more precise: the mark-up times the hazard rate of the demand shock must be non-increasing, otherwise pure-strategy SFE cannot exist in pay-as-bid

[5]Since (random) generation from wind farms is dispatched at zero price, it can be regarded as subtracting from demand.

[6]Note that we do not consider financial contracts here. Otherwise, this could be private information that would influence producer's optimal offers (Hortacsu and Puller, 2008).

auctions when costs are common knowledge. In electricity markets, marginal costs are approximately stepped, i.e. locally constant, and demand shocks are approximately normally distributed, which implies an increasing hazard rate. Analogous to Wang and Zender's (2002) model of treasury auctions, Bertrand NE with zero mark-ups can occur in pay-as-bid markets without capacity constraints, including the case when hazard rate is increasing. But when the largest firm in the market is pivotal, which is often the case in electricity markets, then there is always a profitable deviation from this potential equilibrium (Genc, 2009; Holmberg, 2009). We can summarise this discussion by saying that the existence of pure-strategy SFE in discriminatory electricity auctions with costs that are common knowledge is very much in doubt.

One empirical observation is that offers in wholesale electricity markets with uniform-pricing often have a hockey-stick shape; offers are competitive for low outputs but mark-ups can be huge near the capacity constraint (Hurlbut, 2004). We show that hockey-stick offers are likely to occur in pay-as-bid markets as well.

In the next section we lay out the mathematical model. Section 3 derives optimality conditions for agents offering in a general discriminatory divisible-good setting. In Section 4, we give a new set of necessary conditions for the existence of pure-strategy supply function equilibrium in pay-as-bid auctions. Section 5 studies mixed-strategy equilibria over strictly increasing supply functions and in Section 6 we analyze slope-constrained mixed-strategy equilibria. Section 7 concludes and all proofs are given in the Appendix.

2 Model

In our model each producer has an increasing and convex cost function that is differentiable up to some maximum capacity. There can be several firms in the market. We consider one of these firms. Its output is q, its cost $C(q)$ and its capacity q_m. When calculating Nash equilibria, we will assume that firms have identical costs. The firm in question offers the good with an offer curve $p(q)$, which indicates the amount that the producer is prepared to supply at any given price. As in electricity auctions and similar divisible-good procurement auctions, the offer must be non-decreasing. We assume that $p(q)$ is piecewise smooth. But observe that $p(q)$ may not be continuous and it may not be strictly monotonic (so that there can be intervals on which it is constant). We use the convention that it is left-continuous, so $p(q-) = p(q)$, where $p(q-) := \lim_{\delta \searrow 0} p(q - \delta)$. We will also make use of the supply function $q(p)$, which can be thought of as the inverse of the offer curve when this exists. It is formally defined by

$$q(p) = \sup\{t : p(t) \leq p\}.$$

Thus a supply function is monotonic and right-continuous with respect to price, and may have discontinuities and intervals on which it is constant. Throughout this paper we will use the terminology *supply function* to refer to $q(p)$ and *offer curve* to refer to $p(q)$. Any offer curve may be derived from its supply function $q(p)$ using $p(q) = \inf\{s : q(s) \geq q\}$.

In the model we discuss, demand can be elastic, being represented by a differentiable demand curve $D(p,\varepsilon)$, where the demand shock ε has the probability distribution F. It has a well-defined density function f with support $[\underline{\varepsilon},\overline{\varepsilon}]$, such that $f(\varepsilon)>0\ \forall\varepsilon\in(\underline{\varepsilon},\overline{\varepsilon})$. We let $H(\varepsilon)=f(\varepsilon)/(1-F(\varepsilon))$ be the hazard rate of the demand shock. As in most of the industrial organization literature we neglect consumers' income effect (Tirole, 1988). Hence, as in Federico and Rahman (2003), demand depends on the highest accepted offer in the market.[7] The corresponding assumption in Bertrand models is called parallel rationing (Maskin, 1986).[8] As in other models of electricity markets and Bertrand models we assume that the consumers are non-strategic and that their bid curves are simply determined by their marginal benefit of the commodity. This will be the case when consumers have quasi-linear preferences and pay the average of the producers' accepted offers.[9]

The auctioneer clears the discriminatory auction such that demand can be met at the lowest procurement cost. Thus the auctioneer accepts the cheapest offers first and keeps accepting offers until the accepted quantity equals the demand at the highest accepted offer price. We refer to the latter as the clearing price; this corresponds to the stop-out price in treasury auctions. We assume the clearing price is well-defined (i.e. that the combined capacity for the suppliers is greater than $D(p,\overline{\varepsilon})$ for p chosen large enough). All accepted offers are paid as bid. In particular if a producer with cost function $C(q)$ offers quantity t at price $p(t)$, and the market clears the agent at quantity q, then the agent is paid $\int_0^q p(t)dt$, and achieves a profit of

$$\int_0^q p(t)dt - C(q).$$

We assume that producers are risk-neutral. Thus each producer chooses its offer curve in order to maximize its expected profit given properties of its residual demand curve. Generally, the residual demand curve is uncertain, because of uncertainties in competitors' offer curves and/or uncertainties in the demand curve. As in Anderson and Philpott (2002) we use a market distribution function $\psi(q,p)$ to characterize this uncertainty for the firm under study. We define it to be the probability that the producer's supply offer of quantity q at price p is not accepted in the auction. Thus the market distribution function equals one minus Wilson's (1979) probability distribution of the market price, which was later extended to pay-as-bid auctions by Hortacsu (2002). The expected payoff of a supplier offering

[7] Consumers' total expenditure also depends on prices of inframarginal offers in a pay-as-bid mechanism. But this does not influence demand if it does not have an income effect.

[8] The other common assumption in Bertrand models is called proportional rationing (Maskin, 1986; Edgeworth, 1925). The two consumer rationing assumptions are identical for perfectly inelastic demand.

[9] This assumption implies that consumers' total expenditure equal producers' total revenue, which would not be the case if consumers paid according to their bids and producers were paid according to their offers. More details can be found in Anderson et al. (2009).

a curve $p(t)$ into a pay-as-bid market can now be written as

$$\Pi = \int_0^{q_m} \left(\int_0^q p(t)dt - C(q) \right) d\psi(q, p(q)) \tag{1}$$
$$+ (1 - \psi(q_m, p(q_m))) \left(\int_0^{q_m} p(t)dt - C(q_m) \right).$$

The integral with respect to ψ could be interpreted in the Lebesgue-Stieltjes sense since this formulation would apply even if ψ was not continuous. However, when deriving optimality conditions we will assume that ψ is well-behaved and in fact differentiable at every point where $\psi(q, p) \in (0, 1)$: we only allow the market distribution function to be non-smooth at the ends of this interval.

3 Optimality conditions for pay-as-bid auctions

The optimality conditions we derive here are valid for situations in which a supplier is facing an uncertain residual demand curve and is offering a divisible homogeneous good with a discriminatory price-schedule. We assume that the level curves of the residual demand distribution (the market distribution function) are smooth, but otherwise we do not impose any restrictive assumptions on the uncertainty of the residual demand curve: it can be caused by demand uncertainty and uncertainty in competitors' offer curves (e.g. when competitor costs are unknown or when they randomize their offer curves). The producer may be a monopolist in the market or in the most general case, face competition from other producers offering differentiated goods (as long as product differentiation does not introduce any non-smoothness in the residual demand distribution). Our analysis confirms the Euler condition in Hortacsu (2002); our contribution is to derive second-order conditions for optimal offers and optimality conditions for cases where a monotonicity constraint in the price-schedule binds. This allows us to deal with cases where the discriminatory price schedule includes horizontal segments, and we give a first-order condition that is valid for any Bertrand game where the level curves of the residual demand distribution are smooth.[10]

Having emphasized the generality of the conditions we now turn our focus back to the setting where a producer chooses its offer curve in order to maximize its expected profit given uncertainties in its residual demand that are characterized

[10] The optimality conditions are also applicable to non-linear pricing (Anderson et al., 2009).

by $\psi(q,p)$. Using integration by parts we can rewrite (1) as follows:

$$\begin{aligned}
\Pi &= \left[\left(\int_0^q (p(t)-C'(t))dt\right)\psi(q,p(q))\right]_0^{q_m} - \int_0^{q_m}(p(q)-C'(q))\psi(q,p(q))dq \\
&\quad + (1-\psi(q_m,p(q_m)))\left(\int_0^{q_m}(p(t)-C'(t))dt\right) \\
&= \psi(q_m,p(q_m))\int_0^{q_m}(p(t)-C'(t))dt - \int_0^{q_m}(p(q)-C'(q))\psi(q,p(q))dq \\
&\quad + (1-\psi(q_m,p(q_m)))\left(\int_0^{q_m}(p(t)-C'(t))dt\right),
\end{aligned}$$

whence

$$\Pi = \int_0^{q_m}(p(q)-C'(q))(1-\psi(q,p(q)))dq. \tag{2}$$

This formula has another interpretation. We may consider each increment of capacity dq offered to the market to earn a marginal profit of $(p(q)-C'(q))dq$. The probability of this increment being dispatched is $(1-\psi(q,p(q)))$, and so (2) represents the expected profit.

Now consider the problem of choosing a curve $p(q)$, $q\in[0,q_m]$ to maximize Π for given strategies (mixed or pure) of competitors. In some cases this will not have an optimal solution, but where there is a price cap in operation we can demonstrate the existence of a solution, at least in the space of bounded measurable functions. We let Ω be the set of monotonic increasing functions on $[0,q_m]$ taking values in $[0,P]$. Since monotonic functions are Lebesgue measurable, Ω is a bounded subset of $L^\infty[0,q_m]$, the space of bounded Lebesgue measurable functions on $[0,q_m]$ endowed with the essential supremum norm. By Alaoglu's Theorem (see e.g. Conway (1994) p. 130) Ω is weak* compact in this metric.

Proposition 1 *If there is a price cap P and both $\psi(q,p)$ and $C'(q)$ are continuous for $q\in[0,q_m]$ and $p\in[0,P]$, then there exists an optimal solution for the problem of maximizing profit Π over offer curves in Ω.*

In the absence of any constraints an optimal $p(q)$ must satisfy the Euler equation

$$\frac{\partial}{\partial p}(p(q)-C'(q))(1-\psi(q,p(q)))=0.$$

Thus a producer maximizes its total expected profit by independently maximizing the expected profit from each (infinitesimally small) production unit. This first-order condition may be rewritten

$$1-\psi\left(q,p(q)\right)-\psi_p(q,p(q))(p(q)-C'(q))=0. \tag{3}$$

This condition can be interpreted as follows. Assume that we increase the offer price of a production unit q, then there are two counteracting effects on its expected pay-off. The revenue increases for outcomes when the offer is accepted. This is the price effect and the rate of increase of revenue is equal to the probability that the offer is accepted, $1-\psi$. On the other hand a higher price means that there is a higher risk that offer is rejected. This is the quantity effect. The marginal loss in profit is given by the increased rejection probability ψ_p times the lost mark-up. For an optimal monotonic offer the two counteracting effects are equal. In the analysis we will find it useful to introduce $Z(q,p)$, the difference between the price and quantity effects.

$$\begin{aligned} Z(q,p) &= \frac{\partial}{\partial p}(p(q)-C'(q))(1-\psi(q,p(q))) \\ &= 1-\psi(q,p)-\psi_p(q,p)(p-C'(q)). \end{aligned} \tag{4}$$

Below we verify that (3) must be satisfied for each segment of $p(q)$ that is strictly increasing.

Proposition 2 *On any segment of an optimal curve with $0 < p'(q) < \infty$ we have*

$$Z(q,p(q)) = 1-\psi(q,p(q))-\psi_p(q,p(q))(p(q)-C'(q)) = 0 \tag{5}$$

and

$$\left.\frac{\partial Z(q,p)}{\partial q}\right|_{q=q(p)} \geq 0. \tag{6}$$

The local second-order condition of an optimal offer curve is that the price effect (weakly) dominates the quantity effect just below the offer curve $p(q)$ and the other way around just above the curve. Otherwise the offer curve would locally minimize profits. For a strictly increasing offer curve we can equivalently say that the quantity effect must dominate to the left of the offer curve and that the price effect dominates to the right of the curve, and this is the essence of (6).

Most procurement auctions require offer curves to be non-decreasing, and we consider this constraint in our model. Thus if the Euler curve, i.e. the solution to (3), decreases at some point then it is not a valid supply curve. In this case part of $p(q)$ will be locally slope-constrained, and (3) is not necessarily satisfied at these points. The following result characterizes this situation.

Proposition 3 *Suppose that $p(q)$ is an optimal curve. Then (a) If $p(\cdot)$ is strictly increasing at $q_1 < q_2$ then*

$$\int_{q_1}^{q_2}(1-\psi(q,p(q))-\psi_p(q,p(q))(p(q)-C'(q)))dq \geq 0.$$

(b) If $p(\cdot)$ is strictly increasing at $q_3 > q_2$ then

$$\int_{q_2}^{q_3}(1-\psi(q,p(q))-\psi_p(q,p(q))(p(q)-C'(q)))dq \leq 0.$$

(c) If $p(\cdot)$ is horizontal between q_1 and q_3 and these quantities are the end points of the horizontal segment then

$$\int_{q_1}^{q_3}(1-\psi(q,p(q))-\psi_p(q,p(q))(p(q)-C'(q)))dq = 0.$$

The monotonicity constraint reduces the set of allowed perturbations of a horizontal segment. Condition (c) says that the price and quantity effects are equal on average if we marginally increase the offer price of the whole horizontal segment. The local second-order conditions are (a) that the price effect must (weakly) dominate along the first part of the segment, i.e. $\int_{q_1}^{q_2} Z(q,p)dq \geq 0$, and (b) that the quantity effect must dominate in the last part of the segment, i.e. $\int_{q_2}^{q_3} Z(q,p)dq \leq 0$. Otherwise it would be profitable to decrease the offer price of the first part and to increase the offer price of the last part.

It is possible to use the same proof idea to show that condition (c) in Proposition 3 applies when the auction design requires offers to be stepped with a maximum number of steps. In particular, it is for example straightforward to show that a general first-order condition for Bertrand games, where suppliers are restricted to offer their capacity at one price, is given by:

$$\int_0^{q_m} (1 - \psi(q,p) - \psi_p(q,p)(p - C'(q)))dq = 0.$$

If the Euler curve bends back on itself, so that its slope is first infinite and then negative, then $p(q)$ will have a vertical segment. There is a corresponding condition to condition (c) in Proposition 3 that will apply in the case of a vertical segment (see Anderson et al., 2009). A similar first-order condition has been derived by Kastl (2008).

4 Pure-strategy supply function equilibria

In Sections 4-6 we will use the optimality conditions to calculate equilibrium offers in pay-as-bid auctions. In these derivations we assume

Assumption 1 Costs are common knowledge and the demand shock is additive, so that $D(p,\varepsilon) = D(p) + \varepsilon$.

This is a standard assumptions in the literature on supply function equilibria and electricity auctions (Green and Newbery, 1992; Klemperer and Meyer, 1989). In this section we use the optimality conditions to derive necessary conditions for pure-strategy supply function equilibria in pay-as-bid auctions. Holmberg (2009) has shown that symmetric pure-strategy supply function equilibria can be ruled out in such auctions if producers are pivotal and the shock distribution has a locally increasing hazard rate when marginal costs are locally flat. We re-examine this by means of our general optimality conditions.

Consider a producer i, who submits a supply function $q(p)$. Suppose that its competitors' total quantity offered at price p is given by $q_j(p)$. It is easy to see that

$$\psi(q,p) = F(q + q_j(p) - D(p)).$$

Thus, when q_j is differentiable, we can rewrite (4) as follows:

$$Z(q,p) = 1 - F(q + q_j(p) - D(p)) - (p - C'(q))f(q + q_j(p) - D(p))(q_j'(p) - D'(p)).$$

Now the first- and second-order conditions in Proposition 2 imply that an optimal strictly increasing offer curve $q_i(p)$ must satisfy

$$(1 - F(q + q_j(p) - D(p))) -$$
$$(p - C'(q)) f(q + q_j(p) - D(p))(q_j'(p) - D'(p)) \begin{cases} \leq 0, & q < q_i(p) \\ = 0, & q = q_i(p) \\ \geq 0, & q > q_i(p). \end{cases} \quad (7)$$

Recall that the hazard rate of the demand shock is defined by $H(x) = f(x)/(1 - F(x))$. Thus we can state the results above as follows:

Proposition 4 *Consider a market where Assumption 1 is satisfied. If competitors' total supply is given by* $q_j(p)$*, then any optimal segment with* $0 < q'(p) < \infty$ *must satisfy the first-order condition*

$$1 - (p - C'(q)) H(q + q_j(p) - D(p))(q_j'(p) - D'(p)) = 0$$

and the second-order condition

$$\frac{\partial}{\partial q} \left[(p - C'(q)) H(q + q_j(p) - D(p)) \right] \Big|_{q=q(p)} \leq 0. \quad (8)$$

Note that both the first and second-order condition need to be satisfied at each level of output. Marginal costs in electricity markets are approximately stepped, i.e. locally constant, so that the second-order condition (8) is close to a requirement that $H'(x) \leq 0$ when mark-ups are positive. This is a very strong restriction on the form of F. It means that the density of the demand shock must decrease faster than e^{-x} throughout its range. Thus, requiring a decreasing hazard rate rules out most demand shocks that one would encounter in practice including the normal distribution.[11]

A special case of supply function equilibrium that is not covered by Proposition 4 (because $q'(p) = \infty$) is the pure strategy Bertrand NE with zero markups that occurs for symmetric non-pivotal producers with $C'(q) = c$. Observe that this will not be a Nash equilibrium when players are pivotal.

We can gain some intuition for the problem of finding a pure-strategy SFE by the following argument. Consider the effect of one producer raising the price for the initial quantity δ that it offers, so that the offer becomes perfectly elastic over a range $(0, \delta)$. If this player ends up supplying an amount more than δ then it improves its profit due to the higher price received for this first part of its output. The only loss occurs when demand is very low and the player ends up supplying less than δ. For a supply function that is not horizontal to occur, these considerations must balance. This can only happen when there is approximately equal probability of supplying an amount less than or greater than δ. This demonstrates that we will need a demand function weighting low demand values very highly, and hence the very steeply decreasing density functions implied by a decreasing hazard rate.

[11] Garcia and Kirschen (2006) find that power system imbalances in Britain are approximately normally distributed. Given the central limit theorem, this is not surprising as the system imbalance results from a large number of actions by different market participants acting independently.

5 Mixtures with non-binding slope constraints

In the previous section we concluded that pure-strategy SFE can be ruled out for many pay-as-bid markets that we encounter in practice. Now we begin the main task of this paper which is an analysis of mixed-strategy equilibria for pay-as-bid markets. In this section we analyze cases where duopoly producers mix over a range of offer curves each of which satisfies the Euler condition $Z(q,p)=0$. Each curve corresponds to a supply function with *non-binding slope constraints*. We will show that equilibria of this form may occur, but the conditions for such an equilibrium are very restrictive; they normally only exist when demand is inelastic and producers are non-pivotal (so that demand can still be met even if the largest firm exits the market). A second class of equilibrium mixture with slope-constrained offer curves is analyzed in Section 6.

We consider an equilibrium in which there is mixing over a whole range of solutions each of which falls into a region Γ for which

$$Z(q,p)\equiv 0, \quad (q,p)\in\Gamma. \tag{9}$$

We suppose that the interior of Γ is a non-empty simply connected set.

Substituting for Z in (9) using (4) implies that the function $(p-C'(q))(1-\psi(q,p))$ is independent of p (i.e. a function of q only) in the region Γ. Hence

$$(p-C'(q))(1-\psi(q,p))=\theta(q) \tag{10}$$

for some function θ. Thus the expected profit from a production unit q is the same for any offer curve in the mixture. Hence, we realize that mark-ups must be strictly positive for the lowest (most competitive) offer curve in the mixture. The market distribution function that a mixing producer is facing can now be written

$$\psi(q,p)=1-\frac{\theta(q)}{p-C'(q)}=\frac{p-k(q)}{p-C'(q)}, \tag{11}$$

where $k(q)$ is some function, such that $C'(q)\leq k(q)\leq p$.

In a mixed strategy equilibrium, each producer's strategy can be expressed by means of its offer distribution function $G(q,p)$, which is defined as the probability that the producer offers strictly more than q units at a price p or less. In general there will be many different ways to produce the same offer distribution function by mixing over complex sets of offer curves that typically cross each other. The simplest way for a producer to generate a mixture of offer curves corresponding to G is to first draw γ from a uniform distribution on $(0,1)$ and then to offer the supply curve corresponding to the contour $G(q,p)=\gamma$. In fact this is the only way to obtain G if the mixture is taken over offer curves that do not cross. To see why this is true suppose that the offer curves do not cross and take a point (q,p) on one of these curves. Then the probability that the producer offers more than q units at a price p or less is the probability that the producer offers one of the curves that pass under and to the right of this point. The non-crossing condition ensures that this is the same set of curves for any point (q,p) chosen on the offer curve, and hence G is constant along the offer curve.

Such crossings cannot be ruled out for the unconstrained mixtures analyzed in this section. But note that, if there are multiple mixed-strategy NE with identical offer distribution functions, then they are all payoff equivalent, because the offer distribution functions of the players determine the probability distributions of players' payoffs. For the slope-constrained mixtures analyzed in Section 6, the monotonicity constraint will prevent randomization over crossing supply functions.

Supply functions are monotonic by assumption. Hence, the probability that an offer $p(q)$ in the mixture is below and to the right of the point (p, q) cannot decrease if either p increases or q decreases. Thus we have that $\frac{\partial}{\partial p} G \geq 0$ and $\frac{\partial}{\partial q} G \leq 0$, where these derivatives exist. We will assume that all quantities supplied are non-negative. We write $q^L(p)$ for the infimum of quantities offered by any supply function at price p for producer i. If the mixture has a finite mass on the lowest supply function then G will be discontinuous at this point. But in the case where G is continuous (in its first argument) then from the definition of G we will have $G\left(q^L(p), p\right) = 1$.

Without loss of generality we can assume that the lower boundary of Γ is included in Γ and corresponds to the supply function with the highest output. We write $q^U(p)$ for this highest supply function in the mixture. Then by definition $G\left(q^U(p), p\right) = 0$ even if G has a discontinuity there.

5.1 Mixed supply function equilibria

For simplicity we limit our analysis to a symmetric equilibrium in a symmetric duopoly market, where costs are common knowledge and the demand shock is additive. We consider the offer of producer i and we denote the offer distribution function of its competitor by $G_j(q, p)$. The accepted output of producer i at price p is given by the difference between two independent random variables: the shock outcome ε and $q_j(p)$, the supply of the competitor at the price p. Hence, the probability that an offer of q by producer i is not fully dispatched if offered at the price p is

$$\psi(q, p) = \int_{-\infty}^{\infty} f(\varepsilon)\, G_j(\varepsilon + D(p) - q, p)\, d\varepsilon. \tag{12}$$

Making the substitution $t = \varepsilon + D(p) - q$ yields

$$\psi(q, p) = F(q - D(p)) + \int_{0}^{\infty} f(t + q - D(p))\, G_j(t, p)\, dt, \tag{13}$$

since $G_j(t, p) = 1$ for $t < 0$. Here we have assumed that F is continuous and f is well-defined.

We let $\underline{p}$ be the infimum of clearing prices with a positive output.[12] For a given demand, the lowest clearing price will occur when the producers offer the largest

[12] Note that offer curves may start with a vertical segment up to $\underline{p}$ at zero output. Thus the market may clear at a price below $\underline{p}$ if the cleared total output is zero. However, payoffs are zero for these outcomes and such a vertical segment would not be part of the mixing region, so we are only interested in characterizing outcomes with a positive output.

amount i.e. when the offers are $q^U(p)$. Thus $\underline{p}$ can be defined explicitly as

$$\underline{p} = \inf\{p : q_1^U(p) + q_2^U(p) \geq D(p) + \varepsilon > 0 \text{ for some } \varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}]\}. \tag{14}$$

We are mainly interested in analyzing mixed-strategy NE when pure-strategy SFE do not exist. We consider a distribution of demand shocks satisfying

$$\left.\frac{\partial}{\partial q} \frac{\left(\underline{p} - C'(q)\right) f(q+u)}{1 - F(q - D(\underline{p}))}\right|_{q=0+} > 0, \quad \text{for each } u \in \left[-D(\underline{p}), \overline{\varepsilon}\right], \tag{15}$$

where the case $u = -D(\underline{p})$ ensures that the second-order condition for pure-strategy SFE with positive mark-ups in (8) is not satisfied at zero output. Under the inequality (15) and Assumption 1, it is possible to establish the following properties of mixed-strategy equilibria with non-binding slope constraints. The details can be found in Anderson et al. (2009).

- The most competitive supply function $q^U(p)$ in the mixture offers at least the maximum possible demand at $\underline{p}$, i.e. producers must be non-pivotal. The reason is that the competitor's output needs to be sufficiently uncertain already at the lowest price $\underline{p}$, otherwise (15) enables a profitable deviation at zero output (as in the case of pure-strategy NE).
- Nonbinding slope constraints can only occur in equilibrium if demand is inelastic. It follows from (10) that the expected profit from a unit q is the same for all offer curves in the mixture. Thus it can never be profitable to let an offer curve cross the maximum residual demand curve if this curve is elastic, because the part of the offer curve that is to the right of the maximum residual demand curve is never accepted and does not contribute to the expected profit. This would imply that the total profit from this offer curve is less than that from the offer curve offering all supply at $\underline{p}$. Since all offer curves in the mixture must have the same profit we have a contradiction. With non-pivotal producers there is always an intersection with the maximum residual demand curve at $q < q_m$.
- $G_j\left(q, \underline{p}\right)$ is zero in a symmetric equilibrium, with $\underline{p} > C'(0)$. If there were an accumulation of horizontal offers at this lowest mixing price $\underline{p}$, then producers would have incentives to undercut each other.

Now we calculate an unconstrained mixed-strategy equilibrium for non-pivotal producers and inelastic demand, $D(p) = 0$ and $\underline{\varepsilon} = 0$. From the results above, we know that the equilibrium needs to satisfy $q^U\left(\underline{p}\right) \geq \overline{\varepsilon}$, and the entire horizontal line $(0, \underline{p})$ to $(\overline{\varepsilon}, \underline{p})$ is the lower boundary of the region Γ where mixing takes place. Thus using the equations (11) and (13) as well as $G\left(q, \underline{p}\right) = 0$ we get

$$F(q) = \psi\left(q, \underline{p}\right) = \frac{\underline{p} - k(q)}{\underline{p} - C'(q)},$$

provided $\underline{p} > C'(\bar{\varepsilon})$. This determines the function

$$k(q) = \underline{p} - \left(\underline{p} - C'(q)\right) F(q) \tag{16}$$

and substitution back into (11) shows that

$$\psi(q,p) = \frac{p - \underline{p} + \left(\underline{p} - C'(q)\right) F(q)}{p - C'(q)} \text{ for } q \in (0, \bar{\varepsilon}) \text{ and } p > \underline{p}. \tag{17}$$

If $p \leq \underline{p}$, then any offer of (q, p) by producer i will be fully dispatched with probability $1 - F(q)$, so

$$\psi(q,p) = F(q) \text{ for } q \in (0, \bar{\varepsilon}) \text{ and } p \leq \underline{p}.$$

By substituting (17) into (13) it is easy to verify that

$$\int_0^{\bar{\varepsilon}-q} f(t+q)\, G_j(t,p)\, dt = \frac{p - \underline{p} + \left(\underline{p} - C'(q)\right) F(q)}{p - C'(q)} - F(q)$$

so

$$\int_0^{\bar{\varepsilon}-q} f(t+q)\, G_j(t,p)\, dt = \frac{(p-\underline{p})(1-F(q))}{p - C'(q)}. \tag{18}$$

The integral equation in (18) can be solved for G using Laplace transforms or Fourier transforms (Abdul, 1999). For multiple firms, $G_j(t,p)$ above can be interpreted as being the total offer distribution function of firm i's competitors. Thus many of the results relating to non-binding slope constraints given by Anderson et al. (2009) will hold for multiple firms as well. Symmetric mixtures with non-binding slope constraints require firms to be non-pivotal and demand to be inelastic. Moreover, $G_j\left(q, \underline{p}\right)$ is zero in a symmetric equilibrium, with $\underline{p} > C'(0)$. For multiple firms, $G_j(t,p)$ is a convolution of competitors' individual offer distribution functions similar to the convolution in (18). Thus once $G_j(t,p)$ has been derived, firms' individual offer distribution functions can in their turn be found from $G_j(t,p)$ using Laplace or Fourier transforms.

In the duopoly, we can characterize the form of the mixture analytically when F has a uniform distribution.

Proposition 5 *Consider a symmetric duopoly market. Suppose that demand is inelastic and has an additive uniformly distributed demand shock with support* $[0, \bar{\varepsilon}]$. *If costs are common knowledge, each producer has capacity* $q_m > \bar{\varepsilon}$ *and* $C'''(q) \leq 0$ *for every* $q \in (0, \bar{\varepsilon})$ *then there exists a symmetric mixed-strategy equilibrium with non-binding slope constraints provided the lowest clearing price* $\underline{p}$ *is chosen so that*

$$\underline{p} \geq C'(q) + C''(q)(\bar{\varepsilon} - q) \tag{19}$$

for every $q \in (0, \bar{\varepsilon})$. *All such equilibria are defined by the offer distribution function*

$$G_j(q,p) = \begin{cases} 0, & p < \underline{p},\ q \in (0, \bar{\varepsilon}), \\ (p - \underline{p}) \frac{(p - C'(\bar{\varepsilon}-q)) - qC''(\bar{\varepsilon}-q)}{(p - C'(\bar{\varepsilon}-q))^2}, & p \geq \underline{p},\ q \in (0, \bar{\varepsilon}). \end{cases}$$

Note that Proposition 5 has the implication that whenever $q_m > \overline{\varepsilon}$ and $C'''(q) \leq 0$ then there will be a continuum of unconstrained mixed-strategy NE with different lowest clearing prices that are sufficiently high. In the special case when the marginal cost is constant, our model generates a continuum of mixed-strategy NE over (unconstrained) horizontal offers given by the offer distribution function $G_j(q,p) = \frac{p-\underline{p}}{p-c}$. Thus our result corresponds to the continuum of mixed-strategy NE that have been found in Bertrand games with non-pivotal producers and unbounded prices (Bayea and Morgan, 1999; Hoernig, 2002; Kaplan and Wettstein, 2000).

Example 6 ***Symmetric duopoly with uniform demand.*** *Consider a symmetric duopoly market with $C(q) = q^2/2$, and ε uniformly distributed on $[0,1]$. Proposition 5 gives an equilibrium mixture defined by*

$$G(q,p) = \frac{(p-1)(p-\underline{p})}{(p+q-1)^2}. \tag{20}$$

Thus:

$$q(p,G) = \sqrt{\frac{(p-1)(p-\underline{p})}{G}} + 1 - p.$$

Here the scalar $G \in [0,1]$ parameterizes the supply functions over which mixing takes place. The corresponding offer curves are shown in Figure 1 for $\underline{p} = 2$. By means of the expressions above and the clearing condition that total output should equal the demand shock, it is possible to calculate the clearing (stop-out) price $p(\varepsilon, G_1, G_2)$ and each firm's output for each outcome.[13] *The firms have the same convex costs, so firms should have the same output when production is efficient. But this rarely happens for randomized offers. The expected welfare loss (production inefficiency) is*

$$\begin{aligned} W &= \overbrace{\int_0^1\int_0^1\int_0^1 \frac{q_1^2\left(p\left(\varepsilon,G_1,G_2\right),G_1\right)}{2} + \frac{q_2^2\left(p\left(\varepsilon,G_1,G_2\right),G_2\right)}{2} d\varepsilon dG_1 dG_2}^{\text{Expected production cost}} \\ &- \quad 2\overbrace{\int_0^1 \frac{(\varepsilon/2)^2}{2} d\varepsilon}^{\text{Efficient expected production cost}}, \end{aligned}$$

which can be estimated by numerical integration. Detailed calculations give an expected welfare loss of 79% of the efficient expected production cost when $\underline{p} = 2$.

[13] We assume that each firm's output is non-negative, so when one firm has a low G and the other a high G, it might be the case that the firm with the lowest G meets the whole demand.

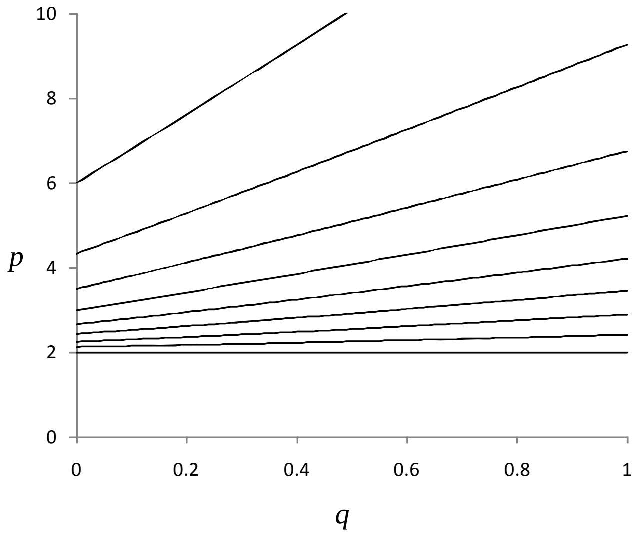

Figure 1: Contours of $G(q, p)$ at levels $0, 0.1, \ldots, 0.8$ for mixed strategy equilibrium with $\underline{p} = 2$.

6 Mixtures over slope-constrained offer curves

We saw in Section 4 that pure-strategy SFE can be ruled out in markets where $(p - C'(q))\, H(q + S_j(p) - D(p))$ is locally increasing. In Section 5, we were able to find equilibria with mixtures over supply functions with non-binding slope constraints under such circumstances when demand is inelastic and firms have sufficiently large capacities. In markets with pivotal producers or elastic demand, we know that equilibria with mixtures over supply functions with non-binding slope constraints will in general not exist.

In this section we will show that mixtures over slope-constrained offer curves can exist under such circumstances. This takes us back to the discussion in Section 3 in which we provided optimality conditions for solutions having this type of structure. For horizontal segments of an offer curve it is now enough that the price and quantity effects are equal on average along the segment. We analyze two cases. We start with mixtures over horizontal offers, i.e. offer curves are slope-constrained along the whole output. We show that there exist equilibria where all offer curves in the mixture are of this type if the price cap is sufficiently high. For lower price caps, there is another mixed-strategy equilibrium, which we call a hockey-stick mixture. Offers in this mixture also start with horizontal segments at the top. But in this case, the supply slopes upwards at high outputs for the lowest offer curves in the mixture.

6.1 Mixtures over horizontal offers

In this subsection we will consider mixtures over horizontal offer curves, i.e. the curves are slope-constrained for the whole output. We use the term *slope-*

constrained horizontal offer curves to distinguish from mixtures over horizontal offers with non-binding slope constraints that we mentioned in the previous section. We consider the case with two players, with capacities q_i^m, $i = 1, 2$, where demand may exceed $\max(q_1^m, q_2^m)$ but not $q_1^m + q_2^m$. As before we let $\underline{p}$ be the lowest clearing price and let $\overline{p}$ be the highest clearing price, where this exists. We consider a situation in which producer $j \neq i$ offers its capacity at a price p or below with probability $G_j(p)$. From (13) we have the resulting market distribution function for firm i :

$$\begin{aligned}\psi_i(q_i, p) &= F(q_i - D(p)) + \int_0^{q_j^m} f(t + q_i - D(p))\, G_j(p)\, dt \\ &= (1 - G_j(p))F(q_i - D(p)) + G_j(p)\, F(q_i + q_j^m - D(p)).\end{aligned} \tag{21}$$

Note that $F(\varepsilon) = 1$ for $\varepsilon > \overline{\varepsilon}$. The payoff of a horizontal offer at price p is given by (2)

$$\Pi_i(p) = \int_0^{q_i^m} (p - C_i'(q))(1 - \psi_i(q, p))dq.$$

In equilibrium we require the offer of q_i^m at any price p in the support of G_j to yield the same expected profit, and we let K_i be the value of $\Pi_i(p)$ in this region. After substituting for ψ_i this gives

$$K_i = \int_0^{q_i^m} (p - C_i'(q))(1 - (G_j(p)F(q + q_j^m - D(p)) + (1 - G_j(p))F(q - D(p)))dq.$$

After rearranging we get

$$G_j(p) = \frac{\int_0^{q_i^m} (p - C_i'(q))(1 - F(q - D(p)))dq - K_i}{\int_0^{q_i^m} (p - C_i'(q))(F(q + q_j^m - D(p)) - F(q - D(p)))dq}. \tag{22}$$

This generalizes the necessary first-order condition for mixed-strategy Nash equilibria in discriminatory auctions given by Anwar (2006), Fabra et al. (2006) and Son et al. (2004), who consider cases with constant marginal costs and vertical demand. Moreover, the condition generalizes the previous conditions that have been used to calculate mixed-strategy Nash equilibria in Bertrand-Edgeworth games, which assume zero marginal costs and certain linear demand as in Beckmann (1967) and Levitan and Shubik (1972).

In the remainder of this section we will focus on discriminatory divisible-good auctions where demand is inelastic up to a reservation price/price cap. Without loss of generality we assume that $D(p) = 0$. First we show that one can eliminate the possibility of a slope-constrained horizontal mixture when the smallest producer is non-pivotal, i.e. when $\max(q_1^m, q_2^m) > \overline{\varepsilon}$.

Proposition 7 *Consider a symmetric duopoly market where Assumption 1 is satisfied. In an equilibrium where a player i mixes over slope-constrained horizontal offer curves in some price interval (p_1, p_2) and $D(p) = 0$, it must be the case that $\overline{\varepsilon} > q_j^m$.*

Although first-order conditions for mixtures over horizontal offers are similar in our framework and in the game analyzed by Fabra et al. and Bertrand-Edgeworth games, our strategy space is less constrained as it allows for strictly increasing supply functions. Thus one would expect sufficiency conditions to be different in our framework. For example, there would be profitable deviations from any potential equilibrium with horizontal mixtures where $\underline{p} < C'(q_m)$.

Our analysis is in a more general setting than Anwar (2006) and Genc (2009), who consider constant marginal costs, but we will restrict attention to the case where firms are symmetric, so that $C_1'(q) = C_2'(q) = C'(q)$ and $q_1^m = q_2^m = q_m$, and we look for an equilibrium in which both firms offer the same mixture of offers, $G_1(p) = G_2(p) = G(p)$. As a consequence of Proposition 7 we only consider cases where $q_m < \overline{\varepsilon} < 2q_m$.

It will be convenient to introduce some notation to shorten the expressions we deal with. We let

$$A = \int_0^{q_m} (1 - F(q))dq, \qquad B = \int_0^{q_m} (1 - F(q + q_m))dq, \tag{23}$$

and

$$L(p) = \int_0^{q_m} (p - C'(q))(F(q + q_m) - F(q))dq. \tag{24}$$

It is also convenient to define

$$J = -L(0) = \int_0^{q_m} C'(q)(F(q + q_m) - F(q))dq. \tag{25}$$

Mixed-strategies where firms only use horizontal offers are referred to as horizontal mixtures. When such mixtures are Nash equilibria we call them horizontal mixed-strategy NE. We can establish the following general result for such equilibria, when there is a price cap, P. As in Genc (2009) and Fabra et al. (2006) the existence of a price cap singles out a unique equilibrium.

Proposition 8 *Consider a symmetric duopoly market where Assumption 1 is satisfied. When $D(p) = 0$ and $\overline{\varepsilon}/2 \leq q_m < \overline{\varepsilon}$ then a horizontal mixed-strategy equilibrium, where producers only mix over horizontal offers, can only exist if the price cap P satisfies*

$$P \geq \frac{1}{B^2}\left(A^2 C'(q_m) - (A + B)J\right) \tag{26}$$

Moreover, the equilibrium is uniquely determined by P. It has a highest price of P, a lowest price of

$$\underline{p} = (PB + J)/A \tag{27}$$

and a distribution of prices given by:

$$G(p) = \frac{(p - \underline{p})A}{L(p)}. \tag{28}$$

The condition

$$\frac{\partial}{\partial q}\left[(\underline{p} - C'(q))\frac{(F(q + q_m) - F(q))}{1 - F(q)}\right] > 0, \quad \text{for } q \in [0, \overline{\varepsilon} - q^m) \tag{29}$$

is sufficient to ensure that this is an equilibrium.

Thus it follows from (27) that horizontal mixed-strategy NE can always be found for sufficiently high price caps if $\frac{\partial}{\partial q}\left[\frac{(F(q+q_m)-F(q))}{1-F(q)}\right] > 0$ for $q \in [0, \overline{\varepsilon} - q^m)$. The sufficiency condition in (29) ensures that the price effect dominates in the first part of each horizontal offer and that the quantity effect dominates in the last part of each horizontal offer, otherwise firms would have incentives to deviate by decreasing offers near zero output and increasing them near the capacity constraint.

Using the definitions (23) and (25), we can write (26) as

$$P \geq C'(q_m) + \frac{(A+B)}{B^2} \int_0^{q_m} (C'(q_m) - C'(q)) (F(q+q_m) - F(q)) dq. \tag{30}$$

For pivotal producers $B > 0$, and this inequality is satisfied if the price cap is high relative to the slope of the marginal cost curve. In particular, for constant marginal costs c and uniformly distributed demand shocks, which satisfies (29), we can deduce that there is always an equilibrium of this form for pivotal producers and any price cap larger than c, which confirms Genc's (2009) result.

Observe that from (28) we can calculate the density $g(p)$ of the mixture distribution: it turns out that

$$g'(p) = \frac{-2L(\underline{p})A(A-B)}{[L(p)]^3} \leq 0$$

if $p \geq \underline{p} \geq C'(q_m)$. Thus the density of the mixture is weighted towards lower prices.

Vives (1986) characterizes Bertrand-Edgeworth NE for multiple symmetric firms. He shows that the mixed-strategy equilibrium occurs in markets where firms are pivotal.[14] The equilibrium becomes more competitive with more firms in the market, but otherwise it has similar properties as for duopoly markets. Thus it is reasonable to conjecture that a similar horizontal mixture will occur in a pay-as-bid market with uncertain demand also for multiple firms.

6.2 Hockey-stick mixture

For increasing marginal costs, (30) shows that the price cap needs to be sufficiently high for horizontal mixtures to exist. The problem for low price caps is that when mark-ups are sufficiently low then the price effect will dominate the quantity effect for high outputs, which would violate the second-order condition in Proposition 3. The proposition below shows that in this case we get another type of equilibrium where the lowest offer curves in the mixture are horizontal and slope-constrained in the quantity interval $[0, q_A(p))$ and then strictly increasing and unconstrained along the curve $q_A(p)$ where $Z(q_A(p), p) = 0$, i.e. the price and quantity effects are equal. We call this a hockey-stick offer. The highest offers in this hockey-stick mixture are still horizontal along the whole output.

[14] In Vives' model with elastic demand it is also possible to get Cournot related pure-strategy NE when firm's capacities are sufficiently small. But they do not occur in our setting with inelastic demand.

Proposition 9 *Consider a symmetric duopoly market where Assumption 1 is satisfied. Assume that $D(p) = 0$, $\overline{\varepsilon}/2 < q_m < \overline{\varepsilon}$ and $C'(q_m) \leq P$. Then a hockey-stick mixture has the following form:*

1. *There is some p_m such that for $p \in [p_m, P]$ producers mix over horizontal offers and the mixture is defined from*

$$G(p) = \frac{Ap - J - BP}{Ap - J - Bp}. \tag{31}$$

2. *There is some $\underline{p}$ such that for $p \in \left[\underline{p}, p_m\right]$ producers mix over hockey stick offers, where the individual offer, which can be parameterized by a price p, is defined by $p(q) = p$, for $q \in [0, q_A(p)]$ and $p(q) = q_A^{-1}(q)$ for $q \in [q_A(p), q_m]$. Moreover the functions $G(p)$ (defining the mixture) and $q_A(p)$ in the range $\left[\underline{p}, p_m\right]$ satisfy the linked differential equations:*

$$g(p) = G'(p) = \frac{1 - G(p)}{(p - C'(q_A(p)))}, \tag{32}$$

$$q_A'(p) = \frac{\int_0^{q_A(p)} 1 - F(q) - \frac{[p - C'(q) - G(p)(C'(q_A(p)) - C'(q))]}{(p - C'(q_A(p)))}(F(q + q_A(p)) - F(q))dq}{G(p)\int_0^{q_A(p)}(p - C'(q))f(q + q_A(p))dq}, \tag{33}$$

provided that $q_A\left(\underline{p}\right) > \overline{\varepsilon}/2$. The initial conditions for these differential equations are

$$\underline{p} = \frac{\int_0^{q_A\left(\underline{p}\right)}\left[C'\left(q_A\left(\underline{p}\right)\right)(1 - F(q)) - C'(q)(F(q + q_A\left(\underline{p}\right)) - F(q))\right]dq}{\int_0^{q_A\left(\underline{p}\right)}(1 - F(q + q_A\left(\underline{p}\right)))dq}, \tag{34}$$

$$G(\underline{p}) = 0, \tag{35}$$

and, in addition,

$$q_A'\left(\underline{p}\right) = \frac{2\int_0^{q_A\left(\underline{p}\right)}\left[C'\left(q_A\left(\underline{p}\right)\right) - C'(q)\right](F(q + q_A\left(\underline{p}\right)) - F(q))dq}{\left(\underline{p} - C'\left(q_A\left(\underline{p}\right)\right)\right)\int_0^{q_A\left(\underline{p}\right)}\left[\begin{array}{c} 2(\underline{p} - C'(q))f(q + q_A\left(\underline{p}\right)) \\ +(1 - F(q))\,C''\left(q_A\left(\underline{p}\right)\right)\end{array}\right]dq}. \tag{36}$$

3. *The value of $\underline{p}$ is chosen so that a solution to the differential equations satisfies*

$$G(p_m) = \frac{Ap_m - J - BP}{Ap_m - J - Bp_m}, \tag{37}$$

$$q_A(p_m) = q_m. \tag{38}$$

A sufficient condition for a hockey-stick mixture satisfying conditions 1-3 above to be a Nash equilibrium is that

$$\frac{\partial}{\partial q}\left[\left(\underline{p}-C'\left(q\right)\right)\frac{f(q+u)}{1-F(q)}\right]>0, \tag{39}$$

for every pair $(u,q)\in\{u\geq 0, 0\leq q\leq q_m : u+q<\overline{\varepsilon}\}$. *Under this condition any hockey-stick mixture has* $q_A\left(\underline{p}\right)>\overline{\varepsilon}/2$.

The condition (39) ensures that the second-order conditions in Proposition 3 are not violated. We compare it with (8) and conclude that it can only be satisfied when pure-strategy Nash equilibria with positive mark-ups do not exist. When the hockey-stick mixture is a Nash equilibrium, we refer to it as a mixed-strategy hockey-stick NE. The proposition below provides sufficient conditions for such an equilibrium to exist.

Proposition 10 *Consider a symmetric duopoly market where Assumption 1 is satisfied. Assume that* $D\left(p\right)=0$, $\overline{\varepsilon}/2<q_m<\overline{\varepsilon}$ *and* $\underline{p}\in\left[\underline{p}^{\min},\underline{p}^{\max}\right]$, *where*

$$\underline{p}^{\min}=\frac{\int_0^{\overline{\varepsilon}/2}\left[C'\left(\overline{\varepsilon}/2\right)\left(1-F(q)\right)-C'(q)(F(q+\overline{\varepsilon}/2)-F(q))\right]dq}{\int_0^{\overline{\varepsilon}/2}(1-F(q+\overline{\varepsilon}/2))dq} \tag{40}$$

and

$$\underline{p}^{\max}=\frac{C'\left(q_m\right)A-J}{B}. \tag{41}$$

Then there is a price cap P *for which a mixed-strategy hockey-stick NE exists if for some* $\delta>0$

$$\frac{\partial}{\partial q}\left[\left(\underline{p}-C'\left(q\right)\right)\frac{f(q+u)}{1-F(q)}\right]>\delta \tag{42}$$

$$\text{for } (u,q)\in\{u\geq 0, 0\leq q\leq q_m : u+q<\overline{\varepsilon}\}.$$

This result establishes that under a derivative condition (42) there exists a mixed-strategy hockey-stick NE for a range of lowest prices provided that the price cap is chosen appropriately. It is interesting to ask, firstly, for what range of price caps will there be an equilibrium solution which is a hockey stick mixture, and, secondly, if such a mixture exists will it be unique? To answer these questions requires a detailed consideration of the characteristics of the solutions to the differential equations defining $q_A(p)$.

We continue to consider the case $D\left(p\right)=0$, $\overline{\varepsilon}/2<q_m<\overline{\varepsilon}$ and we need to assume that the condition on the distribution of the demand shock in Proposition

10 applies with $\underline{p} = \underline{p}^{\min}$ as determined by (40). In other words we assume there is some $\delta > 0$ with

$$\frac{\partial}{\partial q}\left[\left(\underline{p}^{\min} - C'(q)\right)\frac{f(q+u)}{1-F(q)}\right] > \delta \qquad (43)$$
$$\text{for } (u,q) \in \{u \geq 0, 0 \leq q \leq q_m : u+q < \overline{\varepsilon}\}.$$

In this case there will be a unique mixed-strategy hockey-stick NE for a range of price caps from $P_{\min}$ (as determined by the mixture starting at $\underline{p}^{\min}$) to

$$P_{\max} = \frac{1}{B^2}\left(A^2 C'(q_m) - (A+B)J\right).$$

which, from (26), is the lowest price cap for horizontal mixtures. The details of this result can be found in Anderson et al. (2009).

Note that $\underline{p}^{\min}$ depends on properties of the cost function and demand distribution, so this provides an existence condition based on primitives of our model. Whenever (43) is satisfied it is also satisfied for $\underline{p} > \underline{p}^{\min}$. Thus integration of (43) with respect to u implies that (29) must be satisfied. Hence we can apply Proposition 8 to see that, when the price cap is in the range $(P_{\min}, P_{\max})$, then there is a unique mixed-strategy hockey-stick NE and no horizontal mixture, and when the price cap is greater than $P_{\max}$ there is a unique horizontal mixed-strategy NE and no mixed-strategy hockey-stick NE. It is easy to see from (30) that $P_{\max} = c$ for constant marginal costs c, so hockey-stick mixtures cannot exist in this case. This confirms Genc (2009), who proves that mixed-strategy equilibria over partly increasing supply functions do not exist for pivotal producers and constant marginal costs when demand is uniformly distributed.

We also know from Proposition 10 that hockey-stick mixtures satisfying (39) have horizontal offers at the top, i.e. $P > p_m$. By means of Proposition 7 we can therefore also rule out hockey-stick mixtures satisfying (39) for non-pivotal producers.

6.3 Examples

We conclude this section with two examples of equilibria with mixed strategies. In both examples we assume a symmetric duopoly with each player having capacity $q_m \in (\frac{1}{2}, 1)$ and $C(q) = \frac{1}{2}q^2$, and inelastic demand that is uniformly distributed on $[0, 1]$.

Example 11 ***Horizontal mixtures***[15]
With a uniform demand shock and quadratic costs we get

$$A = \frac{1}{2}q_m(2 - q_m),$$

[15] In this example we have assumed that demand is inelastic. Using the same approach, Anderson et al. (2009) are able to construct similar examples in which $D(p) \neq 0$, as long as $q_m < D(P) + \overline{\epsilon}$.

$$B = \frac{1}{2}(1 - q_m)^2,$$

and

$$J = \frac{1}{6}(-1 + 3q_m - q_m^3).$$

From Proposition 8 we get the necessary condition

$$P \geq \frac{1}{B^2}\left(A^2 C'(q_m) - (A + B)J\right) = \frac{13q_m^3 - 12q_m^4 + 3q_m^5 + 1 - 3q_m}{3(1 - q_m)^4} \tag{44}$$

for an equilibrium. We can see that this is sufficient because using this value of P gives, from $\underline{p} = (PB + J)/A$,

$$\underline{p} \geq \frac{-2q_m^3 + 6q_m^2 - 3q_m + 1}{3(1 - q_m)^2}$$

which is easily seen to be strictly greater than 1 for $q_m \in (\frac{1}{2}, 1)$. This means that

$$\frac{\partial}{\partial q}\left[(\underline{p} - C'(q))\frac{(F(q + q_m) - F(q))}{1 - F(q)}\right] = q_m \frac{\underline{p} - 1}{(q - 1)^2} > 0, \quad \text{for } q \in [0, 1 - q_m)$$

which is sufficient for an equilibrium by Proposition 8. So for every value of $q_m \in (\frac{1}{2}, 1)$ and price cap P satisfying (44) there is a horizontal mixed-strategy NE. The equilibrium can be uniquely determined by

$$\underline{p} = \frac{-\frac{1}{6}q_m^3 - \frac{1}{6} + \frac{1}{2}q_m + \frac{1}{2}P - Pq_m + \frac{1}{2}Pq_m^2}{q_m - \frac{1}{2}q_m^2}$$

and

$$G(p) = \frac{3q_m(2 - q_m)\left(p - \underline{p}\right)}{-3p - 3q_m + 12pq_m + q_m^3 - 6pq_m^2 + 1}.$$

The lowest bidder meets all demand up to its capacity. Thus the social welfare loss (production inefficiency) in this example is given by

$$\begin{aligned} W &= \overbrace{\int_0^{q_m} \frac{\varepsilon^2}{2} d\varepsilon + \int_{q_m}^1 \left(\frac{q_m^2}{2} + \frac{(\varepsilon - q_m)^2}{2}\right) d\varepsilon}^{\textit{Expected production cost}} - \overbrace{2\int_0^1 \frac{(\varepsilon/2)^2}{2} d\varepsilon}^{\textit{Efficient expected cost}} \\ &= -\frac{1}{2}q_m^3 + q_m^2 - \frac{1}{2}q_m + \frac{1}{12}. \end{aligned} \tag{45}$$

It is straightforward to see that the welfare loss increases with capacity from 25% of the optimal production cost when $q_m = 0.5$ up to 100% of the optimal production cost when $q_m = 1$ (provided the price cap is sufficiently high).

Example 12 ***Hockey-stick mixtures***
The choice $q_m = \frac{3}{4}$ *makes the bound given by (44) take the value 98.083. Hence if we choose a price cap* $P < 98.083$*, then by Proposition 8 there does not exist a horizontal mixed-strategy NE. However for every* $P \in (10.734, 98.083)$ *there are mixed-strategy hockey-stick NE. The most competitive equilibrium where* $q_A\left(\underline{p}\right) = \frac{1}{2}$ *is plotted in Figure 2. The lines shown are at contours of G taking values* 0, 0.1, 0.2, *up to* 1.0. *It has* $q_A(p)$ *starting at* $\underline{p} = 1$*, and shows a typical hockey-stick offer using a solid line. All the hockey-stick curves in the mixture meet* $q_m = 0.75$ *at* $p_m = 4.5$ *(approximately), and correspond to a price cap* $P = 10.734$*. Observe that the offers in the mixtures are horizontal for* $p > p_m = 4.5$*. As a comparison, the most competitive mixture is compared with another mixture in Figure 3. The higher priced equilibrium starts at* $\underline{p} = \frac{29}{15}$*, and gives* $q_A(\underline{p}) = \frac{6}{10}$ *and* p_m *approximately 5.54. The lowest priced hockey stick offer in this mixture is shown as a dashed line in Figure 3, along with the lowest priced horizontal offer in this mixture (at* p_m *=5.54). The highest priced offer in this mixture (at* $P = 24.6$*) is not shown. If we replace* q_m *by* $q_A(p)$*, we can use (45) to calculate the welfare loss when the highest of the two firms' chosen hockey-stick offers has its horizontal segment at* p*. It follows from the theory of order statistics that* $G^2(p)$ *is the probability distribution and* $2G(p)\,g(p)$ *is the probability density of this highest offer. Thus*

$$\begin{aligned} W &= \int_{\underline{p}}^{\overline{p}} 2G(p)\,g(p)\left(\int_0^{q_A(p)} \frac{\varepsilon^2}{2}d\varepsilon + \int_{q_A(p)}^1 \left(\frac{q_A^2(p)}{2} + \frac{(\varepsilon - q_A(p))^2}{2}\right) d\varepsilon\right) dp \\ &\quad - \int_0^1 (\varepsilon/2)^2\, d\varepsilon. \end{aligned}$$

From this expression we can calculate (by numerical integration) the welfare loss to be approximately 52% *of the optimal production cost when* $q_m = 0.75$ *and* $P = 10.734$*. The welfare loss will be higher for less competitive equilibria (higher* P*). When* $P > 98.083$ *and* $q_m = 0.75$*, we have horizontal mixtures where the welfare loss is approximately* 72% *of the optimal production cost.*

7 Conclusions

In this paper we derive general optimality conditions for pay-as-bid procurement auctions that are valid for any uncertainty in a producer's residual demand curve, i.e. for any combination of demand uncertainty, uncertainty in competitors' costs or randomization of competitors' offer curves. We use these conditions to derive necessary conditions for pure-strategy equilibria in electricity auctions, i.e. when costs are common knowledge and demand uncertain. We show that they fail to exist whenever the market clears at a point where a producer's mark-up times the hazard rate of the demand shock is increasing. Hence, it is of great interest to analyze mixed strategy equilibria under those circumstances.

We consider a symmetric duopoly market and characterize three different types of mixed-strategy equilibria: 1) Mixtures over increasing supply curves,

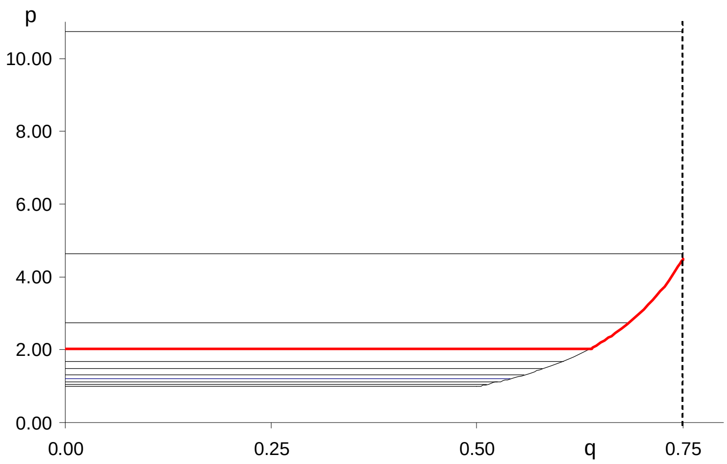

Figure 2: Mixed-strategy equilibrium for $\underline{p} = 1$ and $P = 10.734$, showing hockey-stick bids.

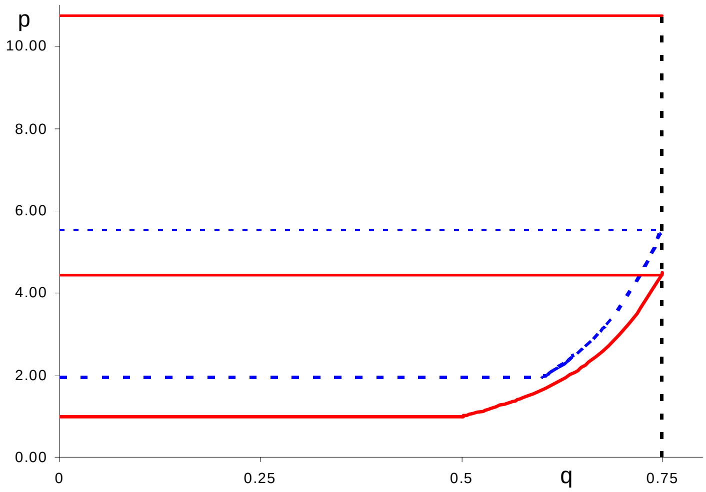

Figure 3: Comparison of two hockey-stick mixtures with $\underline{p} = 1$ (solid) and $\underline{p} = \frac{29}{15}$ (dashed). The price cap (24.6) for the dashed mixture is not shown.

where slope-constraints are not binding. 2) Mixtures over horizontal offers, where the whole output is offered at the same price. 3) Mixtures with hockey-stick offers, where some offers are slope-constrained for low outputs and strictly increasing for high outputs. We show that mixtures over strictly increasing supply functions can only occur in markets with non-pivotal producers, inelastic demand and no price cap. Under those circumstances we get a continuum of equilibria.

Mixtures with slope-constraints, however, only occur for pivotal producers. This type of equilibrium is uniquely determined by the price cap when demand is inelastic. If price caps are sufficiently high relative to the slope of the marginal cost, we realize from (30) that all offer curves of the producers are slope-constrained along the whole output. These one-dimensional mixtures correspond to mixed-strategy equilibria previously analyzed by Anwar (2006), Fabra et al. (2006), Genc (2009) and Son et al. (2004), and they are also Nash equilibria in corresponding Bertrand-Edgeworth games with uncertain demand. When the price cap becomes sufficiently low, this equilibrium is continuously transformed into a mixed-strategy hockey-stick Nash equilibrium, so that the top of the mixture has horizontal offers and the bottom of the mixture has hockey-stick shaped offers. More offers in the mixture will have a hockey-stick shape as the price cap decreases.

Mixed-strategy equilibria are a nuisance for agents in the market, as they cause additional uncertainty. Another disadvantage is that mixed strategies will lead to welfare losses due to inefficient production. With symmetric producers and convex increasing costs, production is most efficient if all producers have the same output, which is the case for symmetric pure-strategy SFE of a uniform-price auction (Klemperer and Meyer, 1989). But with the randomized offer curves occurring in pay-as-bid auctions, the realized production will typically be asymmetric. The welfare loss is significant, in our examples with two symmetric firms that have quadratic costs, it is between 25% and 100% of the optimal production cost.

With our model it becomes possible to quantitatively compare strategic bidding in uniform-price and pay-as-bid electricity auctions for previously unexplored but still very relevant cases, for example when producers are pivotal, stepped marginal costs are common knowledge and demand shocks are normally distributed.

Hortacsu (2002) uses the Euler equation to empirically analyse strategic bidding in pay-as-bid treasury auctions. Kastl (2008) extends this empirical analysis to settings with stepped offers/bids. Our contribution to the empirical analysis of pay-as-bid auctions with pivotal producers is to show that offers in such markets are expected to have horizontal segments due to slope-constraints, even if there are no restrictions on the number of steps in an offer. Moreover, our optimality conditions can be used in empirical studies of such offers. Our analysis also provides second-order conditions which can be important in establishing that an empirical model gives rise to a Nash equilibrium (and not only a first-order solution).

Hockey stick bids have often been observed in practice in wholesale electricity markets with uniform pricing. Our work gives a theoretical justification for the use of hockey stick mixtures also in discriminatory price auctions, so it would be interesting to look for offers with such shapes in for example the discriminatory electricity auction used in Britain.

As explained by Harsanyi (1973), mixed-strategy NE can correspond to pure-strategy Bayesian NE where small private cost fluctuations significantly influence the offers. Thus this is the result that we would expect in case a structural empirical model with private costs was used to analyse offers in a pay-as-bid electricity market. But we want to stress that our results rely on the assumption that cost uncertainties, private or affiliated, are small relative to the demand uncertainty, so that costs can be regarded as being approximately common knowledge. It would be very valuable to test this key assumption empirically for electricity markets and related markets.

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Appendix

Proposition 1

Proof. We show that when ψ is continuous then Π defined from (2) is a continuous function of the offer curve using the essential supremum metric. Suppose $p_1(q)$ and $p_2(q)$ are two functions in Ω with profits $\Pi(p_1)$ and $\Pi(p_2)$ respectively. Then

$$\begin{aligned}
\Pi(p_2) - \Pi(p_1) &= \int_0^{q_m} (p_2(q) - C'(q))(1 - \psi(q, p_2(q)))dq \\
&\quad - \int_0^{q_m} (p_1(q) - C'(q))(1 - \psi(q, p_1(q)))dq \\
&= \int_0^{q_m} (p_2(q) - p_1(q))(1 - \psi(q, p_2(q)))dq \\
&\quad + \int_0^{q_m} (p_1(q) - C'(q))(\psi(q, p_1(q)) - \psi(q, p_2(q)))dq
\end{aligned}$$

Since $\psi(q,p)$ and $C'(q)$ are continuous then this is enough to establish that Π is a continuous function in the essential supremum metric. Hence in this case we have the required continuity property for Π which together with compactness of Ω establishes the result. ■

Proposition 2 and Proposition 3

Proof. We prove these two Propositions together. **Part (a) of Proposition 3:** Consider a vertical perturbation of the curve downwards by $\delta > 0$, between limits q_1 and q_2, to give a new curve

$$r(q) = \begin{cases} \max\{p(q_1), p(q) - \delta\} & q_1 \le q \le q_2 \\ p(q), & \text{otherwise.} \end{cases}$$

This formulation will ensure that the result of the perturbation is still a monotonic curve. Since $p(\cdot)$ is increasing at q_1 we let $\eta > 0$ be the derivative there. Hence

$p(q) > p(q_1) + \delta$ for $q > q_1 + 2\delta/\eta$ and δ small enough. Then

$$\begin{aligned}\Pi(r) - \Pi(p) &= \int_0^{q_m} (r(q) - C'(q))(1 - \psi(q, r(q))dq \\ &- \int_0^{q_m} (p(q) - C'(q))(1 - \psi(q, p(q))dq \\ &= -\delta \int_{q_1}^{q_2} (1 - \psi(q, r(q))dq + \delta \int_{q_1}^{q_2} \psi_p(q, p(q))(p(q) - C'(q))dq + O(\delta^2).\end{aligned}$$

The $O(\delta^2)$ term takes account both of the difference between δ times the derivative of ψ and $\psi(q, p(q)) - \psi(q, r(q))$ and also the fact that in the range $(q_1, q_1 + 2\delta/\eta)$ we may have $r(q) = p(q_1) > p(q) - \delta$.

From optimality we must have $\Pi(r) - \Pi(p) \leq 0$, and so, letting δ approach zero, we obtain

$$\int_{q_1}^{q_2} (1 - \psi(q, p(q)) - \psi_p(q, p(q))(p(q) - C'(q)))dq \geq 0, \tag{46}$$

which gives the result.

Part (b) of Proposition 3: The argument for the second part is very similar, but now we consider a perturbation upwards by $\delta > 0$, to give a new curve

$$r(q) = \begin{cases} \min\{p(q_3), p(q) + \delta\} & q_2 \leq q \leq q_3 \\ p(q), & \text{otherwise.} \end{cases}$$

Letting δ approach zero, we obtain

$$\int_{q_2}^{q_3} (1 - \psi(q, p(q)) - \psi_p(q, p(q))(p(q) - C'(q)))dq \leq 0. \tag{47}$$

Proposition 2: The conditions in (46) and (47) hold on any part of the strictly increasing segment, so for sufficiently small $\delta > 0$ we must have

$$Z = 1 - \psi(q, p) - \psi_p(q, p)(p - C'(q)) \begin{array}{ll} \leq 0 & \text{if } p = p(q) + \delta \\ = 0 & \text{if } p = p(q) \\ \geq 0 & \text{if } p = p(q) - \delta \end{array},$$

which gives (5). We have $Z(q, p) \leq 0$ just above the optimal curve $p(q)$ and $Z(q, p) \geq 0$ just below the curve. We can therefore conclude that $Z(q, p)$ is negative for (q, p) above and to the left of the increasing offer curve and positive for (q, p) below and to the right of the offer curve, which gives (6).

Part (c) of Proposition 3: Choose $q_0 < q_1$ with $p(q)$ increasing in (q_0, q_1) and let q_2 be an arbitrary point in the range (q_1, q_3). Then from part (a) of Proposition 3 we have

$$\int_{q_0}^{q_2} (1 - \psi(q, p(q)) - \psi_p(q, p(q))(p(q) - C'(q)))dq \geq 0.$$

Now using Proposition 2 we know that the integrand is zero over (q_0, q_1), and letting q_2 approach q_3 shows that

$$\int_{q_1}^{q_3} (1 - \psi(q, p(q)) - \psi_p(q, p(q))(p(q) - C'(q)))dq \geq 0.$$

In the same way we can choose $q_4 > q_3$ with $p(q)$ increasing in (q_3, q_4) and use part (b) of Proposition 3 and Proposition 2 to establish that the integral from q_2 to q_3 is non-positive. Then letting q_2 approach q_1 shows

$$\int_{q_1}^{q_3} (1 - \psi(q, p(q)) - \psi_p(q, p(q))(p(q) - C'(q)))dq \leq 0.$$

Finally we combine the two inequalities for the result we require. ■

Proposition 5

Proof. The formula for $G_j(q,p)$ can be obtained by setting $f(q) = \frac{1}{\overline{\varepsilon}}$, and $F(q) = \frac{q}{\overline{\varepsilon}}$ and differentiating (18). This gives

$$\begin{aligned} -\frac{1}{\overline{\varepsilon}} G_j(\overline{\varepsilon} - q, p) &= \frac{\partial}{\partial q} \left(\frac{(p - \underline{p})(1 - \frac{q}{\overline{\varepsilon}})}{p - C'(q)} \right) \\ &= (p - \underline{p}) \frac{(1 - \frac{q}{\overline{\varepsilon}})C''(q) - (p - C'(q))\frac{1}{\overline{\varepsilon}}}{(p - C'(q))^2} \end{aligned}$$

so

$$G_j(\overline{\varepsilon} - q, p) = (p - \underline{p}) \frac{(p - C'(q)) - (\overline{\varepsilon} - q)C''(q)}{(p - C'(q))^2}$$

as required.

From writing

$$G_j(\overline{\varepsilon} - q, p) = \frac{p - \underline{p}}{p - C'(q)} \frac{(p - C'(q)) - (\overline{\varepsilon} - q)C''(q)}{p - C'(q)}$$

we see that (19) implies

$$0 \leq G_j(\overline{\varepsilon} - q, p) \leq 1, \quad \text{for every } q \in [0, \overline{\varepsilon}], \text{ and } p \geq \underline{p}.$$

We next check that $G_j(q,p)$ is monotonic.

$$\begin{aligned} \frac{\partial}{\partial p} G_j(\overline{\varepsilon} - q, p) &= \frac{\partial}{\partial p} \left((p - \underline{p}) \frac{(p - C'(q)) - (\overline{\varepsilon} - q)C''(q)}{(p - C'(q))^2} \right) \\ &= \frac{(p - C'(q))(\underline{p} - C'(q)) + C''(q)(\overline{\varepsilon} - q)(p + C'(q) - 2\underline{p})}{(p - C'(q))^3} \end{aligned}$$

which is nonnegative for every $q \in [0, \overline{\varepsilon}]$ when

$$(p - C'(q))(\underline{p} - C'(q)) + C''(q)(\overline{\varepsilon} - q)(p + C'(q) - 2\underline{p}) \geq 0.$$

The left-hand side of this inequality is increasing in p, and so this is equivalent to requiring

$$(\underline{p} - C'(q))(\underline{p} - C'(q)) + C''(q)(\overline{\varepsilon} - q)(\underline{p} + C'(q) - 2\underline{p}) \geq 0,$$

which follows from (19).

We also require

$$\frac{\partial}{\partial q}G_j\left(\overline{\varepsilon}-q,p\right)\geq 0$$

which is equivalent to

$$\left(p-C'\left(q\right)\right)2C''\left(q\right)\geq\left(2C''\left(q\right)^2+\left(p-C'\left(q\right)\right)C'''\left(q\right)\right)\left(\overline{\varepsilon}-q\right)$$

or

$$2C''\left(q\right)\geq\left(\frac{2C''\left(q\right)^2}{p-C'\left(q\right)}+C'''\left(q\right)\right)\left(\overline{\varepsilon}-q\right).$$

The right-hand side of this inequality is decreasing in p, so this is equivalent to

$$2C''\left(q\right)\geq\left(\frac{2C''\left(q\right)^2}{\underline{p}-C'\left(q\right)}+C'''\left(q\right)\right)\left(\overline{\varepsilon}-q\right).$$

We note that this inequality is satisfied if (19) is satisfied and $C'''\left(q\right)\leq 0$.

It follows from the construction of G that every offer in the mixing region has the same profit, and offers at $\underline{p}$ are accepted with the same probability as offers at lower prices, so it is never profitable to undercut $\underline{p}$. ■

Proposition 7

Proof. From Proposition 3 we know that a necessary characteristic of an optimal horizontal offer is that $Z(q,p)$ is non-negative for small values of q, otherwise we can obtain an improvement by reducing the first section of the offer. We will show that this property fails for player i when $\overline{\varepsilon}$ is equal to or less than the competitor's capacity. In this case $F(q_j^m)=1$ and hence we can write (22) as

$$G_j(p)=1-\frac{K_i}{W(p)}, \tag{48}$$

where

$$W(p)=\int_0^{q_i^m}(p-C_i'(q))(1-F(q))dq,$$

and we have

$$W'(p)=\int_0^{q_i^m}(1-F(q))dq. \tag{49}$$

Observe that if $p\in(p_1,p_2)$ then it follows from (48) that $W(p)>0$ and so $p>C_i'(0)$. Now, from (21),

$$\psi_i\left(q,p\right)=(1-G_j(p))F(q)+G_j\left(p\right)$$

and thus it follows from (4)

$$Z(q,p)=(1-G_j\left(p\right))(1-F(q))-(p-C_i'(q))G'_j(p)(1-F(q)).$$

From the above relations we have

$$
\begin{aligned}
Z(0,p) &= (1-F(0))[(1-G_j(p))-(p-C_i'(0))G_j'(p)] \\
&= (1-F(0))[\frac{K_i}{W(p)}-(p-C_i'(0))\frac{K_iW'(p)}{W(p)^2}] \\
&= \frac{K_i(1-F(0))}{W(p)^2}[W(p)-(p-C_i'(0))W'(p)].
\end{aligned} \tag{50}
$$

Substituting for $W(p)$ and $W'(p)$ we can show

$$
Z(0,p) = \frac{K_i(1-F(0))}{W(p)^2}\int_0^{q_i^m}(C_i'(0)-C_i'(q))(1-F(q))dq \leq 0. \tag{51}
$$

If marginal costs are not constant then $C_i'(0) < C_i'(q_i^m)$ and inequality (51) is strict, which gives the contradiction we need.

For the special case in which $C_i(x) = cx$ we can simplify (22) to

$$
G_j(p) = 1 - \frac{K_i}{(p-c)\int_0^{q_i^m}(1-F(q))dq} = 1 - \frac{\underline{p}-c}{p-c},
$$

because $G_j(\underline{p}) = 0$. Combined with (21) this gives

$$
\begin{aligned}
\psi_i\left(q_i,p\right) &= (1-G_j(p))F(q_i)+G_j\left(p\right) \\
&= 1-\frac{\left(\underline{p}-c\right)\left(1-F(q)\right)}{p-c}
\end{aligned}
$$

which satisfies the condition in (11) ensuring that this situation has $Z=0$ throughout the region in which offers are made. Thus we are again in the non-slope-constrained case, even though the equilibrium offers are horizontal. ■

Proposition 8

Proof. We start with establishing the general form that an equilibrium solution must have. Under these assumptions we can rewrite (21) and (22) as

$$
\psi(q,p) = F(q)+G(p)(F(q+q_m)-F(q)) \tag{52}
$$

and

$$
G(p) = \frac{\int_0^{q_m}(p-C'(q))(1-F(q))dq-K}{L(p)}, \tag{53}
$$

where

$$
\begin{aligned}
K &= \int_0^{q_m}(p-C'(q))(1-(G(p)F(q+q_m)+(1-G(p))F(q)))dq && (54) \\
&= \int_0^{q_m}(\underline{p}-C'(q))(1-F(q))dq && (55) \\
&= \int_0^{q_m}(\overline{p}-C'(q))(1-F(q+q_m))dq, && (56)
\end{aligned}
$$

because K is equal to the pay-off for all price levels, including the highest and lowest clearing price. Hence, we can rewrite (53) in the form (28).

Writing $g(p) = G'(p)$ for the density function, we obtain

$$g(p) = \frac{L(\underline{p})A}{[L(p)]^2}, \tag{57}$$

which implies that $g(p) \geq 0$ as long as $p \geq \underline{p} \geq C'(q_m)$.

A relation between the minimum and maximum prices in the horizontal mixture can be calculated from (55) and (56)

$$\underline{p} = (\overline{p}B + J)/A. \tag{58}$$

Observe that the existence of $\underline{p}$ guarantees that there is a finite maximum price (at which $G(p)$ reaches 1) if firms are pivotal so that $B > 0$.

By means of (52) and (4) we can calculate

$$\begin{aligned} Z(q,p) &= 1 - F(q) - [G(p) + (p - C'(q))g(p)](F(q+q_m) - F(q)) \\ &= [1 - F(q)]\,\Lambda(q,p), \end{aligned} \tag{59}$$

where

$$\Lambda(q,p) = 1 - [G(p) + (p - C'(q))g(p)]\,\frac{(F(q+q_m) - F(q))}{1 - F(q)}. \tag{60}$$

We begin by establishing the first part of the proposition relating to the necessary conditions for an equilibrium. It follows from Proposition 3 that a necessary condition for a mixed-strategy equilibrium with horizontal offers is:

$$\int_0^{q_m} Z(t,p)dt = 0, \tag{61}$$

so that the marginal profit from increasing the offer of the whole segment is zero. This implies that (28) is satisfied, since this is the condition to ensure the same payoff for horizontal offers at any price $p \in [\underline{p}, \overline{p}]$. Due to Proposition 3 we also require that

$$\textstyle\int_0^q Z(t,p)dt \geq 0, \quad q \leq q_m. \tag{62}$$

Otherwise the producer would find it profitable to deviate by reducing the price of the first part of the segment.

Now suppose $p > \overline{p}$ then from (59) we get

$$Z(q,p) = 1 - F(q+q_m), \tag{63}$$

because the competitor has $g(p) = 0$ and $G(p) = 1$ in this price range. Hence, $\int_0^{q_m} Z(t, \overline{p}+)dt > 0$ and producers have incentives to raise their highest offers unless it is prevented by the price cap: so we require that $\overline{p} = P$.

Using (58) we can show (after some algebra) that the condition (26) is equivalent to

$$\underline{p} \geq p^* = \frac{C'(q_m)A - J}{B},$$

which we will now prove. This inequality follows trivially when marginal costs are constant, so that $p^* = c$. But we need a more complicated argument for cases when $C'(q_m) > C'(0)$. Note that since $2q_m \geq \bar{\varepsilon}$ we have from (60) that

$$\Lambda\left(q_m, \underline{p}\right) = 1 - (\underline{p} - C'(q_m))g(\underline{p}) \tag{64}$$
$$= 1 - \frac{(\underline{p} - C'(q_m))A}{\underline{p}(A-B) - J}.$$

When $\underline{p} = p^*$ we obtain

$$\Lambda\left(q_m, \underline{p}\right) = 1 - \frac{\left(C'(q_m)(A-B) - J\right)A}{\left(C'(q_m)A - J\right)(A-B) - BJ} = 0.$$

From differentiation of (64) it is easy to see that $\Lambda\left(q_m, \underline{p}\right)$ is decreasing in $\underline{p}$. Hence if $\underline{p} < p^*$ then $\Lambda\left(q_m, \underline{p}\right) > 0$ and hence $Z\left(q_m, \underline{p}\right) > 0$. But this would imply that $\int_0^q Z(t, \underline{p})dt$ is increasing in q as q approaches q_m from below, which leads to a contradiction from either (61) or (62). Thus we have established the condition we require that $\underline{p} \geq p^*$, which in turn leads to the condition (26).

Now we want to establish that under (29) this mixture over horizontal offers is a Nash equilibrium. Our assumption (29) implies that, for $q \in [0, \bar{\varepsilon} - q^m)$,

$$(\underline{p} - C'(q))\frac{\partial}{\partial q}\left\{\frac{(F(q+q_m) - F(q))}{1 - F(q)}\right\} > C''(q)\,\frac{(F(q+q_m) - F(q))}{1 - F(q)}.$$

As the right hand side is non-negative this establishes that $\underline{p} > C'(\bar{\varepsilon} - q^m)$ and that $\frac{\partial}{\partial q}\left\{\frac{(F(q+q_m) - F(q))}{1-F(q)}\right\} > 0$ in this interval. Observe that

$$\begin{aligned}
\Lambda_q(q,p) &= -\frac{\partial}{\partial q}\left\{[G(p) + (p - C'(q))g(p)]\,\frac{(F(q+q_m) - F(q))}{1 - F(q)}\right\} \\
&= -G(p)\frac{\partial}{\partial q}\left\{\frac{(F(q+q_m) - F(q))}{1 - F(q)}\right\} \\
&\qquad - \frac{\partial}{\partial q}\left\{(p - C'(q))g(p)\frac{(F(q+q_m) - F(q))}{1 - F(q)}\right\} \\
&= -\left(G(p) + g(p)(p - \underline{p})\right)\frac{\partial}{\partial q}\left\{\frac{(F(q+q_m) - F(q))}{1 - F(q)}\right\} \\
&\quad - g(p)\frac{\partial}{\partial q}\left\{(\underline{p} - C'(q))\frac{(F(q+q_m) - F(q))}{1 - F(q)}\right\}.
\end{aligned}$$

Thus $\Lambda_q(q,p) < 0$ for $p \geq \underline{p}$ and $q \in [0, \bar{\varepsilon} - q^m)$. On the other hand, from (60) we note that $\Lambda_q(q,p) = C''(q)\,g(p) \geq 0$ if $q > \bar{\varepsilon} - q_m$.

The results for $\Lambda_q(q,p)$ imply that $Z(q,p)$ cannot be always zero. We know that $Z(0,p) \geq 0$, $Z(q_m,p) \leq 0$ and $\int_0^{q_m} Z(q,p)dq = 0$ for $p \geq \underline{p}$. As the derivative of Λ changes from negative to positive at $\bar{\varepsilon} - q^m$ we can deduce that Λ (and therefore Z) has a single zero crossing moving from positive to negative at some point $q^*(p) \in (0, \bar{\varepsilon} - q^m)$ for $p \geq \underline{p}$.

Suppose that one player uses the mixture we have defined and consider the best response offer by the other player. It is clear that the lowest price used is $\underline{p}$. Now an optimal offer cannot contain a segment with $0 < p'(q) < \infty$ since from Proposition 2 this only happens when $Z = 0$ and the optimal solution cannot follow $q^*(p)$ since $Z(q,p)$ is positive for $q < q^*(p)$ and negative for $q > q^*(p)$, which would contradict the second-order condition (6). Hence an optimal solution can only consist of horizontal and vertical segments. First consider a step function offer with at least two horizontal segments. As $\int_0^{q_m} Z(q,p)dq = 0$ and as Z has a single zero crossing moving from positive to negative, we know that the integral of Z on the first horizontal segment (which must start at $q = 0$) will be positive if it finishes before q_m. Hence its payoff can be improved by raising this first horizontal segment until its height matches the next segment in the step function. Thus any optimal solution must consist of a single horizontal segment, and thus is in the set of solutions already considered as part of our Nash equilibrium. ■

Proposition 9

Proof. We begin by showing that the conditions of the Proposition statement are necessary for a hockey stick mixture to be an equilibrium. We start by considering the range above p_m (the first condition) where we have from (52)

$$\psi(q,p) = F(q) + G(p)(F(q+q_m) - F(q))$$

and from (59) and (60) that $Z(q,p) = [1 - F(q)]\,\Lambda\,(q,p)$ where

$$\Lambda\,(q,p) = 1 - [G(p) + (p - C'\,(q))g(p)]\,\frac{(F(q+q_m) - F(q))}{1 - F(q)}.$$

Now we note from our previous discussions that we can derive the equation for G in (31) from (53) and (56)

$$G(p) = \frac{\int_0^{q_m}(p - C'(q))(1 - F(q))dq - K}{\int_0^{q_m}(p - C'(q))(F(q+q_m) - F(q))dq}$$

where

$$K = \int_0^{q_m}(P - C'(q))(1 - F(q+q_m))dq,$$

which is the profit achieved by offering at the price cap. Thus the competitor mixing in this way is exactly what is required to ensure that all the horizontal offers with $p \geq p_m$ achieve the same profit for a producer, and hence, by construction, $\int_0^{q_m} Z(t,p)dt = 0$ for $p \in [p_m, P]$.

We also have P as the highest price offered, as in Proposition 8. Otherwise producers would have incentives to increase their highest offer, because of the result in (63).

Now we turn to the part of the solution below p_m. We will show that the differential equations (32) and (33) arise from the requirement that $\int_0^{q_A(p)} Z(t,p)dt = 0$ and $Z(q_A(p),p) = 0$ for $p \in \left[\underline{p}, p_m\right]$. We have

$$\psi(q,p) = F(q) + G(p)(F(q+q_A\,(p)) - F(q))$$

for $p \in [\underline{p}, p_m]$. This follows from the fact that with probability $G(p)$ the other player offers one of the hockey stick offers with price below p, and each of these offer curves coincides at the quantity $q_A(p)$ at price p. By means of (4), we can now calculate

$$\begin{aligned} Z(q,p) &= 1 - \psi(q,p) - (p - C'(q))\,\psi_p(q,p) \\ &= 1 - F(q) - [G(p) + (p - C'(q))g(p)]\,(F(q + q_A(p)) - F(q)) \\ &- (p - C'(q))q_A'(p)\,f(q + q_A(p))G(p) \\ &= [1 - F(q)]\,\Lambda(q,p)\,, \end{aligned} \tag{65}$$

where

$$\begin{aligned} \Lambda(q,p) = 1 - [G(p) + (p - C'(q))g(p)]\,&\frac{(F(q + q_A(p)) - F(q))}{1 - F(q)} \\ - (p - C'(q))q_A'(p)\,&\frac{f(q + q_A(p))}{1 - F(q)}G(p). \end{aligned} \tag{66}$$

As we assume that $q_A(\underline{p}) > \bar{\varepsilon}/2$, we also have $q_A(p) > \bar{\varepsilon}/2$ and we can simplify (65) at $q = q_A(p)$ to obtain

$$Z(q_A(p), p) = 1 - F(q_A(p)) - [G(p) + (p - C'(q_A(p)))g(p)]\,(1 - F(q_A(p))). \tag{67}$$

Since increasing supply functions must follow a $Z(q,p) = 0$ curve, we require $Z(q_A(p), p) = 0$. This implies (32), as required for $p \in [\underline{p}, p_m]$.

From (32) and (65) we also have

$$\begin{aligned} Z(q,p) = 1 - F(q) - \left(\frac{p - C'(q) - G(p)(C'(q_A(p)) - C'(q))}{p - C'(q_A(p))}\right)(F(q + q_A(p)) - F(q)) \\ - (p - C'(q))q_A'(p)\,f(q + q_A(p))G(p). \end{aligned} \tag{68}$$

Since $\int_0^{q_A(p)} Z(t,p)dt = 0$ we can deduce (33) as required.

By definition the lowest price $\underline{p}$ has $G(\underline{p}) = 0$. So, we can deduce from (32) that

$$g(\underline{p}) = \frac{1}{(\underline{p} - C'(q_A(\underline{p})))}. \tag{69}$$

Thus, when $p = \underline{p}$ we can simplify (65) to

$$Z(q,\underline{p}) = 1 - F(q) - \frac{(\underline{p} - C'(q))}{\underline{p} - C'(q_A(\underline{p}))}(F(q + q_A(\underline{p})) - F(q)).$$

Hence, the condition $\int_0^{q_A(p)} Z(t,p)dt = 0$ yields:

$$0 = \int_0^{q_A(\underline{p})} \left[1 - F(q) - \frac{(\underline{p} - C'(q))}{\underline{p} - C'(q_A(\underline{p}))}(F(q + q_A(\underline{p})) - F(q))\right] dq. \tag{70}$$

from which we obtain (34).

Now consider the value of $q_A'(\underline{p})$ as given by the expression (33) evaluated at $\underline{p}$. Using (70) and the fact that $G(\underline{p}) = 0$ shows that both the numerator and denominator are equal to zero. Hence, we must calculate $q_A'(\underline{p})$ from (33) by means of l'Hôpital's rule. Let $q_A'(p) = N(p)/M(p)$ where

$$\begin{aligned} N(p) &= (p - C'(q_A(p))) \int_0^{q_A(p)} [1 - F(q)]\, dq \\ &\quad - \int_0^{q_A(p)} (p - C'(q) - G(p)[C'(q_A(p)) - C'(q)])(F(q + q_A(p)) - F(q))dq \end{aligned}$$

and

$$M(p) = (p - C'(q_A(p)))G(p) \int_0^{q_A(p)} (p - C'(q))f(q + q_A(p))dq.$$

In order to apply l'Hôpital's rule we calculate $N'(\underline{p})$ and $M'(\underline{p})$ using $G(\underline{p}) = 0$, $Z(q_A(\underline{p}), \underline{p}) = 0$ and Leibniz' rule. Note that the contribution from differentiating the integration limits are zero in both cases.

$$\begin{aligned} N'(\underline{p}) &= (1 - C''(q_A(\underline{p}))\, q_A'(\underline{p})) \int_0^{q_A(\underline{p})} (1 - F(q))\, dq \\ &\quad - \int_0^{q_A(\underline{p})} \left(1 - g(\underline{p}) \left[C'(q_A(\underline{p})) - C'(q)\right]\right) (F(q + q_A(p)) - F(q))dq \\ &\quad - q_A'(\underline{p}) \int_0^{q_A(\underline{p})} (\underline{p} - C'(q))f(q + q_A(\underline{p}))dq \end{aligned}$$

and

$$M'(\underline{p}) = (\underline{p} - C'(q_A(\underline{p})))g(\underline{p}) \int_0^{q_A(\underline{p})} (\underline{p} - C'(q))f(q + q_A(\underline{p}))dq.$$

Since $q_A'(\underline{p}) = N'(\underline{p})/M'(\underline{p})$ and using (69) we have

$$q_A'(\underline{p}) \int_0^{q_A(\underline{p})} (\underline{p} - C'(q))f(q + q_A(\underline{p}))dq = N'(\underline{p}).$$

We can now collect all terms with $q_A'(p))$ and use (69), so that

$$q_A'(\underline{p}) = \frac{\int_0^{q_A(\underline{p})} \left\{[1 - F(q)] - \left(1 - \frac{C'(q_A(\underline{p})) - C'(q)}{\underline{p} - C'(q_A(p))}\right)(F(q + q_A(\underline{p})) - F(q))\right\} dq}{\int_0^{q_A(\underline{p})} \left\{2(\underline{p} - C'(q))f(q + q_A(\underline{p})) + [1 - F(q)]\, C''(q_A(\underline{p}))\right\} dq}. \tag{71}$$

The relationship (36) now follows from (70).

The condition (37) follows from (31) and the fact that $G(p)$ is continuous at p_m which is the price at which the supply function $q_A(p)$ hits the right-hand boundary q_m.

In order to show sufficiency we will need to establish a number of different things, we start by showing that $q_A(\underline{p}) > \overline{\varepsilon}/2$ follows from the assumption $\frac{\partial}{\partial q}\left[(\underline{p} - C'(q))f(q+u)/(1-F(q))\right] > 0$. Under this assumption we can deduce

$$\frac{\partial}{\partial q}\left[(\underline{p} - C'(q))\frac{(F(q+q_A(p)) - F(q))}{1-F(q)}\right] = \frac{\partial}{\partial q}\left[\frac{(\underline{p} - C'(q))}{1-F(q)}\int_0^{q_A(p)} f(q+u)du\right]$$
$$= \int_0^{q_A(p)} \frac{\partial}{\partial q}\left[(\underline{p} - C'(q))\frac{f(q+u)}{1-F(q)}\right] du > 0 \tag{72}$$

if $q + q_A(\underline{p}) < \overline{\varepsilon}$. Hence, if $2q_A(\underline{p}) \leq \overline{\varepsilon}$ then (66) and $G(\underline{p}) = 0$ implies

$$\Lambda_q(q, \underline{p}) = -g(\underline{p})\frac{\partial}{\partial q}\left\{(\underline{p} - C'(q))\frac{(F(q+q_A(\underline{p})) - F(q))}{1-F(q)}\right\} < 0$$

if $q < q_A(\underline{p})$. This would imply that $Z(q, \underline{p})$ reaches 0 from a positive Z-value as $q \to q_A(\underline{p})$. But it would violate the necessary conditions from Proposition 3 that $\int_0^{q_A} Z(t,p)dt = 0$ and $\int_0^q Z(t,p)dt \geq 0$ for $q \leq q_A(p)$. Hence, we have established $2q_A(p) \geq 2q_A(\underline{p}) > \overline{\varepsilon}$, as required.

Next we consider the case $p \in (\underline{p}, p_m)$, and we show that the conditions are enough to guarantee that $G(p)$ and $q_A(p)$ are non-decreasing as functions of p. First observe from (34) that

$$\underline{p} \geq \frac{C'(q_A(\underline{p}))\int_0^{q_A(\underline{p})}\left[(1-F(q)) - (F(q+q_A(\underline{p})) - F(q))\right]dq}{\int_0^{q_A(\underline{p})}(1-F(q+q_A(\underline{p})))dq} = C'(q_A(\underline{p})).$$

Note that the condition (39) of the Proposition statement can be written more explicitly as

$$(\underline{p} - C'(q))\frac{\partial}{\partial q}\left[f(q+u)/(1-F(q))\right] - f(q+u)/(1-F(q))\,C''(q) > 0$$

for $u \geq 0$, $0 \leq q \leq q_m$ with $u + q < \overline{\varepsilon}$. This assumption would be violated for $q = q_A(\underline{p}) \leq q_m < \overline{\varepsilon}$ if $\underline{p} \leq C' q_A(\underline{p})$. Hence, we can establish that $\underline{p} > C'(q_A(\underline{p}))$, so that the above inequality is strict. So $G'(p) > 0$, because of (32), and from (36) we observe that $q_A'(\underline{p}) \geq 0$. Now consider differentiating the identity $\int_0^{q_A(p)} Z(t,p)dt = 0$ with respect to p under the assumption that $q_A'(p) = 0$ and we make use of (32) and (68).

$$\int_0^{q_A(p)}(p - C'(q))q_A''(p)\,f(q+q_A(p))G(p)dq$$
$$= -\int_0^{q_A(p)}\frac{\partial}{\partial p}\left[\frac{(p - C'(q)) + G(p)(C'(q) - C'(q_A(p)))}{(p - C'(q_A(p)))}\right](F(q+q_A(p)) - F(q))dq$$
$$= \int_0^{q_A(p)}\left[\frac{2\left[1-G(p)\right](C'(q_A(p)) - C'(q))}{(p - C'(q_A(p)))^2}\right](F(q+q_A(p)) - F(q))dq \geq 0.$$

Hence, $q''_A(p) \geq 0$ whenever $q'_A(p) = 0$. Thus the derivative $q'_A(p)$ can move from negative to positive but not the other way around as p increases. Since $q'_A(\underline{p}) \geq 0$ and this value is defined by continuity from above, there can be no changes of sign in q'_A between $\underline{p}$ and p_m and so q_A is increasing throughout this range.

We know that $Z(q_A(p), p) = 0$. Now we will analyze other potential zero-crossings where $Z(q, p) = 0$ in the interval $q \in (0, q_A(p))$. From (66) we note that $\Lambda_q(q, p) = C''(q) g(p) \geq 0$ if $q > \bar{\varepsilon} - q_A(p)$, and so Λ is non-decreasing in this range and zero at $q = q_A(p)$, which implies that $\int_{\bar{\varepsilon}-q_A(p)}^{q_A(p)} Z(q, p) dq \leq 0$ and $Z(\bar{\varepsilon} - q_A(p)+, p) \leq 0$. Together with the condition $\int_0^{q_A(p)} Z(q, p) dq = 0$ this implies that there must be at least one $q^* \in [0, \bar{\varepsilon} - q_A(p)]$ with $Z(q^*, p) = 0$.

From (66) we have

$$\begin{aligned}\Lambda_q(q, p) = &-\frac{\partial}{\partial q}\left[[G(p) + (p - C'(q))g(p)]\frac{(F(q + q_A(p)) - F(q))}{1 - F(q)}\right] \\ &- q'_A(p)\, G(p)\frac{\partial}{\partial q}\left[(p - C'(q))\frac{f(q + q_A(p))}{1 - F(q)}\right].\end{aligned}$$

Notice that the first term can be rewritten

$$\begin{aligned}&\frac{\partial}{\partial q}\left[[G(p) + (p - C'(q))g(p)]\frac{(F(q + q_A(p)) - F(q))}{1 - F(q)}\right] \\ &= \left(\frac{G(p)}{p - C'(q)} + g(p)\right)\frac{\partial}{\partial q}\left[(p - C'(q))\frac{(F(q + q_A(p)) - F(q))}{1 - F(q)}\right] \\ &+ \frac{G(p)}{p - C'(q)}C''(q)\frac{(F(q + q_A(p)) - F(q))}{1 - F(q)}.\end{aligned} \tag{73}$$

It is straightforward to show that condition (39) is satisfied for all prices $p > \underline{p} > C'(q)$ if it is satisfied for $p = \underline{p}$. Thus we can conclude from (72) and (73) that the first term in $\Lambda_q(q, p)$ is negative for $q < \bar{\varepsilon} - q_A(p)$. As $q'_A(p) \geq 0$ the second term in $\Lambda_q(q, p)$ is non-positive from (39) and so we have established that $\Lambda_q(q, p) < 0$ for $q < \bar{\varepsilon} - q_A(p)$. This implies that there is exactly one point at which $\Lambda(q, p) = 0$ in this range. Since Λ changes sign at q^* the same must be true for Z.

But there might be a range of points where $Z(q, p) = 0$ for $q > \bar{\varepsilon} - q_A(p)$. The relations $Z(q_A(p), p) = 0$ and $\Lambda_q(q, p) = C''(q) g(p)$ if $q > \bar{\varepsilon} - q_A(p)$ imply that if $C''(q) = 0$ for q in some range (q_B, q_C) and $q_A(p) \in [q_B, q_C]$ then $Z(q, p) = 0$ for $q \in (\max[q_B, \bar{\varepsilon} - q_A(p)], q_C)$. In cases where marginal costs are non-constant around $q_A(p)$ we let $q_B = q_C = q_A(p)$, and we can sum up the situation as follows:

$$\begin{aligned}&Z(q, p) > 0, \text{ for } q \in (0, q^*) \text{ and } q \in (q_C, q_m), \\ &Z(q, p) < 0, \text{ for } q \in (q^*, \max[q_B, \bar{\varepsilon} - q_A(p)]), \\ &Z(q, p) = 0, \text{ for } q = q^* \text{ and } q \in (\max[q_B, \bar{\varepsilon} - q_A(p)], q_C).\end{aligned}$$

Now we want to establish that this hockey stick mixture is a Nash equilibrium. Suppose that one player uses the mixture we have defined and consider the optimal choice of offer by the other player. As in Proposition 8 it is clear that the other player has no incentives to bid below $\underline{p}$. We know that Z has a single zero crossing in the interval $[0, \overline{\varepsilon} - q_A(p)]$ moving from positive to negative at some point $q^*(p) \in (0, \overline{\varepsilon} - q_A(p))$ for $p \in (\underline{p}, p_m)$. Now an optimal offer cannot follow the $q^*(p)$ curve since this would contradict the condition (6). So the only place where an optimal offer can have $0 < p'(q) < \infty$ is along the curve defined by $q_A(p)$ (or in a region surrounding $q_A(p)$ where $Z = 0$ in the case when marginal costs are constant). We can change the offer curves within the region where $Z = 0$ without changing their optimality. Hence we may suppose that an optimal response is adjusted to lie along the curve $q_A(p)$ as much as possible. Thus where $0 < p'(q) < \infty$ it follows the curve $q_A(p)$ and it does not begin a horizontal segment or end a vertical segment within this $Z = 0$ region. Apart from the segment along the curve defined by $q_A(p)$, an optimal offer can only consist of horizontal and vertical segments. Consider the final horizontal segment, say from q_X to q_Y at price p. Since there are no more horizontal segments we must have either $q_Y = q_m$ if $p \geq p_m$, or $q_Y \geq q_A(p)$ if $p \in (\underline{p}, p_m)$. Suppose that this horizontal segment does not start at zero, so $q_X > 0$.

Now, consider the case $p \in (\underline{p}, p_m)$ and suppose that $q_Y = q_A(p)$. Then $\int_{q_X}^{q_A(p)} Z(t,p)dt < 0$ since $\int_0^{q_A(p)} Z(q,p)dq = 0$ and because of the single-crossing property of Z we have either $Z(t,p) < 0$ throughout the interval $t \in (q_X, q_B)$ or $Z(t,p) > 0$ for $t \in (0, q_X)$. Hence this solution can be improved by moving the horizontal segment slightly downwards (as in Proposition 3) contradicting the claimed optimality. Thus we must have $q_Y > q_A(p)$. But we know that $Z(q,p) \geq 0$ below the $q_A(p)$ curve, so without lost profit we can increase the price of the units $q \in (q_A(p), q_Y)$ up to this curve. Thus it can never be a profitable deviation to have $q_Y > q_A(p)$.

The argument for the case when $p \geq p_m$ is easier. As in Proposition 8 we simply establish that $\int_{q_X}^{q_m} Z(t,p)dt < 0$ when $q_X > 0$ and hence use Proposition 3 to show the deviation is not optimal.

Thus we have established that there is exactly one horizontal segment starting at zero and finishing on the q_A curve or at q_m. So any optimal response is already represented in the mixture and this is enough to show that the solution is a Nash equilibrium. ■

Proposition 10

Proof. From the result of Proposition 9 it is enough to find a set of price cap values P such that there will be choices of $\underline{p}$ and p_m and a solution of the differential equations (32) and (33) with initial conditions (34), (35) and (36) satisfying the conditions (37) and (38). We will generate these solutions by showing that each of a range of possible starting points is matched to a final price cap value P. We do this by starting with one of the initial points given by (34), (35) and constructing a solution to the differential equations (32) and (33) from this point (which will automatically satisfy (36)).

We need to establish that the set of possible starting points is non-empty. First recall the defining relationship for $q_A\left(\underline{p}\right)$ in (70)

$$0 = \int_0^{q_A\left(\underline{p}\right)} \left[1 - F(q) - \frac{(\underline{p} - C'(q))}{\underline{p} - C'\left(q_A\left(\underline{p}\right)\right)}(F(q + q_A\left(\underline{p}\right)) - F(q))\right] dq. \tag{74}$$

We can differentiate both sides of (74) with respect to $\underline{p}$. The calculations are simplified since the integrand is $Z\left(q, \underline{p}\right)$ and $Z\left(q_A\left(\underline{p}\right), \underline{p}\right) = 0$.

$$\begin{aligned}
0 = &- \int_0^{q_A\left(\underline{p}\right)} \frac{(F(q + q_A\left(\underline{p}\right)) - F(q))}{\underline{p} - C'\left(q_A\left(\underline{p}\right)\right)} dq \\
&+ \int_0^{q_A\left(\underline{p}\right)} \left[\frac{(\underline{p} - C'(q))(F(q + q_A\left(\underline{p}\right)) - F(q))\left(1 - C''\left(q_A\left(\underline{p}\right)\right)\frac{\partial q_A(\underline{p})}{\partial \underline{p}}\right)}{\left[\underline{p} - C'\left(q_A\left(\underline{p}\right)\right)\right]^2}\right] dq \\
&- \int_0^{q_A\left(\underline{p}\right)} \frac{(\underline{p} - C'(q))f(q + q_A\left(\underline{p}\right))\frac{\partial q_A(\underline{p})}{\partial \underline{p}}}{\underline{p} - C'\left(q_A\left(\underline{p}\right)\right)} dq \\
= &\int_0^{q_A\left(\underline{p}\right)} \left[\frac{(C'\left(q_A\left(\underline{p}\right)\right) - C'(q))(F(q + q_A\left(\underline{p}\right)) - F(q))}{\left[\underline{p} - C'\left(q_A\left(\underline{p}\right)\right)\right]^2}\right] dq \\
&- \frac{\partial q_A\left(\underline{p}\right)}{\partial \underline{p}} \int_0^{q_A\left(\underline{p}\right)} (\underline{p} - C'(q)) \left[\frac{f(q + q_A\left(\underline{p}\right))}{\underline{p} - C'\left(q_A\left(\underline{p}\right)\right)} + \frac{(F(q + q_A\left(\underline{p}\right)) - F(q))C''\left(q_A\left(\underline{p}\right)\right)}{\left[\underline{p} - C'\left(q_A\left(\underline{p}\right)\right)\right]^2}\right] dq,
\end{aligned}$$

which implies that $\frac{\partial q_A(\underline{p})}{\partial \underline{p}} \geq 0$, because $C'\left(q_A\left(\underline{p}\right)\right) - C'(q) \geq 0$. So the highest value of $\underline{p}$ occurs when $q_A\left(\underline{p}\right)$ is q_m. With this value we get (41) from (34). Moreover, the lowest value of $\underline{p}$ occurs when $q_A\left(\underline{p}\right) = \overline{\varepsilon}/2$, which gives us (40).

We show that the capacity constraint q_m must bind at some price p_m where $G\left(p_m\right) < 1$. We know from (32) that

$$\frac{G'(p)}{1 - G(p)} = \frac{1}{(p - C'\left(q_A\left(p\right)\right))}.$$

Since $G(\underline{p}) = 0$, integration gives

$$-\ln(1 - G(p)) = \int_{\underline{p}}^{p} \frac{dp}{p - C'\left(q_A\left(p\right)\right)} = \alpha(p),$$

so

$$G(p) = 1 - e^{-\alpha(p)}.$$

The assumption that the inequality (39) holds for

$$(u, q) \in \{u \geq 0, 0 \leq q \leq q_m : u + q < \overline{\varepsilon}\}$$

shows that $\underline{p} - C'(q)$ is bounded away from zero as q approaches q_m. Hence there exists some δ such that

$$p - C'\left(q_A\left(p\right)\right) \geq \underline{p} - C'\left(q_m\right) \geq \delta > 0$$

for every $p \in \left[\underline{p}, p_m\right]$. Thus $\alpha(p)$ is finite for finite p, whence $G(p) < 1$, i.e. a mixture with hockey-stick offers never reaches $G(p) = 1$ at a finite price.

Next we will show that $q_A(p) > q_m$ as $p \to \infty$. We make the contradictory assumption that $q_A(p) \leq q_m$ in this limit. For $p > \underline{p}$ we then have from (33) that

$$\begin{aligned} q_A'(p) &= \frac{\int_0^{q_A(p)} 1 - F(q) - \left[G(p) + \frac{(p-C'(q))[1-G(p)]}{(p-C'(q_A(p)))}\right](F(q+q_A(p)) - F(q))dq}{\int_0^{q_A(p)}(p - C'(q))f(q+q_A(p))G(p)dq} \\ &> \frac{\int_0^{1-q_m}\left[1 - F(q+q_m)\right]dq}{p} + O\left(\frac{1}{p^2}\right) = \frac{k}{p} + O\left(\frac{1}{p^2}\right), \end{aligned}$$

where k is some positive constant. Thus

$$q_A(p) = q_A(\underline{p}) + \int_{\underline{p}}^{p} q_A'(p)\,dp > q_A(\underline{p}) + k\ln p + O\left(\frac{1}{p}\right).$$

Hence, $q_A(p) > q_m$ for sufficiently large p, which is a contradiction. Hence, the capacity constraint q_m must bind at some finite price p_m where $G(p_m) < 1$.

Finally we define the price cap P from (37). ■