Finding Supply Function Equilibria with Asymmetric Firms

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Abstract Firms compete in supply functions when they offer a schedule of prices and quantities into a market; for example, this occurs in many wholesale electricity markets. We study the equilibrium behaviour when firms differ, both with regard to their costs and their capacities. We characterize the types of equilibrium solution that can occur. If the demand can be low enough for it to be met economically with supply from just one firm, then the supply function equilibria are ordered in a natural way. Moreover, there can be at most one supply function equilibrium with the property that all but one of the firms are at their capacity limits when demand is at its highest level. We also propose a new numerical approach to find asymmetric supply function equilibria. We use a scheme which approximates general supply function equilibria using piecewise linear supply functions and a discretization of the demand distribution. We show that this approach has good theoretical convergence behaviour. Finally we present numerical results from an implementation of this method using GAMS, to demonstrate that the approach is effective in practice. Our method is superior to approaches based on numerical methods for ordinary differential equations, or on iteration procedures in the function space of admissible supply functions.

1 Introduction

In many situations firms compete by offering a schedule of quantities and prices into a market rather than using a single strategic variable of quantity or price. In this paper we analyse supply function equilibria which occur in this environment. The first complete analysis of such equilibria was carried out by Klemperer and Meyer [25] who recognized that supply function equilibria can exist when there is demand uncertainty, so that firms specify their supply functions before they know the demand. Supply function equilibria (SFE) have been used extensively in the analysis of wholesale electricity markets as well as in other environments ([27, 1]). However there are two significant problems with this approach. First in general there may be a whole family of different equilibria, giving rise to an equilibrium selection problem. Second it has often proved hard to produce an equilibrium solutions, either analytically or numerically, even when they are known to exist. We make a contribution to both these problems here. We will show that though in general there may be a whole family of alternative equilibrium solutions, these will be ordered and in many cases there will be a unique solution to the SFE problem with capacity constraints. Moreover, we demonstrate a new robust numerical technique for solving the SFE problem for an oligopoly, and in so doing we hope to make this approach more useful in practice. Our methods go beyond prior work in this area since we need make very few assumptions on the characteristics of the demand and cost functions.

In our model each firm specifies their supply quantity as a function of the price. A supply function is always required to be non-decreasing, so that the higher the market price, the more commodity a firm is willing to sell. After the simultaneous announcement of supply functions a stochastic demand occurs and the market clears at a single spot price which is then the price paid to each firm for the quantity they supply. In actual markets there are often other restrictions: for example there may be a restriction on the range of prices for which the supply function is specified (e.g. a price cap).

Wholesale electricity markets have been the main focus for research on supply function equilibria, with Green and Newbery [19] being the first to apply this approach to firm offers in the England and Wales market. Despite the problems of dealing with equilibria in supply functions, most authors regard this as the most appropriate framework for an equilibrium analysis. Cournot models have been used by a number of authors, (e.g. [8, 12]), but this has been primarily because of their flexibility and tractability. The case for preferring a supply function approach rests on what happens in actual electricity markets in which firms bid schedules of prices and quantities, rather than quantities alone. For more discussion on some of the modelling issues see [24].

The existence and computation of supply function equilibria in general cases has attracted a great deal of research attention. Klemperer and Meyer discussed a case with identical convex cost functions for each firm and a concave demand function. They showed the existence of a family of SFE in this case. Electricity demand is almost completely insensitive to spot market price in the short term and [28] and [3] addressed the case of SFE with stochastic demand which is independent of price, where each firm has an identical cost function. Both papers provide closed-form formulae for strictly increasing supply function equilibria when there are no capacity constraints or other operating constraints. Holmberg [20] has shown that, when demand is inelastic and capacity constraints bind with a positive probability, there can only be one SFE. Anderson and Xu [6] extended the work of Klemperer and Meyer to deal with hedging contracts, capacity constraints and price caps, which are all commonplace in electricity markets. However all of this work is restricted to symmetric solutions in which the firms are identical and each offer the same supply function. Challenges remain for the supply function equilibrium models when the firms are asymmetrical (having different cost functions) and have operating constraints such as capacity limits.

The simplest form of asymmetric problem arises when firms have identical cost functions but have different capacities. Problems of this form, in which marginal cost is constant, are considered by Genc and Reynolds [16] and by Holmberg [21]. The only case in which an asymmetric supply function equilibrium can be easily found is one in which the supply functions are linear (strictly affine). This form of SFE can be found whenever the cost functions are quadratic (linear marginal cost) and the demand is linear. However there may also be non-linear solutions under these conditions, which will be difficult to find analytically. Green(1996) [17] and Green (1999) [18] have shown how affine SFE can be used in analyzing different aspects of the England and Wales market when it operated under a pool system.

It is natural to consider piecewise linear supply function equilibria in which different price ranges have different linear solutions. This approach can be used to approximate more general functional forms. Unfortunately such solutions cannot always be found, since the supply functions determined in this way may decrease in moving from one price interval to the next. Moreover capacity limits are difficult to incorporate in this framework. Nevertheless the approach has been used successfully by Baldick et al [9] and forms the basis of the numerical approach of Rudkevich [30]

When we come to consider algorithms for the numerical estimation of supply function equilibria there are a number of options. As we will see later the equilibrium conditions produce coupled first order differential equations, and we can use the standard approaches for the solution of such equations. However there are some difficulties. The primary problem is that a solution can be derived from initial conditions, but infeasibilities in the solution can easily occur, either through a supply function starting to decrease or because the quantity exceeds a capacity limit. Thus we may need to search among potential initial conditions to locate a feasible solution. Another difficulty arises because the equations are poorly conditioned, in particular an ODE solver may encounter difficulties when the system becomes close to singular, which happens at the lowest point of the supply function where the quantity is close to zero and we expect the price to approach the marginal cost of supply. Finally when using the ODE approach we need to be careful that the solution of the ODE system is actually an equilibrium point rather than being just locally stationary. That is we need to check that the supply function offered by the i'th firm is optimal given the offers from the other firms.

There are a number of alternative approaches that have been suggested. For asymmetric firms, Baldick and Hogan [10] develop an iterative scheme based on piecewise affine solutions. At each stage the existing set of supply functions are moved towards the best piecewise affine response to the other firms' offers. Day and Bunn [13] use a similar approach to computationally determine equilibrium solutions, but they use a step length of 1 in their iterative procedure which may be responsible for the cycling that they observe.

Baldick and Hogan [11] investigate the use of polynomial approximations. They show that there are significant difficulties with this approach, particularly if it is required that the equilibria are stable - by which they mean that a small perturbation by one player will still lead to convergence if each player repeatedly responds according to its best response to the current offers of other players. As has been shown by Rudkevich [29], when players are restricted to affine supply functions and use a Cournot adjustment process there is a rapid convergence to the equilibrium solution.

Rudkevich [30] shows how a piecewise linear optimal response can be developed allowing for both horizontal and vertical segments in a grid framework. He also discusses the difficulties of finding an SFE and proposes a method in which each player observes prices and from these develops a piecewise linear approximation to the supply function of its rivals. This approach has been found effective in practice. Holmberg [22] gives a numerical procedure for finding a solution. His approach uses numerical integration to solve the system of ODEs and searches for feasible solutions, by varying the values for the prices at which the capacity constraints are reached for each firm.

In this paper we begin in Section 2 by discussing some fundamentals of the equilibrium model and characterizing the equilibrium solutions that arise. We show that discontinuities in the supply functions can only occur at a price at which one of the other firms begins to supply. We also show that if the lowest demand level has only one firm supplying, then there is an ordering between equilibrium solutions, and at most one equilibrium solution where all but one of the firms reach their capacity for the highest level of demand. Section 3 formulates the approximation scheme, in which we take a discretization over demand shocks, rather than over price or quantity. We establish a convergence result for this discretization. Section 4 gives some implementation details for the approximation scheme and presents numerical examples.

2 Characterizing equilibrium solutions

Our basic model has n firms. Each firm i has a maximal capacity denoted by $\bar{q}_i$. We let $c_i(x)$ be the cost for firm i of producing an amount x and we assume that $c_i(x)$ is convex and differentiable. Furthermore, we assume that the marginal cost is always positive, so $c_i'(0) \geq 0$, for $i = 1, \ldots, n$. It is convenient to assume that each firm has a different initial marginal cost of supply, and by reordering the firms if necessary we will suppose that $c_1'(0) < c_2'(0) < ... < c_n'(0)$.

We suppose that demand is given by a function of the form $D(p, \varepsilon) = D(p) + \varepsilon$. The price sensitive element of the demand, $D(p)$, is strictly decreasing, smooth and concave. The demand shock ε, which is the stochastic element in the demand, has some distribution with cumulative distribution function $F(\varepsilon)$. Moreover we assume that this distribution has positive density throughout the interval on which it is defined: specifically we suppose that there is a density function f with $f(x) > 0$ for $x \in (\varepsilon_{\min}, \varepsilon_{\max})$, where $\varepsilon_{\min}$ is the lowest demand shock that may occur and $\varepsilon_{\max}$ is the largest.

We will assume that there is a given price cap $\bar{p}$, but if there is no explicit price cap we may take $\bar{p}$ to be the price where $D(\bar{p}) + \varepsilon_{max} = 0$. At this price the maximum demand is zero and so the market can only clear at this price in the trivial case that all the supply functions are identically zero. We will suppose that prices are always greater than zero (even though in some electricity markets negative prices can occur).

The supply function for firm i is a function $s_i : [0, \bar{p}] \to [0, \bar{q}_i]$. We will assume that supply functions are non-negative, non-decreasing and piecewise smooth (specifically we assume each $s_i(\cdot)$ has both left and right derivatives for all $p \in (0, \bar{p})$).

In fact, once we consider supply function equilibria, some forms of 'bad' behaviour will be ruled out. For example we will show (Theorem 2) that the supply functions can have only finitely many jumps. Thus it may be possible to relax the requirement of piecewise smooth supply functions and instead use the conditions arising from the equilibrium to establish good behaviour. For example, for an initial value problem, $x'(t) = f(t, x(t))$, $x(t_0) = x_0$, if $x(t)$ is a solution in the distribution sense, $x(t)$ is square-integrable, and $f(t, x)$ is locally Lipschitz in x, then $x(t)$ is at least continuously differentiable (see [26]). However we choose the simpler approach of assuming piecewise smoothness, which is not a restriction in practice. Notice that our assumption is weaker than Klemperer and Meyer [25] who consider only supply functions that are twice continuously differentiable.

Throughout this paper we will use the notation $g(x^-)$ and $g(x^+)$ for $\lim_{\delta\downarrow 0} g(x-\delta)$ and $\lim_{\delta\downarrow 0} g(x+\delta)$ when these exist; both quantities are well defined when g is a supply function since this is monotonic. In the same way we will write $g'(p^-) = \lim_{\delta\downarrow 0} g'(p-\delta)$ and $g'(p^+) = \lim_{\delta\downarrow 0} g'(p+\delta)$ where these exist.

After the supply functions have been chosen, the stochastic demand is realized. Thus a particular value ε^* of the demand shock ε becomes known and the market clears at a price p^* such that $D(p^*) + \varepsilon^* = \sum_j s_j(p^*)$ and each firm i supplies an amount $s_i(p^*)$. Thus firm i earns a profit of $p^* s_i(p^*) - c_i(s_i(p^*))$.

We need to be careful however if the supply functions are discontinuous (which corresponds to a range of quantities of supply being offered all at the same price). In this case the market clears at a price p with $\sum_j s_j(p^-) \le D(p) + \varepsilon^* \le \sum_j s_j(p^+)$. If just one supply function, say s_i, is discontinuous at p then the other firms meet their part of the demand $s_j(p)$, $j \neq i$, and firm i meets the residual demand $D(p) + \varepsilon^* - \sum_{j\neq i} s_j(p)$. When more than one firm has a discontinuous supply function at the same price then there is no longer a single allocation of demand to firms. Later we will show that in the equilibrium of concern to us this cannot happen, but in case it does happen we need to specify a rule for sharing demand between firms offering at the same price (see [7]). In our case we assume that there is a preference order between the firms with demand at any price being met first from firm 1, then firm 2, etc.

To find the best response function for firm i , given the supply functions for the other firms s_j, $j \neq i$, we can ask what choice of price will give the highest profit to firm i? If the market clears at a price p then the amount supplied by i is given by $s_i(p) = D(p, \varepsilon) - \sum_{j\neq i} s_j(p)$ and this gives a profit of

$$\pi_i(p, \varepsilon) = p[D(p) + \varepsilon - \sum_{j\neq i} s_j(p)] - c_i(D(p) + \varepsilon - \sum_{j\neq i} s_j(p)).$$

If the supply functions $\{s_i(p)\}_{i=1}^n$ in equilibrium are smooth, and there are no other constraints, then the optimal choice of price for firm i, is achieved when

$$\begin{aligned} \partial\pi_i(p,\varepsilon)/\partial p &= [p - c_i'(D(p)+\varepsilon - \sum_{j\neq i} s_j(p))][D'(p) - \sum_{j\neq i} s_j'(p)] + [D(p)+\varepsilon - \sum_{j\neq i} s_j(p)] \qquad (1) \\ &= [p - c_i'(s_i(p))][D'(p) - \sum_{j\neq i} s_j'(p)] + s_i(p) = 0. \end{aligned}$$

Thus we obtain the following first order optimality conditions (see also [25, p.1251])

$$s_i(p) = [p - c_i'(s_i(p))]\,[\sum_{j\neq i} s_j'(p) - D'(p)],\ i = 1,\ldots,n. \qquad (2)$$

If the s_j are continuous at p, but not smooth, then we need to replace (2) with

$$[p - c_i'(s_i(p))]\,[\sum_{j\neq i} s_j'(p^-) - D'(p)] \le s_i(p) \le [p - c_i'(s_i(p))]\,[\sum_{j\neq i} s_j'(p^+) - D'(p)].$$

If $s_i(p)$ satisfies these inequalities for a given p, then (subject to a global optimality condition) firm i earns the maximum profit possible when the demand shock is the ε which corresponds to this market clearing price. If the condition holds for all p, then the the profit for firm i is optimal whatever the demand shock, and consequently whatever the distribution $F(\cdot)$.

The optimal choice for $s_i(p)$ can be constrained either by the allowable range of prices or by a capacity constraint, so the first order optimality conditions hold only when, at the solution, $0 < p < \bar{p}$ and $0 < s_i(p) < \bar{q}_i$. An important case arises when all but one of the supply functions are either zero or at their capacity limits in a range $(p-\delta, p+\delta)$, then if s_i is the supply function that is not constrained, its value is given by

$$s_i(p) = -\left(p - c_i'(s_i(p))\right) D'(p). \qquad (3)$$

In effect firm i is acting as a monopoly over this range of prices.

There is a further complication which arises from the restriction that the supply functions are non-decreasing. When the best response function has negative slope we are no longer able to solve the problem locally for each p. Instead we will need to use a supply function which is constant over an interval. In this case the choice of supply function is no longer optimal for any ε, and instead we must weigh up losses and benefits for different demand shock realizations. The consequence is that we will seek a supply function that is optimal for the expected profit: the choice of supply function is no longer independent of the demand shock distribution, F, and the first order optimality conditions given above will not hold. Solving the best response problems which then arise is more complex, and is fully described in Anderson and Philpott [4] and Anderson and Xu [5].

We say that a supply function $s_i(p)$ is *strongly optimal*, as a response to a set of supply functions $s_j(p)$, $j \neq i$, if given any demand realization, the amount supplied by firm i under this demand realization gives firm i the highest achievable profit subject to constraints on price and capacity. We use the term *strong equilibrium* to refer to an equilibrium in which each supply function is piecewise smooth and strongly optimal given the other supply functions. In this paper we will only consider strong equilibria.

The number of possible supply function equilibria will be constrained according to the range of the supply shocks: in general there are fewer equilibrium solutions possible when the demand has a wide range of possible values. For a specified supply function equilibrium we can find the minimum and maximum values of price which occur - we call these $p_{\min}$ and $p_{\max}$ respectively (corresponding to demand shocks $\varepsilon_{\min}$ and $\varepsilon_{\max}$). The values of the supply functions outside the range $(p_{\min}, p_{\max})$ are irrelevant and we cannot say anything about the characteristics of the supply functions outside this range.

We will proceed in stages in our characterization of the SFE. We begin by showing that in a strong equilibrium no supply is offered at prices below marginal cost - which translates into the following statement about supply functions.

Lemma 1 *In a strong equilibrium,*

$$\begin{aligned} s_i(p) &= 0 \text{ if and only if } p_{\min} < p \leq c_i'(0); \\ c_i'(s_i(p)) &< p \text{ for } c_i'(0) < p \leq p_{\max}. \end{aligned}$$

Proof Choose ε such that the market clears at price p with $s_i(p) > 0$ and $p \leq c_i'(0)$. Then consider increasing the choice of p for this ε fixed. We have

$$\pi_i'(p^+) = [p - c_i'(s_i(p))][D'(p) - \sum_{j \neq i} s_j'(p^+)] + s_i(p) > 0$$

since $p \leq c_i'(0) \leq c_i'(s_i(p))$, using the convexity of c, and $D'(p) < 0$. Hence increasing p increases profits, which contradicts the strong optimality of s_i. The conclusion is the same even if there is a discontinuity in some s_j at p. On the other hand if there is some demand shock ε such that the market clears at price p with $s_i(p) = 0$ and $p > c_i'(0)$, then $\pi_i'(p^-) < 0$ showing that decreasing p will be advantageous; again giving a contradiction.

The second part follows similarly. If $c_i'(0) < p \leq c_i'(s_i(p))$, then $s_i(p) > 0$ and $\pi_i'(p^+) > 0$ contradicting the optimality of this choice of p. ■

The next step is to show that the solutions are continuous (with some specific exceptions). The argument is roughly as follows. Suppose that $s_i(p_A)$ jumps from q_a to q_b. The consequence is that there is range of demands at which the market will clear at p_A. Hence there is a non-zero probability of a demand occurring in this range and there is a jump in the profit for some other firm by offering at a price just below p_A in comparison to just above p_A. This is enough to show that this is not an equilibrium (this type of situation is described in more detail in [7]). The result below makes this more precise.

Theorem 2 *In a strong equilibrium, if* $s_i(p_A^-) < s_i(p_A^+)$ *then (a) all firms* k, $k \neq i$, *for which* $s_k(p_A) > 0$ *are at their capacity limit, i.e.* $s_k(p_A) = \bar{q}_k$ *and (b) there is some firm* $j \neq i$ *with* $p_A = c_j'(0)$

Proof We begin by showing that there cannot be a strong equilibrium with two supply functions discontinuous at the same price. Suppose otherwise and let $\{s_j(p)\}_{j=1}^n$ be a strong equilibrium in which both s_i and s_k are discontinuous at p_A. Now consider a demand shock ε with $\sum s_j(p_A^-) - D(p_A) < \varepsilon < \sum s_j(p_A^+) - D(p_A)$ so that there is an ambiguity in the amount supplied by i and k. Then from Lemma 1, both $c_k'(s_k(p_A)) < p_A$ and $c_i'(s_i(p_A)) < p_A$ and so both firms would prefer to have the largest possible supply. The sharing rule gives preference to one of the firms, say i. Then firm k will improve its profits by choosing a slightly lower price than p_A, which contradicts the fact that this is a strong equilibrium.

Thus to prove part (a) we suppose that we have $s_i(p_A^-) < s_i(p_A^+)$ and $0 < s_k(p_A) < \bar{q}_k$, with s_k continuous at p_A. Thus, from Lemma 1, we have $c_k'(s_k(p_A)) < p_A$. Since supply function s_i has a jump at p_A, the price will be p_A for a range of values of demand shock ε, specifically for ε between $\sum s_j(p_A^-) - D(p_A)$ and $\sum s_j(p_A^+) - D(p_A)$. For ε in this range we let

$$\Delta(\varepsilon) = D(p_A) + \varepsilon - \sum_j s_j(p_A^-) > 0.$$

We select a demand shock ε_A in this range with $\Delta(\varepsilon_A)$ small enough so that $\Delta(\varepsilon_A) < \bar{q}_k - s_k(p_A)$ and $c_k'(s_k(p_A) + \Delta(\varepsilon_A)) < p_A$ (using the continuity of c_k').

We claim that p_A is not an optimal price for firm k for demand shock ε_A given the other firms' supply functions. That is, $\{s_j(p)\}_{j=1}^n$ does not form a strong supply function equilibrium. In fact, if firm k chooses a price $p_A - \delta$, instead of p_A, with this demand shock, then its supply changes from $s_k(p_A)$ to

$$D(p_A - \delta) + \varepsilon_A - \sum_{j \neq k} s_j(p_A - \delta).$$

As $\delta \to 0$ this quantity approaches $s_k(p_A) + \Delta(\varepsilon_A)$ (which is less than $\bar{q}_k$ by construction). Hence the change in profit for firm k in moving from p_A to $p_A - \delta$ is

$$\begin{aligned}
&(p_A - \delta)[D(p_A - \delta) + \varepsilon_A - \textstyle\sum_{j \neq k} s_j(p_A - \delta)] - c_k\big(D(p_A - \delta) + \varepsilon_A - \textstyle\sum_{j \neq k} s_j(p_A - \delta)\big) \\
&\quad - p_A s_k(p_A) + c_k(s_k(p_A)) \\
&= p_A \Delta(\varepsilon_A) + [c_k(s_k(p_A)) - c_k(s_k(p_A) + \Delta(\varepsilon_A))] + O(\delta) \\
&\geq [p_A - c_k'(s_k(p_A) + \Delta(\varepsilon_A))]\Delta(\varepsilon_A) + O(\delta)
\end{aligned}$$

since c_k is convex. Therefore, this change in profit is positive for δ chosen small enough. This contradicts the fact that this is a strong equilibrium. Therefore, $s_i(p)$ cannot have a jump at p_A.

To prove part (b) we show first that there must be at least one firm $j \neq i$ with $0 < s_j(p)$ as p approaches p_A from above or $s_j(p) < \bar{q}_j$ as p approaches p_A from below (i.e. s_j increases from zero to the right of p_A or s_j hits its capacity bound at p_A). If not then there is some interval $(p_A - \delta, p_A + \delta)$ over which the other firms offer a constant quantity (either 0 or $\bar{q}_j$). So over this interval firm i chooses a price p to maximize $\pi_i(p, \varepsilon) = p[D(p) + \varepsilon - \Delta] - c_i(D(p) + \varepsilon - \Delta)$ where $\Delta = \sum_{j \neq i} s_j(p)$. This expression has first and second derivatives with respect to p given by

$$\begin{aligned}
\pi_i' &= pD'(p) + D(p) + \varepsilon - \Delta - c_i'(D(p) + \varepsilon - \Delta)D'(p) \\
\pi_i'' &= [p - c_i'(s_i(p))]D''(p) + 2D'(p) - c_i''(s_i(p))(D'(p))^2.
\end{aligned}$$

As $D' < 0$, $D'' \leq 0$, $c_i'' \geq 0$ it is easy to see that $\pi_i'' < 0$ and so π_i is concave and will have a single optimizing choice of p. Since the functions D and c_i are smooth this choice of p will depend continuously on the demand shock ε, which shows that the supply function will also be continuous in this interval, giving a contradiction.

So now we suppose that there is no firm $j \neq i$ with $0 < s_j(p) < \bar{q}_j$ as p approaches p_A from above, and therefore there is some set of firms (excluding i), with $s_j(p) < \bar{q}_j$ as p approaches p_A from below and $s_j(p_A) = \bar{q}_j$. We call this set $I(p_A)$.

Using (2), we can write the expressions for the values of s_i either side of the jump as follows

$$s_i(p_A^-) = -[p_A - c_i'(s_i(p_A^-))]\,[D'(p_A) - \sum_{j \in I(p_A)} s_j'(p_A^-)] \tag{4}$$

$$s_i(p_A^+) = -[p_A - c_i'(s_i(p_A^+))]\, D'(p_A). \tag{5}$$

Since c_i is convex, $c_i'(s_i(p_A^-) \leq c_i'(s_i(p_A^+))$, and moreover $s_j'(p_A^-) \geq 0$. Using $p_A - c_i'(s_i(p_A^+)) > 0$, we can deduce

$$-[p_A - c_i'(s_i(p_A^-))]\,[D'(p_A) - \sum_{j\in I(p_A)} s_j'(p_A^-)] \geq -[p_A - c_i'(s_i(p_A^+))]\, D'(p_A).$$

But this contradicts our assumption that s_i jumps up at p_A.

Hence we have shown that there is a firm $j \neq i$ with $0 < s_j(p)$ as p approaches p_A from above, and $s_j(p_A^+) = 0$ by (a). Using Lemma 1 establishes that $p_A = c_j'(0)$ as required. ■

One implication of Theorem 2 is that the supply function of each firm in a strong supply function equilibrium is continuous if all the firms have identical cost functions (see [25], and for identical cost functions, but different capacity limits, [21]).

Given a strong supply function $\{s_i(p)\}_{i=1}^n$, by Theorem 2, there are at most only finitely many price points at which some $s_i(\cdot)$ is not continuous.

We define $H(p) = \{i : 0 < s_i(p) < \bar{q}_i\}$, the set of the unconstrained firms at p. Since an individual firm can only enter $H(p)$ at one price and then leave it at a higher price, if $H(p_1) = H(p_2)$ then H must be constant over the interval $[p_1, p_2]$. We define $H_-(p_0) = \lim_{p_0>p\to p_0} H(p)$ and $H_+(p_0) = \lim_{p_0<p\to p_0} H(p)$. It is easy to see that $H_+(p)$, $H_-(p)$ are well-defined (provided $\{s_i(p)\}_{i=1}^n$ is well-defined). Moreover, $H(p)$ is a subset of a set with finitely many elements, and so there are a finite set of points at which the set function H changes. Notice, however, that we could have $H_+(p)$, $H_-(p)$ and $H(p)$ all different if some supply function s_i hits its capacity limit at the same price, $c_j'(0)$, at which s_j leaves zero. Using Lemma 1 we can see that $i \notin H(c_i'(0))$, which is the price at which i enters $H(p)$. Hence $H(p) \subset H_-(p)$.

At a smooth point p of a strong supply function equilibrium $\{s_i(\cdot)\}$, we can rewrite (2) as

$$\sum_{j\in H(p)} s_j'(p) - s_i'(p) = \frac{s_i(p)}{p - c_i'(s_i(p))} + D'(p) \text{ for } i \in H(p). \tag{6}$$

This is well defined except when $p - c_i'(s_i(p)) = 0$. From Lemma 1 we know that this only occurs when $p = c_i'(0)$.

Summing this over $i \in H(p)$ we have

$$(m-1)\sum_{j\in H(p)} s_j'(p) = \sum_{i\in H(p)} \frac{s_i(p)}{p - c_i'(s_i(p))} + mD'(p)$$

where $m = |H(p)|$, the number of firms in $H(p)$. Thus when $m > 1$, we have

$$s_i'(p) = \frac{1}{(m-1)}\left(\sum_{j\in H(p)} \frac{s_j(p)}{p - c_j'(s_j(p))} + D'(p)\right) - \frac{s_i(p)}{p - c_i'(s_i(p))} \text{ for } i \in H(p). \tag{7}$$

It will be convenient to write $h_i(p,x)$ for the expression $x/(p-c_i'(x))$, so we can rewrite (7) as

$$s_i'(p) = \frac{1}{(m-1)}\left(\sum_{j\in H(p)} h_j(p,s_j(p)) + D'(p)\right) - h_i(p,s_i(p)) \text{ for } i\in H(p). \tag{8}$$

Lemma 3 *In a strong equilibrium, each supply function s_i is smooth with all derivatives except at prices $p\in Q$, where Q is the set of prices at which a firm either starts to supply or reaches its capacity limit, i.e.*

$$Q = \{p_X : s_j(p_X) = \bar{q}_j \text{ and } s_j(p) < \bar{q}_j \text{ for } p < p_X\} \cup \{c_1'(0),...,c_n'(0)\}.$$

Proof By assumption each s_i has continuous derivatives except at a finite set of points. Suppose that one of these points p_z is not in the set Q. Then from Theorem 2 each supply function is continuous at p_z. Moreover $H_+(p_z) = H_-(p_z)$. As the supply functions are smooth at prices p which approach p_z from either the left or right, the equation (8) applies as p approaches p_z. Since the s_j are all continuous and the set H does not change, this implies that $s_i'(p_z^-) = s_i'(p_z^+)$, which is a contradiction. Thus the only points at which the derivative of an s_i is discontinuous are points in Q. Moreover, given equation (8) we can take derivatives to obtain $s_i''(p)$ in terms of $s_j'(p)$ and $s_j(p)$, $j=1,2...n$. The process can be repeated to obtain all the derivatives of s_i. ■

Now suppose that p_A and p_B are adjacent points in Q and we know $s_i(p_A^+)$, $i=1,2,...,n$. Will this be enough to determine the solution throughout the interval (p_A,p_B)? To answer this question we have to know whether the solutions are well behaved near p_A. We shall address this question in a series of lemmas. The first shows that where an ordering between two different equilibria exists at some price, that ordering persists (at least while the supply functions involved are continuous.)

Lemma 4 *Let $\{s_i(p)\}_{i=1}^n$ and $\{\tilde{s}_i(p)\}_{i=1}^n$ be two supply function equilibria. Suppose that there is some price p_0 with $s_i(p_0)\geq \tilde{s}_i(p_0)$ for all i with strict inequality for $i\in \widetilde{H}(p_0)\neq\emptyset$. If both $\{s_i(p)\}_{i=1}^n$ and $\{\tilde{s}_i(p)\}_{i=1}^n$ are continuous in the range (p_0,p_1), then, for $p\in(p_0,p_1)$, $s_i(p)\geq\tilde{s}_i(p)$ for all i with strict inequality for $i\in\widetilde{H}_-(p)$.*

Proof Note that the ordering $s_i(p)\geq\tilde{s}_i(p)$ for all i and $p_0\leq p\leq p_1$ implies that $H(p)\subseteq \tilde{H}(p)$ for $p_0<p<p_1$.

Suppose that p_X is the supremum of values such that the result holds for $p_0\leq p\leq p_X$ and $p_X<p_1$. Then there are two possibilities: either (a) $s_i(p)<\tilde{s}_i(p)$ for some i for p just larger than p_X and $s_j(p_X)>\tilde{s}_j(p_X)$ for all $j\in\widetilde{H}(p_X)$, or (b) $s_i(p_X)=\tilde{s}_i(p_X)$, for some $i\in\widetilde{H}_-(p)$.

Suppose first that (a) holds but not (b). In this case we can deduce that $s_i(p_X) = \tilde{s}_i(p_X)$ and $s_i'(p_X^+) \leq \tilde{s}_i'(p_X^+)$. If both are at their capacity limit $\bar{q}_i$ then both s_i and $\tilde{s}_i$ stay at this level contradicting the supposition. Since we assume (b) does not hold, they must both be zero, and hence $p_X = c_i'(0)$.

We consider separately two situations. First, assume that $\widetilde{H}_+(p_X) = H_+(p_X)$. Then $m = |H_+(p_X)| = |\widetilde{H}_+(p_X)| > 1$. Otherwise firm i is a monopoly for $p > p_X$ near p_X, its supply function is uniquely determined by (3) and (a) cannot occur. So by taking limits in (8) we have

$$s_i'(p_X^+) - \tilde{s}_i'(p_X^+) = \frac{1}{(m-1)} \sum_{j \in H_+(p_X), j \neq i} \left(h_j(p_X, s_j(p_X)) - h_j(p_X, \tilde{s}_j(p_X)) \right).$$

But $h_j(p, x)$ is increasing in x and so the right hand side is positive which contradicts our assumption.

So now consider the case that $\widetilde{H}_+(p_X) \neq H_+(p_X)$, which implies that there is some firm l with $s_l(p_X) = \bar{q}_l > \tilde{s}_l(p_X)$. Considering the limits of (6) for $\{s_i(\cdot)\}, \{\tilde{s}_i(\cdot)\}$ as $p \to p_X$ and noting that $s_i'(p_X^+) \leq \tilde{s}_i'(p_X^+)$, we have

$$\sum_{j \in H_+(p_X), j \neq i} s_j'(p_X^+) = \frac{s_i'(p_X^+)}{1 - c_i''(0) s_i'(p_X^+)} + D'(p_X) \leq \frac{\tilde{s}_i'(p_X^+)}{1 - c_i''(0) \tilde{s}_i'(p_X^+)} + D'(p_X) = \sum_{j \in \widetilde{H}_+(p_X), j \neq i} \tilde{s}_j'(p_X^+)$$

by applying the l'Hopital rule to $h_i(p, s_i(p))$ and $h_i(p, \tilde{s}_i(p))$. Moreover for each $k \in H_+(p_X)$, $k \neq i$, we have $s_k(p_X^+) \geq \tilde{s}_k(p_X^+)$, and therefore the following inequality by considering limits in (6):

$$\sum_{j \in H_+(p_X), j \neq k} s_j'(p_X^+) \geq \sum_{j \in \widetilde{H}_+(p_X), j \neq k} \tilde{s}_j'(p_X^+).$$

Combining the above two inequalities, we have $s_i'(p_X^+) - s_k'(p_X^+) \geq \tilde{s}_i'(p_X^+) - \tilde{s}_k'(p_X^+)$ for any $k \in H_+(p_X)$, $k \neq i$. Since we are supposing $s_i'(p_X^+) \leq \tilde{s}_i'(p_X^+)$ this implies $s_k'(p_X^+) \leq \tilde{s}_k'(p_X^+)$ for each $k \in H_+(p_X)$. Now observe that the fact that $s_l(p_X) = \bar{q}_l$ implies, from the optimality conditions,

$$\sum_{j \in H_+(p_X)} s_j'(p_X^+) \geq h_l(p_X, \bar{q}_l) + D'(p_X),$$

and the right hand side of this inequality is strictly greater than

$$h_l(p_X, \tilde{s}_l(p_X)) + D'(p_X) = \sum_{j \in \widetilde{H}_+(p_X), j \neq l} \tilde{s}_j'(p_X^+).$$

But since $s_j'(p_X^+) \leq \tilde{s}_j'(p_X^+)$ for each $j \in H_+(p_X)$ and each such j also appears in $\widetilde{H}_+(p_X) \backslash \{l\}$, we have a contradiction.

Thus we are reduced to case (b) $s_i(p_X) = \tilde{s}_i(p_X)$, for some $i \in \widetilde{H}_-(p_X)$. (This includes the case where $s_i(p)$ and $\tilde{s}_i(p)$ both approach $\bar{q}_i$ as $p \to p_X$ from below.) From this we can deduce that $s_i'(p_X^-) \leq \tilde{s}_i'(p_X^-)$. We begin by dealing with the case that $\widetilde{H}_-(p_X) = H_-(p_X)$. We suppose first that for some $k \in H_-(p_X)$, $s_k(p_X) > \tilde{s}_k(p_X)$, so that not all the supply functions become equal at the same time. From (8) (here $m = |H_-(p_X)| = |\widetilde{H}_-(p_X)| \geq 2$ since the sets include i, k)

$$s_i'(p_X^-) - \tilde{s}_i'(p_X^-) = \frac{1}{(m-1)} \sum_{j \in H_-(p_X), j \neq i} \left(h_j(p_X, s_j(p_X)) - h_j(p_X, \tilde{s}_j(p_X)) \right).$$

Then each term in the right hand side is non-negative with the term corresponding to firm k being strictly positive, and so $s_i'(p_X^-) > \tilde{s}_i'(p_X^-)$, giving a contradiction.

Hence all the supply functions become equal at this point, i.e. $s_j(p_X) = \tilde{s}_j(p_X)$ for each j. Now by our assumption that $\widetilde{H}_-(p) = H_-(p)$, we have $\widetilde{H}(p) = H(p)$ for $p \in (p_X - \delta, p_X)$ for some $\delta > 0$. Hence by the standard uniqueness result for the ODE system of (2), s and $\tilde{s}$ are the same for $p \in (p_X - \delta, p_X)$, which gives a contradiction.

Hence we are left with the case that $\widetilde{H}_-(p_X) \neq H_-(p_X)$. Then there exist $\delta > 0$ and l such that $s_l(p) = \bar{q}_l > \tilde{s}_l(p) > 0$ for $p \in (p_X - \delta, p_X)$. Furthermore, we choose $\delta > 0$ small enough so that $H(p)$ and $\widetilde{H}(p)$ are constant in $(p_X - \delta, p_X)$. Using the same argument as in the proof of (a) (though without needing to appeal to l'Hopital's rule) we can deduce from $s_i'(p_X^-) \leq \tilde{s}_i'(p_X^-)$ that $s_k'(p_X^-) \leq \tilde{s}_k'(p_X^-)$ for each $k \in H_-(p_X)$.

Now consider three possibilities:

(i) Suppose that there is some firm l with $s_l(p_X) = \bar{q}_l > \tilde{s}_l(p_X)$. In this case the proof is the same as the last part in the proof of (a). The optimality conditions for $s_l(p_X^-)$ gives

$$\sum_{j \in H_-(p_X)} s_j'(p_X^-) \geq h_l(p_X, \bar{q}_l) + D'(p_X)$$

and the right hand side of this inequality is greater than

$$h_l(p_X, \tilde{s}_l(p_X)) + D'(p_X) = \sum_{j \in \widetilde{H}(p_X^-), j \neq l} \tilde{s}_j'(p_X^-)$$

which induces a contradiction as before.

(ii) Second, suppose that there exists $k \in H_-(p_X)$ such that $s_k(p_X) > \tilde{s}_k(p_X) > 0$. Then

$$\sum_{j \in H_-(p_X), j \neq k} s_j'(p_X^-) = h_k(p_X, s_k(p_X^-)) + D'(p_X) > h_k(p_X, \tilde{s}_k(p_X^-)) + D'(p_X) = \sum_{j \in \widetilde{H}_-(p_X), j \neq k} \tilde{s}_j'(p_X^-)$$

which contradicts the facts that $s_k'(p_X^-) \leq \tilde{s}_k'(p_X^-)$ for each $k \in H_-(p_X)$ and that $\tilde{s}_k'(p_X^-) \geq 0$.

(iii) Finally, consider the case that neither (i) nor (ii) holds. Thus $s_j(p_X) = \tilde{s}_j(p_X)$ for all $j \in \widetilde{H}_-(p_X)$. Then for $p \in (p_X - \delta, p_X)$,

$$\sum_{j \in H_-(p_X)} s_j'(p) \geq h_l(p, \bar{q}_l) + D'(p) > h_l(p, \tilde{s}_l(p)) + D'(p) = \sum_{j \in \widetilde{H}_-(p_X), j \neq l} \tilde{s}_j'(p),$$

which gives

$$\sum_{j \in H_-(p_X)} [s_j(p) - \tilde{s}_j(p)]' + \sum_{j \in \widetilde{H}_-(p_X) \setminus H_-(p_X), j \neq l} [\bar{q}_j - \tilde{s}_j(p)]' > 0$$

for $p \in (p_X - \delta, p_X)$. But from the definition of p_X, we have

$$\sum_{j \in H_-(p_X)} [s_j(p) - \tilde{s}_j(p)] + \sum_{j \in \widetilde{H}_-(p_X) \setminus H_-(p_X), j \neq l} [\bar{q}_j - \tilde{s}_j(p)] > 0 \tag{9}$$

for $p \in (p_X - \delta, p_X)$. Therefore, the inequality (9) must also hold for $p = p_X$. However this contradicts $s_j(p_X) = \tilde{s}_j(p_X)$ for all $j \in \widetilde{H}_-(p_X)$.

We have shown that neither case (a) or (b) can occur, and so we have established that the result holds throughout (p_0, p_1). ■

We will say that a set of different equilibria is an ordered family if any two members satisfy the conditions of Lemma 4 from the point at which they differ. i.e. there is some price p_0 with $s_i(p) \geq \tilde{s}_i(p)$ for all i with strict inequality for $i \in \widetilde{H}(p)$, for each $p_0 < p \leq \min\{p_{\max}, \tilde{p}_{\max}\}$.

Now we are ready to prove the uniqueness of the strong supply function equilibrium at prices $p = c_l'(0)$ for $l = 2, \ldots, n$.

Lemma 5 *Let $p_0 = c_l'(0)$ for some $2 \leq l \leq n$. Then, for given values of $\{s_i(p_0^+)\}_{i=1}^n$, there is at most one strong supply function equilibrium $\{s_i(p)\}_{i=1}^n$ in $(p_0, p_0 + \Delta)$ for some $\Delta > 0$.*

Proof Any strong supply function equilibrium satisfies the ODE system (8) and so to prove the Lemma we show that there is at most one solution to the initial value problem given by:

$$\begin{aligned} s_i'(p) &= \tfrac{1}{(m-1)} \left(\textstyle\sum_{j \in H} \tfrac{s_j(p)}{p - c_j'(s_j(p))} + D'(p) \right) - \tfrac{s_i(p)}{p - c_i'(s_i(p))} \text{ for } i \in H, \\ s_k(p_0) &= x_k, \ k = 1, \ldots, n. \end{aligned} \tag{10}$$

where $H = \{i = 1, \ldots, n : c_i'(0) < p_0, 0 \leq x_i < \bar{q}_i\} \cup \{l\}$; $0 < x_i \leq \bar{q}_i$ for $i < l$ and $x_i = 0$ for $i \geq l$.

We may assume that $|H| > 1$, since otherwise firm l is an effective monopoly and there is a unique solution. We will ignore those firms not in H in the rest of the proof, We suppose that there are two equilibria $\{s_i(p)\}_{i\in H}$, $\{\tilde{s}_i(p)\}_{i\in H}$ satisfying (10). Let $g_i(p) = s_i(p) - \tilde{s}_i(p)$. We shall prove that $g_i(p)$ is identically zero (i.e. $s_i(p) = \tilde{s}_i(p)$) for $p \geq p_0$ and $i \in H$ under the following two complementary cases.

Case 1. Suppose there exists a sequence $p_m > p_0$, $m = 1, 2, 3, \ldots$ such that $\lim_{m\to\infty} p_m = p_0$ and $s_i(p_m) - \tilde{s}_i(p_m) > 0$ for all m (or $s_i(p_m) - \tilde{s}_i(p_m) < 0$ for all m, but the proof is the same by interchanging $s_i(p)$ and $\tilde{s}_i(p)$) Then by Lemma 4, we have $s_i(p) > \tilde{s}_i(p)$ for $p \geq p_m$ and $i \in H$. Therefore we have $s_i(p) > \tilde{s}_i(p)$ for $p > p_0$ and $i \in H$. Then, we have

$$h_i(p, s_i(p)) - h_i(p, \tilde{s}_i(p)) \leq L[s_i(p) - \tilde{s}_i(p)], \text{ for } i \in H, i \neq l, p \geq p_0 \tag{11}$$

for some constant $L > 0$ since $h_i'(p, y)$ (the derivative with respect to y) is uniformly bounded for any $y \in (\tilde{s}_i(p), s_i(p))$ (independent of p near p_0).

We have a weaker inequality for the l'th component. First note that there is a constant $M > 0$ independent of p near p_0 such that

$$\frac{1}{p - c_l'(s_l(p))} \leq M \frac{1}{p - c_l'(0)}.$$

(and the same inequality holds for $\tilde{s}_l(p)$). Otherwise, there exist $t_k \to c_l'(0)$ for $k = 1, 2, 3, \ldots$, such that $1/(t_k - c_l'(s_l(t_k)) > k/(t_k - c_l'(0)$. Then we have

$$\begin{aligned} \frac{s_l(t_k)}{t_k - c_l'(s_l(t_k))} &> \frac{k s_l(t_k)}{t_k - c_l'(0)} \\ &= k\left(\frac{t_k - c_l'(s_l(t_k))}{s_l(t_k)} + \frac{c_l'(s_l(t_k)) - c_l'(0)}{s_l(t_k)}\right)^{-1} \\ &> k\left(\frac{t_k - c_l'(s_l(t_k))}{s_l(t_k)} + c_l''(y_k)\right)^{-1} \end{aligned}$$

for some $0 \leq y_k \leq s_l(t_k)$. Since $s_l(t_k)/[t_k - c_l'(s_l(t_k))]$ has a finite, positive limit as $t_k \to c_l'(0)$ by l'Hopital's rule, this inequality gives a contradiction. Hence

$$\begin{aligned} h_l(p, s_l(p)) - h_l(p, \tilde{s}_l(p)) &= [\frac{1}{p - c_l'(y^*)} + \frac{y^* c_l''(y^*)}{(p - c_l'(y^*))^2}][s_l(p) - \tilde{s}_l(p)] \\ &\leq \frac{M}{p - c_l'(0)}[s_l(p) - \tilde{s}_l(p)] \end{aligned}$$

for some constant $M > 0$ independent of p, where $y^* \in (\tilde{s}_l(p), s_l(p))$.

Replacing $h_i(p, s_i(p))$ and $h_i(p, \tilde{s}_i(p))$ by $\sum_{j\in H, j\neq i} s_j'(p) + D'(p)$ and $\sum_{j\in H, j\neq i} \tilde{s}_j'(p) + D'(p)$ respectively (from (2)), we have

$$\sum_{j\in H, j\neq i} g_j'(p) \leq L g_i(p), \text{ for } i \neq l$$

and

$$\sum_{j\in H, j\neq l} g_j'(p) \leq \frac{M}{p - c_l'(0)} g_l(p)$$

Summing we have

$$(m-1)\sum_{j\in H} g_j'(p) \leq L \sum_{j\in H, j\neq l} g_j(p) + \frac{M}{p - c_l'(0)} g_l(p).$$

Since $g_j(p_0) = 0$ for any $j \in H$, we can integrate this from p_0 to $p > p_0$ to obtain

$$\sum_{j\in H} g_j(p) \leq \frac{L}{m-1}\int_{p_0}^{p} \sum_{j\in H, j\neq l} g_j(t)dt \; + \; \frac{M}{m-1}\int_{p_0}^{p} \frac{g_l(t)}{t-p_0}dt$$

Now, integrating (10) and applying (11), we have

$$g_l(t) \leq \frac{L}{m-1}\int_{p_0}^{t} \sum_{j\in H, j\neq l} g_j(p).$$

Thus

$$\begin{aligned} \int_{p_0}^{p} \frac{g_l(t)}{t-p_0}dt &\leq \frac{L}{m-1}\int_{p_0}^{p} [\frac{1}{t-p_0}\int_{p_0}^{t} \sum_{j\in H, j\neq l} g_j(s)ds]dt \\ &\leq \frac{L}{m-1}\int_{p_0}^{p} \sum_{j\in H, j\neq l} g_j(t)dt \end{aligned}$$

since $\sum_{j\in H, j\neq l} g_j(t)$ is nondecreasing in $t > p_0$ due to $\sum_{j\in H, j\neq l} g_j'(t) = [h_l(t, s_l(t)) - h_l(t, \tilde{s}_l(t))] > 0$ as $s_l(t) \geq \tilde{s}_l(t)$.

Therefore, we have

$$\begin{aligned} \sum_{i\in H} g_i(p) &\leq \frac{L}{m-1}[1 + \frac{M}{m-1}]\int_{p_0}^{p} \sum_{j\in H, j\neq l} g_j(t)dt \\ &\leq \frac{L}{m-1}[1 + \frac{M}{m-1}]\int_{p_0}^{p} \sum_{j\in H} g_j(t)dt. \end{aligned}$$

By Gronwall's Lemma, and the fact that each g_i, $i \in H$, is non-negative, we have

$$\sum_{i\in H} g_i(p) \leq 0.$$

Therefore $g_i(p) = 0$, that is $s_i(p) = \tilde{s}_i(p)$ for any $i \in H, p \geq p_0$.

Case 2. Suppose that there exists $\delta > 0$ such that no $p \in [p_0, p_0 + \delta]$ satisfies the assumption in Case 1 and that $\{s_i(p)\}_{i=1}^n$ and $\{\tilde{s}_i(p)\}_{i=1}^n$ are equilibria which differ somewhere in the interval $(p_0, p_0 + \delta)$. We define a sign function as follows $I(x) = x/|x|$, for $x \neq 0$, and $I(0) = 0$, and let $T(p) = \sum_{i\in H} I(g_i(p))$, and $\theta(p) = \sum_{i\in H} |g_i(p)|$. As before we let $m = |H|$.

By assumption we have $|T(p)| \leq m-1$ for $p \in [p_0, p_0+\delta]$. It is easy to show, using the monotonicity of h_i, that $I(h_i(p, s_i(p)) - h_i(p, \tilde{s}_i(p))) = I(g_i(p))$. From (2) we have

$$h_i(p, s_i(p)) - h_i(p, \tilde{s}_i(p)) = \sum_{j \neq i} g_j'(p),\ \ i \in H$$

and thus

$$I(g_i(p)) \sum_{j \neq i} g_j'(p) \geq 0,\ \ i \in H.$$

Now observe that $(m-1)I(g_i(p)) - T(p)$ has the same sign as $I(g_i(p))$ or is zero, whenever $I(g_i(p))$ is not zero. On the other hand if $I(g_i(p)) = 0$ then $\sum_{j \neq i} g_j'(p) = 0$. Hence we have the inequalities

$$[(m-1)I(g_i(p)) - T(p)] \sum_{j \neq i} g_j'(p) \geq 0,\ \ i \in H.$$

Summing these inequalities we obtain

$$\begin{aligned} \sum_{i \in H} \left([(m-1)I(g_i(p)) - T(p)] \sum_{j \neq i} g_j'(p) \right) &= \sum_{i \in H} \left(g_i'(p) \sum_{j \neq i} [(m-1)I(g_j(p)) - T(p)] \right) \\ &= \sum_{i \in H} g_i'(p)[(m-1)(T(p) - I(g_i(p))) - (m-1)T(p)] \\ &= -(m-1) \sum_{i \in H} g_i'(p) I(g_i(p)) \geq 0. \end{aligned}$$

Now $\theta(p)$ is Lipschitz continuous, since it is a composite of the absolute value function and continuously differentiable functions $g_i(p)$. Therefore it is differentiable almost everywhere in the interval $[p_0, p_0+\delta)$. When it is differentiable at p we have $\theta'(p) = \sum_{i=1}^n g_i'(p) I(g_i(p)) \leq 0$. This is enough to show that it is non-increasing on the interval (see e.g. Theorem 21.10 in [33]). Since $\theta(p) \geq 0$ and $\theta(p_0) = 0$, we have established that $\theta(\cdot)$ is identically zero on the interval $[p_0, p_0+\delta)$, and thus $\{s_i(p)\}_{i=1}^n$ and $\{\tilde{s}_i(p)\}_{i=1}^n$ do not differ on this interval.

Therefore, the initial value problem (10) cannot have more than one solution. This completes the proof of the lemma. ■

Lemma 6 *Let $\{s_i(p)\}_{i=1}^n$ and $\{\tilde{s}_i(p)\}_{i=1}^n$ be two different supply function equilibria. Let $p_0 \neq c_j'(0)$ for all $j = 1, \ldots, n$ be chosen so that $p_0 < \min\{p_{max}, \tilde{p}_{max}\}$, and that $\{s_i(p_0)\}_{i=1}^n$ and $\{\tilde{s}_i(p_0)\}_{i=1}^n$ are defined and not equal. Suppose that $s_i(p_0) \geq \tilde{s}_i(p_0)$ for all i, with strict inequality for all $i \in \widetilde{H}(p_0)$. Then both $\widetilde{H}(p_0)$ and $H(p_0)$ have two or more elements.*

Proof. First we show that $H(p_0)$ has at least one element. Suppose, on the contrary, that $H(p_0) = \varnothing$. Then we may choose i with $s_i(p_0) = \bar{q}_i > \tilde{s}_i(p_0)$. But the optimality conditions for s_i imply

$$0 \geq h_i(p_0, \bar{q}_i) + D'(p_0) > h_i(p_0, \tilde{s}_i(p_0)) + D'(p_0) \geq h_i(p_0, \tilde{s}_i(p_0)) - \sum_{j \in \widetilde{H}(p_0), j \neq i} \tilde{s}'_j(p) + D'(p_0) = 0$$

which is a contradiction.

(a) Suppose first that $\widetilde{H}(p_0) \subset \{i\}$ ($\widetilde{H}(p_0) \neq \emptyset$ following from the assumptions). Because there is no firm starting to supply at p_0, the other supply functions in the equilibrium $\tilde{s}_j(p)$, $j \neq i$ are either 0 or at their capacity limits for $p \in (p_0, p_0 + \delta)$ for small $\delta > 0$. Because of the ordering assumed and using Lemma 1, the same is true for $s_j(p)$, $j \neq i$. Hence firm i is the monopoly firm in both equilibria and the supply function is determined from the monopoly conditions (3) for $p \in (p_0, p_0 + \delta)$ for both s_i and $\tilde{s}_i$. Thus $s_i(p_0) = \tilde{s}_i(p_0)$ and this contradicts the inequality of the lemma statement.

(b) Now we suppose that $H(p_0) = \{i\}$. Then we can determine $s_i(p_0)$, using (3). We will show that there is at most one element in $\widetilde{H}(p_0)$. Suppose otherwise, noting that $\widetilde{H}(p_0) \subseteq \widetilde{H}_+(p_0)$ since there are no jumps at p_0, we have

$$\sum_{j \in \widetilde{H}_+(p_0), j \neq i} \tilde{s}'_j(p_0^+) = h_i(p_0, \tilde{s}_i(p_0)) + D'(p_0) < h_i(p_0, s_i(p_0)) + D'(p_0) = 0.$$

The strict inequality here comes from our assumption that $s_i(p_0) > \tilde{s}_i(p_0)$. However, as all the slopes are non-negative, this gives a contradiction. But now we have established that $\widetilde{H}(p_0)$ has no more than one element, then part (a) establishes a contradiction. ■

Now we come to the main result describing the equilibria that may exist.

Theorem 7 *If*

$$-D(c'_1(0)) < \varepsilon_{\min} < -D(c'_2(0)), \tag{12}$$

then any equilibrium is part of an ordered family, and only the lowest (smallest offers at any given price) can have the property that all but one of the firms reach their capacity limits prior to the maximum price.

Proof The condition (12) simply means that at price $c'_1(0)$ there is always some demand (i.e. even when $\varepsilon = \varepsilon_{\min}$) but when the price reaches $c'_2(0)$ there may be no demand for the lowest demand shock levels. Therefore any equilibrium solution has $c'_1(0) < p_{\min} < c'_2(0)$.

Suppose that there are two equilibria $\{s_i(p)\}_{i=1}^n$ and $\{\tilde{s}_i(p)\}_{i=1}^n$. We take p_V to be the supremum of prices at which these equilibria are the same. From our previous result on uniqueness of solutions we have $p_V = c'_k(0)$ for some k and at least one of the equilibria has a jump at p_V. Since the two solutions are identical at prices lower than p_V they both have just one firm in the set $H(p_V)$, say i. Swapping the labels for the two equilibria if necessary, we may suppose that $s_i(p_V^+) > \tilde{s}_i(p_V^+)$ and hence that $s'_k(p_V^+) > \tilde{s}'_k(p_V^+)$. So the conditions for Lemma 4 apply for p_0 just above p_V. We will show that these conditions continue to apply while both supply function equilibria are defined (i.e. up to a price $\min(p_{\max}, \tilde{p}_{\max})$).

Let p_W be the supremum of prices above p_V at which the ordering holds. Suppose that $p_W < \min\{p_{\max}, \tilde{p}_{\max}\}$. By the result of Lemma 4 some element of one of the equilibria $\{s_i(p)\}_{i=1}^n$ or $\{\tilde{s}_i(p)\}_{i=1}^n$ has a jump at p_W. Moreover if only $s_i(\cdot)$ jumps up at p_W for some i, then it is easy to see that the ordering conditions of Lemma 4 continue to apply. Hence there must be some $\tilde{s}_i(p_W)$ which is discontinuous, and from Theorem 2, $\widetilde{H}(p_W) = \{i\}$. But applying Lemma 6 to $p_W - \delta$ with sufficiently small $\delta > 0$ we have at least two elements in $\widetilde{H}_-(p_W)$ and so there must be some j with $\tilde{s}_j(p) < \bar{q}_j$ for $p_W > p \to p_W$ and $\tilde{s}_j(p_W) = \bar{q}_j$. But by Lemma 4, we have $s_j(p_W) > \tilde{s}_j(p_W)$ which gives a contradiction.

We need to show that $\{s_i(p)\}_{i=1}^n$ cannot have the property of all but one of the firms reaching their capacity limits prior to the maximum price. Suppose that $p_{\max} > \tilde{p}_{\max}$. Then

$$D(\tilde{p}_{\max}) + \varepsilon_{\max} = \sum_{i=1}^n \tilde{s}_i(\tilde{p}_{\max}) < \sum_{i=1}^n s_i(\tilde{p}_{\max}) \leq \sum_{i=1}^n s_i(p_{\max}) = D(p_{\max}) + \varepsilon_{\max}$$

which contradicts D decreasing, and hence $p_{\max} \leq \tilde{p}_{\max}$.

Now suppose that $s_i(p) = \bar{q}_i$, for $i \neq k$ for some k, for $p \in (p_{\max} - \delta, p_{\max})$. Then for this choice of p,

$$0 = h_k(p, s_k(p)) + D'(p) > h_k(p, \tilde{s}_i(p)) + D'(p_0) \geq h_k(p, \tilde{s}_i(p)) - \sum_{j \in \widetilde{H}(p_0), j \neq i} \tilde{s}'_j(p) + D'(p_0) = 0$$

which is a contradiction. This establishes the result. ■

In the case that the lowest level of demand involves more than one firm, so $D(c'_2(0)) + \varepsilon_{\min} > 0$, then we can no longer deduce that the equilibria belong to an ordered family.

Our numerical experiments show that typical problems either have a range of equilibria each of which has at least two firms below their capacity limits for prices below the maximum, or they have a single equilibrium with all but one firm reaching their capacity limits prior to the maximum price. However, as is illustrated in Figure 1. it is possible for a single problem to have both types of equilibria. The figure shows an equilibrium (solid lines) in which firm 2 reaches its capacity limit, and another equilibrium (dashed lines) in which neither firm reaches its capacity limit. This latter equilibrium will be one of a family of nearby equilibria.

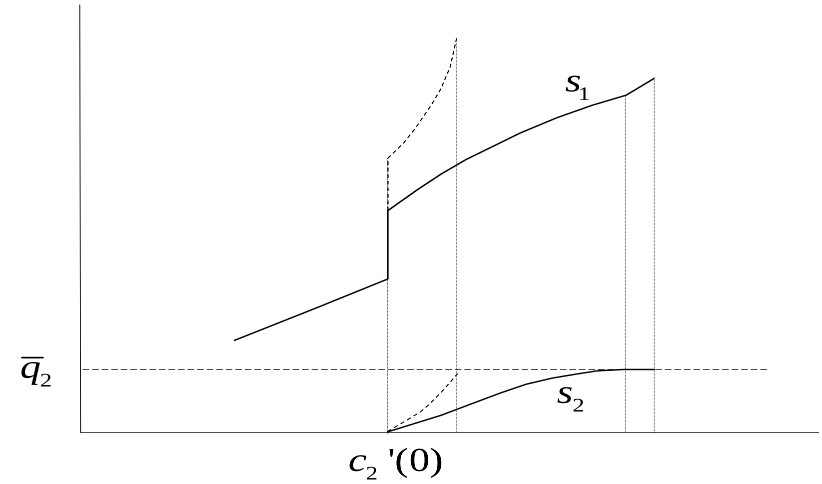

Figure 1: Possible multiple equilibria when all firm but one have capacity limits

Notice that all of our results are for the case of strictly decreasing demand curves (an assumption that was used in the proof of Lemma 1). When demand is insensitive to price a strong equilibrium must always have at least two elements in the set of unconstrained firms $H(p)$, since in a monopoly situation prices are set as high as possible in the best response (see (1)). However it can be shown that the ordered family result still holds with each strong supply function equilibrium having $s_1(p) = 0$, $p < c_2'(0)$ and $s_1(c_2'(0)) \geq D + +\varepsilon_{\min}$.

Holmberg [21] shows the uniqueness of supply function equilibrium under conditions: 1) firms have identical constant marginal cost, but asymmetric capacities; 2) demand is inelastic and exceed the market capacity with a positive probability. Klemperer and Meyer [25, Proposition 4] show uniqueness when there are identical firms with linear marginal cost functions, linear demand function and the assumption that the demand shock spreads to infinity. In both these cases, it is possible to find a closed-form solution for the unique supply function equilibrium. Theorem 7 establishes the extent to which these results extend to the much more general situation considered here.

3 Approximation of supply function equilibria

The results we have given allow us to describe the equilibrium solutions that can occur. Now we turn to the numerical calculation of an asymmetric equilibrium. Most researchers have approached this problem through a discretization over prices. We will take a different approach and look instead at an approximation based on a discretization over possible demand shocks.

Thus we approximate the demand shock distribution using a demand shock profile, $\Omega_\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_K\}$ for some given positive integer K. We may choose to take the values $\varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_K$ as given by $\varepsilon_k = F^{-1}(\frac{k-1}{K-1})$, $k = 1, \ldots, K$. This will correspond to an assumption that demand is equal to $D(p) + \varepsilon_k$ with probability $1/K$ for each $k = 1, \ldots, K$. However, since we are seeking strong equilibria, the results are independent of the distribution function F and hence do not depend on the probability with which each of the ε_k occur.

We suppose that the market clears at the price p_k when the demand shock is ε_k and so

$$\sum_{i=1}^{n} s_i(p_k) = D\left(p_k\right) + \varepsilon_k.$$

We write q_{ik} for the amount that firm i is dispatched at this price and β_{ik} for the slope of the supply function for firm i at this point (i.e. $\beta_{ik} = s_i'(p_k)$). We will construct the supply functions in such a way that they are differentiable at p_k, $k = 1, 2, \ldots, K$, so that these slopes are well-defined.

We will consider supply functions that are piecewise linear, with separate pieces corresponding to each of the possible demand shocks ε_k. Thus the k'th piece of the supply function for firm i passes through the point (p_{ik}, q_{ik}) at a slope β_{ik}. (It is convenient to write the price here as p_{ik} rather than p_k since we wish to think of the choice of (p_{ik}, q_{ik}) being made by firm i). In order to define a supply function we require that two consecutive segments of a supply curve, applying at p_{ik} and p_{ik+1}, intersect at some point between p_{ik} and p_{ik+1}. This simplifies the representation of equilibrium conditions discussed later (e.g. (16)). This is illustrated in Figure 2. In fact, to ensure that p_{ik} is a smooth point of the piecewise supply curve, we require that the intersection point is strictly inside the interval (p_{ik}, p_{ik+1}). Hence we require that there exist $\tilde{p}_{ik}$ for $k = 1, \ldots, K-1$ such that

$$q_{ik+1} + \beta_{ik+1}(\tilde{p}_{ik} - p_{ik+1}) = q_{ik} + \beta_{ik}(\tilde{p}_{ik} - p_{ik}), \quad p_{ik} < \tilde{p}_{ik} < p_{ik+1}. \tag{13}$$

Note that it is possible to have two consecutive lines coincide. In this case, there exists infinitely many points satisfying (13). However, the supply curves constructed below are independent of the choices of such points.

Given (p_{ik}, q_{ik}) and $\beta_{ik} \ge 0, k = 1, \ldots, K$, satisfying $p_{ik} < p_{ik+1}, q_{ik} \le q_{ik+1}$ and (13) for

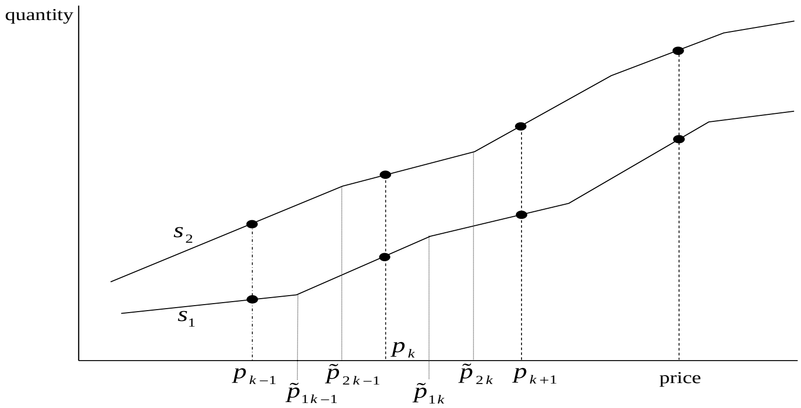

Figure 2: Construction of piecewise linear supply functions

$k = 1, \ldots, K-1$, a supply function for firm i is constructed as follows

$$s_i(p) = \begin{cases} q_{i1} + \beta_{i1}(p - p_{i1}), & \underline{p} \leq p \leq \tilde{p}_{i1} \\ q_{ik} + \beta_{ik}(p - p_{ik}), & \tilde{p}_{ik-1} \leq p \leq \tilde{p}_{ik}, k = 2, \ldots, K-1 \\ q_{iK} + \beta_{iK}(p - p_{iK}), & \tilde{p}_{iK-1} \leq p \leq \bar{p} \end{cases} \tag{14}$$

where $\underline{p}$ and $\bar{p}$ are the price floor and price cap imposed on all firms by market rules.

Firm i determines its optimal price, given the other firms' supply functions $s_j(p)$ for $j \neq i$, by solving the following profit maximization problem for demand shock ε_k:

$$\begin{aligned} \underset{p_{ik}}{\text{maximize}} \quad & [D(p_{ik}) + \varepsilon_k - \textstyle\sum_{j \neq i} s_j(p_{ik})]p_{ik} - c_i(D(p_{ik}) + \varepsilon_k - \textstyle\sum_{j \neq i} s_j(p_{ik})) \\ \text{such that} \quad & \underline{p} \leq p_{ik} \leq \bar{p} \\ & 0 \leq D(p_{ik}) + \varepsilon_k - \textstyle\sum_{j \neq i} s_j(p_{ik}) \leq \bar{q}_i \end{aligned} \tag{15}$$

The optimal choice of q_{ik} follows from the market clearing condition at price p_{ik}: i.e. $q_{ik} = D(p_{ik}) + \varepsilon_k - \sum_{j \neq i} s_j(p_{ik})$.

Note that the objective function problem for (15) is concave on each of the intervals where $\sum_{j \neq i} s_j(p)$ is smooth. A global maximum can be found by comparing the maximum values of the objective function on each of these finitely many pieces.

We aim to approximate a continuous piecewise smooth supply function equilibrium when firms face an uncertain demand which follows a continuous distribution. In fact, as we will show in computational examples, our method will work well for any strong supply function equilibrium which as we have already shown will have at most finitely many jumps. We begin by assuming that the equilibrium price at each of the demand shocks occurs at a smooth point of the supply function (14). So we assemble the equilibrium conditions for $\{(15)\}_{i=1}^n$ for each demand shock realization in the following set of equilibrium conditions (where we now make use of the fact that we are interested in solutions with $p_{ik} = p_k$, $i = 1, \dots, n$) :

$$
\begin{aligned}
& q_{ik} - (p_k - c_i'(q_{ik}))(\textstyle\sum_{j \neq i} \beta_{jk} - D'(p_k)) + \lambda_{ik} - \mu_{ik} = 0 \\
& \textstyle\sum_{j=1}^{n} q_{jk} = D(p_k) + \varepsilon_k \\
& q_{ik+1} - q_{ik} - \beta_{ik+1} p_{k+1} + \beta_{ik} p_k + (\beta_{ik+1} - \beta_{ik}) \tilde{p}_{ik} = 0, \quad \text{void when } k = K \\
& p_k < \tilde{p}_{ik} < p_{k+1}, \ \beta_{ik} \geq 0 \\
& \underline{p} \leq p_k \leq \bar{p}, \quad 0 \leq q_{ik} \leq \bar{q}_i \\
& \lambda_{ik} \geq 0, \ \mu_{ik} \geq 0, \ \lambda_{ik}(\bar{q}_i - q_{ik}) = 0, \ \mu_{ik} q_{ik} = 0 \\
& i = 1, \dots, n, \quad k = 1, \dots, K,
\end{aligned}
\tag{16}
$$

where the pieces $q_{ik} + \beta_{ik}(p - p_k)$ and $q_{ik+1} + \beta_{ik+1}(p - p_{k+1})$ intersect at $\tilde{p}_{ik}$, $k = 1, \dots, K-1$ for $i = 1, \dots, n$, and λ_{ik}, μ_{ik} are the Lagrange multipliers of the constraints on q_{ik}.

A solution to the conditions (16) will determine a piecewise linear approximation to a supply function equilibrium. This gives an alternative to a direct numerical solution of the ODE system (2) if it has at least one smooth solution. A basic ODE method will start from an integration formula [32]:

$$\mathbf{y}(x_j + h) = \mathbf{y}(x_j) + \int_{x_j}^{x_j+h} \mathbf{f}(y(t), t)\, dt$$

at the current point x_j with a forward step $h > 0$, for the initial value problem of ODE:

$$\mathbf{y}'(x) = \mathbf{f}(\mathbf{y}, x), \quad \mathbf{y}(x_0) = \mathbf{y}_0.$$

Various ODE numerical methods are based on an approximate integration in the above formula, for example using a Taylor expansion of $\mathbf{f}$ at $\mathbf{y}(x_j)$ and x_j: there is no equivalent to this integration in the approach we propose. Our method avoids making a fixed discretization of the price range as would be the case with a numerical ODE method. Moreover our approach does not involve any procedure for choosing initial values of the supply functions.

An important advantage of our method, as we show in the theorem below, is that the pairs of prices and quantities of a solution of (16) will lie on the equilibrium supply curves, rather than being an approximation to the curve. To make this more precise, suppose that there exists an equilibrium solution with supply functions $\{s_i^*(p)\}_{i=1}^n$. We would like to show that there is a solution to (16) with the property that $q_{ik} = s_i^*(p_k)$, $i = 1, \dots, n$, $k = 1, \dots, K$ (and then we would expect the slope β_{ik} to approximate the slope of the tangent to s_i^* at p_k).

In some cases there may be a difficulty due to the non-existence of a piecewise linear approximation to s_i^* where the slopes of the pieces match the tangents of s_i^* at certain points

(p_k, q_{ik}). For example if the equilibrium solution were given by the supply function $s_i^*(p) = c + (p - p_0)^3$ and p_1, p_2 (determined by $\varepsilon_1, \varepsilon_2$) were to be two points symmetric about p_0, then the tangent lines of $s_i^*(p)$ at these two points do not intersect. These sorts of problems can only occur when there is a change of sign in the second derivative of s_i^* and can always be resolved by adding or changing the choice of points ε_k.

Theorem 8 *Let $\{s_i^*(p)\}_{i=1}^n$ be a strong supply function equilibrium on $[\underline{p}, \bar{p}]$. Then, for K large enough, there exists a solution ε_k, p_k, q_{ik}, $\tilde{p}_{ik}$, λ_{ik}, μ_{ik}, $i = 1, \ldots, n$, $k = 1, \ldots, K$ to (16), such that $D(p_k) + \varepsilon_k = \sum_j s_j^*(p_k)$, $q_{ik} = s_i^*(p_k)$, $i = 1, \ldots, n$, $k = 1, \ldots, K$. Moreover $\beta_{ik} = {s^*}'_i(p_k)$ for $i = 1, \ldots, n$, and all but a finite number of k.*

Proof We begin by supposing that the supply functions s_i^* are smooth for each $i = 1, \ldots, n$. Given a choice of $p_1 < p_2 < ... < p_K$, we define the corresponding demand shock ε_k such that $D(p_k) + \varepsilon_k = \sum_j s_j^*(p_k)$. By assumption, for $k = 1, \ldots, K$, p_k is a global optimal solution to (15) for each firm i in the following form:

$$\begin{array}{ll} \underset{p}{\text{maximize}} & [D(p) + \varepsilon_k - \sum_{j \neq i} s_j^*(p)]p - c_i(D(p) + \varepsilon_k - \sum_{j \neq i} s_j^*(p)) \\ \text{such that} & \underline{p} \leq p \leq \bar{p} \\ & 0 \leq D(p, \varepsilon_k) - \sum_{j \neq i} s_j^*(p) \leq \bar{q}_i. \end{array}$$

The first order optimality conditions of this problem, when we substitute $q_{ik} = s_i^*(p_k)$ and $\beta_{ik} = {s^*}'_i(p_k)$, gives the first two equation sets in (16). Moreover, since s_i^* is monotonic increasing, $\beta_{ik} \geq 0$. The assumptions on s_i^* imply $\underline{p} \leq p_k \leq \bar{p}$. Since $0 \leq s_i^*(p) \leq \bar{q}_i$, we have $0 \leq q_{ik} \leq \bar{q}_i$, so it only remains to check the conditions involving $\tilde{p}_{ik}$.

Consider the continuous function g, defined by

$$g(z) = q_{ik+1} - q_{ik} - \beta_{ik+1}p_{k+1} + \beta_{ik}p_k + (\beta_{ik+1} - \beta_{ik})z. \tag{17}$$

We will be finished if we can show that there is a zero of this function strictly between p_k and p_{k+1}. Now

$$\begin{aligned} g(p_k) &= \left[s_i^*(p_{k+1}) + (p_k - p_{k+1}){s^*}'_i(p_{k+1})\right] - s_i^*(p_k) \\ g(p_{k+1}) &= s_i^*(p_{k+1}) - \left[s_i^*(p_k) + (p_{k+1} - p_k){s^*}'_i(p_k)\right], \end{aligned}$$

Thus $g(p_k)$ is the difference, at p_k, between the tangent line to s_i^* taken at p_{k+1} and the curve s_i^* (and similarly for $g(p_k)$). If s_i^* is convex (but not affine) on the interval (p_k, p_{k+1}) then $g(p_k) > 0 > g(p_{k+1})$. If s_i^* is concave (and not affine) on this interval then $g(p_k) < 0 < g(p_{k+1})$.

In either case there is a point strictly between p_k and p_{k+1} where g is zero (and the same is true if s_i^* is affine in the interval). With an arbitrary choice of ε_k the only case in which we may not be able to find a point $\tilde{p}_{ik}$ in the required range is when there is a change of sign in the second derivative of s_i^* strictly between p_k and p_{k+1}. In this case we can move the p_k slightly (equivalent to changing ε_k) or add a new price point p_k (equivalent to adding a new demand shock ε_k) so that p_k is at a point where $s_i^{*\prime\prime}(p_k) = 0$. Hence we can arrange to have either concavity or convexity for each supply function s_i^*, in all the intervals (p_k, p_{k+1}).

Now consider the case where the equilibrium supply functions (as in Lemma 3) have finitely many (isolated) points at which they are not smooth and possibly not continuous. First suppose that there is a point, p_z, at which the supply function is continuous but not smooth. We will show that by taking p_k and p_{k+1} approaching p_z from the left and right respectively the condition involving $\tilde{p}_{ik}$ will hold. We take $p_k = p_z - \delta$ and $p_{k+1} = p_z + \delta$ and g_δ to be the function defined in (17) with this choice of p_k and p_{k+1}. Then

$$\begin{aligned} g_\delta(p_z + \delta) &= s_i^*(p_z + \delta) - s_i^*(p_z - \delta) - 2\delta\beta_{ik} \\ &= \delta s_i^{*\prime}(p_z^+) + \delta s_i^{*\prime}(p_z^-) - 2\delta\beta_{ik} + O(\delta^2) = \delta s_i^{*\prime}(p_z^+) - \delta s_i^{*\prime}(p_z^-) + O(\delta^2) \end{aligned}$$

and similarly $g_\delta(p_z - \delta) = \delta s_i^{*\prime}(p_z^-) - \delta s_i^{*\prime}(p_z^+) + O(\delta^2)$. Hence if the derivative of s_i^* changes at p_z the continuous function g will change sign between $p_z - \delta$ and $p_z + \delta$ for δ chosen small enough. This is enough to establish the existence of $\tilde{p}_{ik}$ as required. Notice that from (8) one s_i^* changing slope at p_z will imply that each $s_j^{*\prime}$ is discontinuous at p_z.

Now consider the case where one of the supply functions, say s_i^*, has a discontinuity at p_z. From Theorem 2 other supply functions $s_i^*(p_z)$ are either 0 or at their capacity limits. We choose the ε_k, $k = 1, 2..., K$ so that one of the prices, say p_h, is equal to p_z. We also take p_{h-1} and p_{h+1} sufficiently close to p_h for the following inequality to hold (the left and right hand sides approach $s_i^*(p_z^+)$ and $s_i^*(p_z^-)$ respectively)

$$s_i^*(p_{h+1}) - (p_{h+1} - p_h)s_i^{*\prime}(p_{h+1}) > s_i^*(p_{h-1}) + (p_h - p_{h-1})s_i^{*\prime}(p_{h-1}).$$

Now we need to choose ε_h so that q_{ih} is strictly between the two sides of this inequality. This is enough to ensure that for β_{ih} chosen large enough there will be appropriate intersection points $\tilde{p}_{ih-1}$ and $\tilde{p}_{ih}$. Usually we choose the slopes β_{jk} to match the derivatives $s_j^{*\prime}(p_k)$ in order that the first equation in (16) is satisfied. But this does not happen for β_{ih}, and in fact

$s_i^{*\prime}(p_h)$ is not defined. However observe that the equations:

$$q_{jh} - (p_h - c_j'(q_{jh}))(\sum_{l\neq j}\beta_{lh} - D'(p_h)) + \lambda_{jh} - \mu_{jh} = 0,\ j \neq i \quad (18)$$

are easily satisfied here. When $q_{jh} = \bar{q}_j$ we can take the multiplier λ_{jh} large and positive to achieve equality in (18), and similarly if $q_{jh} = 0$ we can take μ_{jh} large and positive. ■

In general there may be many solutions to the system (16) and not all of these will match a smooth supply function equilibrium. However as we refine our approximation by increasing the number of points ε_k we can expect a limiting solution to the system to approach a true supply function equilibrium.

Hence we investigate the limit of a series of supply function equilibria for finitely many demand scenarios, when these demand shock scenarios approximate a continuous demand shock function. Let $m = 2, 3, \ldots$ and define $\varepsilon_k^{(m)} = F^{-1}(\frac{k-1}{m-1})$, $k = 1, \ldots, m$. Let $p_k^{(m)}$, $q_{ik}^{(m)}$, $\tilde{p}_{ik}^{(m)}$, $\beta_{ik}^{(m)}$, $\lambda_{ik}^{(m)}$, $\mu_{ik}^{(m)}$, $i = 1, \ldots, n$, $k = 1, \ldots, m$, be a solution of (16) and let $\{s_i^{(m)}(p)\}_{i=1}^n$ be the corresponding supply functions constructed from (14). Then we have the following theorem (when there are no jumps or discontinuities).

Theorem 9 *If $p_k^{(m)}$, $k = 1, \ldots, m$ are global optimal solutions of (15) for each firm $i = 1, \ldots, n$ with the supply functions $\{s_j^{(m)}(p)\}_{j=1}^n$, and $\beta_{ik}^{(m)} < \Lambda$, $k = 1, \ldots, m$, $i = 1, \ldots, n$, for some upper bound Λ independent of m, then:*

1) there is a subsequence of $\{s_i^{(m)}(\cdot)\}_{i=1}^n$ which converges to a continuous supply function equilibrium in the sup-norm of continuous functions for the demand shock $F(\cdot)$;

2) moreover, if $\{s_i^{(m)}(\cdot)\}_{i=1}^n$ converges point-wise, then $\{s_i^{(m)}(\cdot)\}_{i=1}^n$ itself converges to a continuous supply function equilibrium in the sup-norm.

Proof Since the slopes β_{ik}^m are between 0 and Λ

$$|s_i^{(m)}(p) - s_i^{(m)}(p')| \leq \Lambda|p - p'|$$

for any $p, p' \in [\underline{p}, \bar{p}]$. Thus $\{s_i^{(m)}(\cdot)\}_{i=1}^n$ is equicontinuous on $[\underline{p}, \bar{p}]$, and applying Áscoli Theorem (see for example [31]) completes the proof of the convergence parts of the theorem.

Now we show that the limit functions form an equilibrium of $\{(15)\}_{i=1}^n$. Let $\{s_i(\cdot)\}_{i=1}^n$ be the limits of the sequence $\{s_i^{(m)}(\cdot)\}_{i=1}^n$ or one of its subsequences (which is denoted by the sequence itself to save notation). It is easy to see that $s_i(\cdot)$ is nondecreasing in p for each $i = 1, \ldots, n$ since $\beta_{ik}^{(m)} \geq 0$. We choose an arbitrary demand shock ε (with $\varepsilon = F^{-1}(r)$ for some $r \in (0,1)$). Then there is a choice of k as a function of m, which we denote by k rather than $k(m)$, such that $\varepsilon_k^{(m)} \to \varepsilon$ as $m \to \infty$.

Let $\pi_i(p, s_{-i}^{(m)}, \varepsilon_k^{(m)})$ denote the profit of firm i given market price p when the other firms have the supply functions $s_{-i}^{(m)} = \{s_1^{(m)}(\cdot), \cdots, s_{i-1}^{(m)}(\cdot), s_{i+1}^{(m)}(\cdot), \cdots, s_n^{(m)}(\cdot)\}$ and the demand shock is $\varepsilon_k^{(m)}$. Since $p_k^{(m)}$ is a global maximizer of $\pi_i(p, s_{-i}^{(m)}, \varepsilon_k^{(m)})$, we have

$$\pi_i(p_k^{(m)}, s_{-i}^{(m)}, \varepsilon_k^{(m)}) \geq \pi_i(p, s_{-i}^{(m)}, \varepsilon_k^{(m)}), \text{ for any } \underline{p} \leq p \leq \bar{p}$$

for $i = 1, \ldots, n$. Letting $m \to \infty$, we have $p_k^{(m)} \to p^*$ for some p^* (if necessary, passing to a subsequence). Thus, from the continuity of π_i, and since $s_{-i}^{(m)} \to \{s_1(\cdot), \cdots, s_{i-1}(\cdot), s_{i+1}(\cdot), \cdots, s_n(\cdot)\}$, and $\varepsilon_k^{(m)} \to \varepsilon$, we know that p^* is a global maximizer for firm i's profit given the other firms' supply functions $s_i(\cdot)$ for demand realization ε for $i = 1, \ldots, n$. ■

Note that when some β_{ik} are unbounded as $k \to \infty$, which means the piecewise linear supply curves constructed from (14) become vertical for those prices p_{ik} at which β_{ik} are unbounded, then we may still have a form of convergence. Specifically the piecewise linear curves may converge to a supply function equilibrium with discontinuities in the Hausdorff metric (i.e. the graphs of the piecewise linear supply functions converge to the graphs of the equilibrium supply functions) see Lemma 2.2 of [5].

4 Implementation and examples

In this section we will describe how the method can be implemented and give some examples of asymmetric equilibria. Briefly, the implementation involves two procedures. The first step is to formulate a sequence of non-linear programming (NLP) problems so that the set of feasible solutions converges to the set of solutions of (16). We use this approach because it provides a convenient way to handle the complementarity constraints between λ_{ik}, μ_{ik} and q_{ik}. It is also helpful in dealing with the strict inequalities $p_k < \tilde{p}_{ik} < p_{k+1}$. We choose the parameters in such a way that a solution to the last iteration is a solution to (16) up to the machine precision. The second step is to check whether the $\{p_k\}$ are the global optimal solution of (15) given the constructed supply functions $s_j(p)$, $j = 1, \ldots, n$.

First we describe the procedure to check global optimality. Let p_k, q_{ik}, $\tilde{p}_{ik}$, β_{ik}, ε_k, λ_{ik}, μ_{ik}, $i = 1, \ldots, n$, $k = 1, \ldots, K$, be the final solution of the sequence of NLP problems. Recall that p_k is already a local optimal solution of (15) for each firm and demand shock ε_k.

For each firm i, we order the points $\tilde{p}_{jk}$, $j = 1, \ldots, n$, $j \neq i$, $k = 1, \ldots, K$. These are the points at which the aggregate supply function from the other firms is non-smooth. Now consider the problem of calculating the optimal choice of price (15) when p is restricted to lie between two neighboring points $\tilde{p}_{jk}$. Given a demand shock ε_l, and the $s_j(p)$ constructed from p_k, q_{jk}, $\tilde{p}_{jk}$, β_{jk}, this is simply a one-dimensional quadratic (concave) profit maximization problem and so the optimal solution of this problem can be calculated very easily. Comparing the profits of these finitely many pieces we can find a global maximizer p^*_{il} for firm i for the given demand shock. If $p^*_{il} = p_l$ for each i and l (in our GAMS implementation, we check whether $|p^*_{il} - p_l| < 10^{-6}$), then we have reached the desired solution. Otherwise, we need to modify the objective function of the NLP problems. The objective functions used in our numerical computation perform well in the sense that if the NLP problems are successfully solved with reasonable accuracy, then this check agrees that p_k, $k = 1, \ldots, K$ are global optimal solutions of (15) for each firm and each demand shock ε_k.

Next we describe the formulation of the NLP minimization problems. First, given K, we choose $\tilde{\varepsilon}_k = \varepsilon_{\min} + (k-1)(\varepsilon_{\max} - \varepsilon_{\min})/(K-1)$ as a discretization of the range of demand shocks in our computation.

Let $\rho > 0$. We replace the equations $\lambda_{ik}(\bar{q}_i - q_{ik}) = 0$ and $\mu_{ik} q_{ik} = 0$ in (16) with the inequalities $\lambda_{ik}(\bar{q}_i - q_{ik}) \leq \rho$ and $\mu_{ik} q_{ik} \leq \rho$ for $i = 1, \ldots, n$, $k = 1, \ldots, K$. This replacement helps the NLP solver handle the complementarity constraints, which violate a standard NLP constraint qualification. We also introduce variables ξ_{ik}, $i = 1, \ldots, n$, $k = 1, \ldots, K-1$ with

$$\xi_{ik} = \frac{p_{k+1} - \tilde{p}_{ik}}{p_{k+1} - p_k},$$

that is, $\xi_{ik}(p_{k+1} - p_k) = p_{k+1} - \tilde{p}_{ik}$. To have $\tilde{p}_{ik}$ lying strictly in (p_k, p_{k+1}), we must have $0 < \xi_{ik} < 1$. We enforce pre-set lower and upper bounds for ξ_{ik}, say $\xi_{ik} \in (0.0001, 0.9999)$ if not otherwise stated in our implementations.

Our computational experience suggests that using the following constraints

$$\xi_{ik}(p_{k+1} - p_k) = p_{k+1} - \tilde{p}_{ik} + \rho \tag{19}$$

enables the NLP solver to more easily find a feasible solution.

To make it easier to find solutions with appropriate values of $\tilde{p}_{ik}$ (as discussed in Theorem 8) we allow ε_k to vary. But we need to ensure that the ε_k are evenly spread through $[0, \bar{\varepsilon}]$, and so we have added the following constraints to the NLPs:

$$0.25(3\tilde{\varepsilon}_k + \tilde{\varepsilon}_{k-1}) \leq \varepsilon_k \leq 0.25(3\tilde{\varepsilon}_k + \tilde{\varepsilon}_{k+1}), \text{ for } 1 < k < K, \varepsilon_1 = \tilde{\varepsilon}_1, \varepsilon_K = \tilde{\varepsilon}_K. \tag{20}$$

Our numerical experience is that there is no difficulty associated with non-existence of $\tilde{p}_{ik}$ when we fix $\varepsilon_k = \tilde{\varepsilon}_k$ for all k, but we have retained the distinction between ε_k and $\tilde{\varepsilon}_k$ in our base implementation.

Thus the basic problem we have solved is the following NLPs for $\rho \to 0^+$:

$$\begin{array}{ll} \underset{p,q,\tilde{p},\beta,\varepsilon,\lambda,\mu}{\text{minimize}} & \sum_{i,k}(\xi_{ik} - 0.5)^2 + \sum_k (\varepsilon_k - \tilde{\varepsilon}_k)^2 \\ \text{such that} & \text{(16) with aforementioned replacement of complementarity constraints} \\ & \text{(19) and (20).} \end{array} \tag{21}$$

We can modify the objective function to explore the existence of various supply function equilibria as seen in the rest of this section.

Our computational experience also suggests that adding $\sum_{i,k} \beta_{ik}$ to the objective function makes it easier for the NLP solver to find a feasible solution and stabilizes the solution procedure. Clearly adding this term to the objective will make it more likely that we end up with an equilibrium with low values for $s_i(p)$, which corresponds to solutions that hold back supply.

In our implementation, we start with $\rho = 10$ and scale it down by a factor of 0.1 at each iteration; solving the problem (21) a total of $N = 14$ times. For some harder problems, it may be helpful to increase the scale factor to, e.g., 0.5 (and at the same time increase the number of iterations, to say $N = 37$ to maintain a final ρ value sufficiently close to zero).

Since (21) is a large-scale problem for multiple firms and a large number of demand shocks, we need to take care with the choice of starting points for the solution of (21). In our examples we have used the following set of starting values : $p_k = (k-1)(\bar{p} - \underline{p})/K$, $\tilde{p}_{ik} = (p_k + p_{k+1})/2$, $\varepsilon_k = \tilde{\varepsilon}_k$, $\xi_{ik} = 0.5$, $q_{ik} = \max\{0, (D(p_k) + \varepsilon_k)/n\}$.

We use the nonlinear programming solver CONOPT (Version 3.0) [15] in the GAMS environment for all the computation in this section. Our numerical experience shows that CONOPT is more capable in solving our problems than other NLP solvers in GAMS. However, we note that CONOPT is sensitive to the choice of objective functions. In particular for some examples, CONOPT may deem a problem infeasible for one objective function, but may find an optimal solution to the problem when a different objective function is used, even though the constraints are unchanged.

We have done many numerical examples to test the scheme (16). We will present only a few results due to limitations of space. In each of the examples we assume quadratic cost functions and a linear demand function. If not specified otherwise, we take $\underline{p} = -1.0$, $\bar{p} = 150$.

We have intentionally set the price cap high enough and price floor low enough so that they are not binding in our examples (even though this may be an important issue in some markets for electricity).

Example 10 We begin with an example with two firms which will illustrate Theorem 7. The firms have cost functions $c_1(q) = q + 0.5q^2, c_2(q) = 10q + q^2$ with capacity limits $\bar{q}_1 = \bar{q}_2 = 80$ respectively, which are large enough so that no firm is constrained. The demand function is $D(p) = 0.5 - 0.5p$ with demand shock ε uniformly distributed over $[0, 100]$.

For this example it is easy to calculate a linear supply function equilibrium: $s_1(p) = 0.4529(p - 1)$ and $s_2(p) = 0.3279(p - 10)$ for $p \geq 10$. This follows from the closed-form solution $s_i(p) = \alpha_i + \beta_i p, i = 1, 2$, with

$$\alpha_i = -C_i\beta_i,\ \beta_i = (BW_i/2)(\sqrt{1 + 4/(D_i BW_i)} - 1)$$

where $W_i = (2D_i + D_1D_2B)/(D_1 + D_2 + D_1D_2B)$. This is now a standard result (see for example, [18]).

The affine supply function equilibrium is confirmed by the numerical models (21) with the objective function $\sum_{i,k}(\xi_{ik} - 0.5)^2 + \sum_{k\geq 30}(\beta_{ik+1} - \beta_{ik})^2$. For the price range $[1.0, 10)$, firm 2 does not supply and so firm 1 is a monopoly with supply curve $s_1(p) = (p - 1)/3$ for $1 \leq p < 10$. The supply function $s_1(\cdot)$ jumps from 3.0 just below $p = 10$ to 4.0764 just above $p = 10$. For prices less than 1.0, firm 1 will supply nothing. Thus we obtain a piecewise linear supply function equilibrium which is plotted in Figure 3. Demand shocks in the interval $[7.5, 8.5764]$ all yield the same price of 10.

The other two nonlinear supply function equilibria, labelled as nonlinear equilibria 1 and 2 in Figure 3, are found using (21) with the objective functions:

$$\sum_{i,k}(\xi_{ik} - 0.5)^2 + \sum_k(\varepsilon_k - \tilde{\varepsilon}_k)^2 + \sum_{i,k}\beta_{ik}$$

for nonlinear equilibrium 1 in Figure 3, and

$$\sum_{i,k}(\xi_{ik} - 0.5)^2 + \sum_k(\varepsilon_k - \tilde{\varepsilon}_k)^2 - \sum_k q_{1k}$$

for nonlinear equilibrium 2 in Figure 3, where the term $-\sum_k q_{1k}$ is used to maximize the q_{1k}'s. We used the constraints $0 \leq \beta_{ik} \leq 10^6$ and $0.0001 \leq \xi_{ik} \leq 0.9999$ and we take $K = 300$. All three equilibria have a jump at $p = 10.0$. and they can be seen to be members of an ordered family.

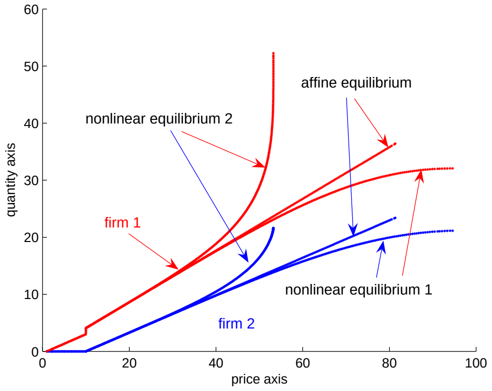

Figure 3: Ordered multiple equilibria with a jump

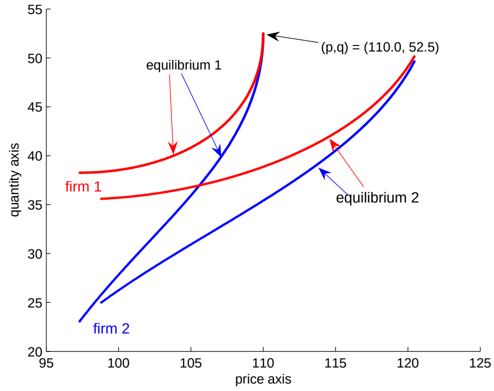

Figure 4: Multiple asymmetic equilibria not in order with symmetric generators

Notice that any two equilibria, say s and $\tilde{s}$, have the property that the values for $s_i(p)$ and $\tilde{s}_i(p)$ move apart from each other as the price increases. This is a general property for problems with just two firms. To see this observe that the equilibrium condition (2) implies

$$\frac{d}{dp}(s_1(p) - \tilde{s}_1(p)) = \frac{s_2(p)}{p - c_2'(s_2(p)} - \frac{\tilde{s}_2(p)}{p - c_2'(\tilde{s}_2(p)} > 0, \;\; \textit{when } p < p_{max}$$

if $s_2(p) > \tilde{s}_2(p)$, and similarly for $(d/dp)(\tilde{s}_2(p) - s_2(p))$.

It is not hard to see that there will be a whole family of solutions in an ordered set between the nonlinear equilibria 1 and 2. This follows from Kneser's Theorem [14, Theorem 2.17].

Example 11 ***(intervals of asymmetric solutions for symmetric suppliers)*** In this example, $c_1(q) = c_2(q) = 5q + q^2$ and the demand function $D(p) = 60 - 0.5p, \bar{q}_1 = \bar{q}_2 = 80, \bar{p} = 150$, and the demand shock range is $[50, 100]$. It has been shown in [25] that there are only symmetric equilibria for symmetric suppliers when demand is low enough so that any supply curve will pass through $(c_i'(0), 0)$. In this example, we show the types of non-symmetric equilibria that occur when the minimum realizable price is above the marginal supply cost at zero supply. To do so, we modify the objective function of (21) to

$$\sum_{i,k}(\xi_{ik} - 0.5)^2 + \sum_k(\varepsilon_k - \tilde{\varepsilon}_k)^2 + \sum_k(q_{2k} - q_{1k}) + (q_{21} - \tau)^2.$$

with $\tau = 20$ (for equilibrium 1 in Figure 4) and

$$\sum_{i,k}(\xi_{ik} - 0.5)^2 + \sum_k(\varepsilon_k - \tilde{\varepsilon}_k)^2 + (q_{21} - \tau)^2.$$

with $\tau = 25$ (for equilibrium 2 in Figure 4). The term $\sum_k(q_{2k} - q_{1k})$ has the effect of maximizing the difference between q_{1k} and q_{2k} in order to force an asymmetrical equilibria.

We also used the bounds $0 \leq \beta_{ik} \leq 10^6$ and $0.0001 \leq \xi_{ik} \leq 0.9999$ and a value $K = 300$ for these solutions. We can see that these two equilibria are not in an ordered family.

Notice that equilibria 1 has both supply functions passing through the point $(110, 52.5)$ at which they have infinite slope (since $p - c_i'(s_i(p)) = 110 - 5 - 2 * 52.5 = 0$, $i = 1, 2$.) This can only be part of a feasible supply function at its end point. In fact this point is at the intersection of the locus of points where the slope is infinite ($p = 5 + 2q$) and the points which, if both supply functions are the same, correspond to the highest demand shock $\varepsilon_{\max}$ (i.e. the points $q = 0.5D(p) + 0.5\varepsilon_{\max}$, which implies $p = 320 - 4q$.).

Example 12 This example has three firms and shows the uniqueness of the supply function equilibrium when the minimum demand shock is low enough and all generators but one reach their capacity limits. The cost functions of the three firms are

$$c_1(q) = 5q + 0.8q^2, \quad c_2(q) = 8q + 1.2q^2, \quad c_2(q) = 12q + 2.3q^2$$

with capacities $\bar{q}_1 = 11, \bar{q}_2 = 8$ and $\bar{q}_3 = 55$ respectively. The demand shock ε is uniformly distributed over $[0, 50]$ and the demand function is $D(p) = 2.5 - 0.5p$. we use (21) with $K = 300, 0 \leq \beta_{ik} \leq 10000$, $0.0001 \leq \xi_{ik} \leq 0.9999$ and the following objective function:

$$\sum_{i,k}(\xi_{ik} - 0.5)^2 + \sum_{k}(\varepsilon_k - \tilde{\varepsilon}_k)^2 + \sum_{i,k}\beta_{ik}.$$

The outcomes are plotted in Figure 5.

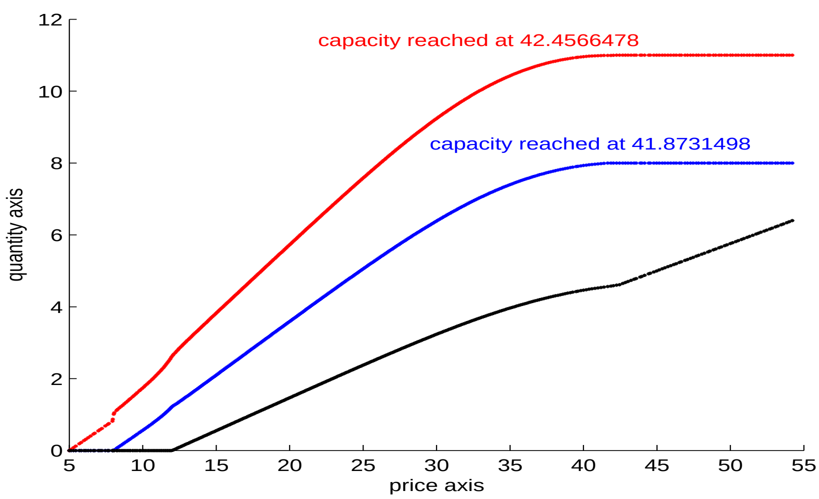

Figure 5: Locally unique SFE with capacitated sppliers and low minimal demand shock

We can see a jump in quantity at the price $p = 8$ where firm 2 starts to supply from zero, but there is no jump at $p = 12$ where the third firm starts to supply from zero. This is in accordance with the result of Theorem 2. In this example we find only one supply function equilibrium in which firms 1 and 2 reach their capacity limits.

Example 13 This example is for five competing electricity generators: it is taken from [9] (see also [10].) The cost functions and capacity limits are given in Table 13. These parameters and the load duration curve are taken from [9]. Here we scale down the lowest demand to force all firms to have zero generation at the lowest demand shock, and then correspondingly increase the largest demand shock so that the largest demand is the same as that in [9, 10].

	firm 1	firm 2	firm 3	firm 4	firm 5
C_i	8.0	8.0	12.0	12.0	12.0
D_i	1.789	1.93	4.615	4.615	2.687
$\bar{q}_i$	10.4482	9.70785	3.35325	3.3609	5.70945

Table 1: Cost function and capacity caps for the five generators

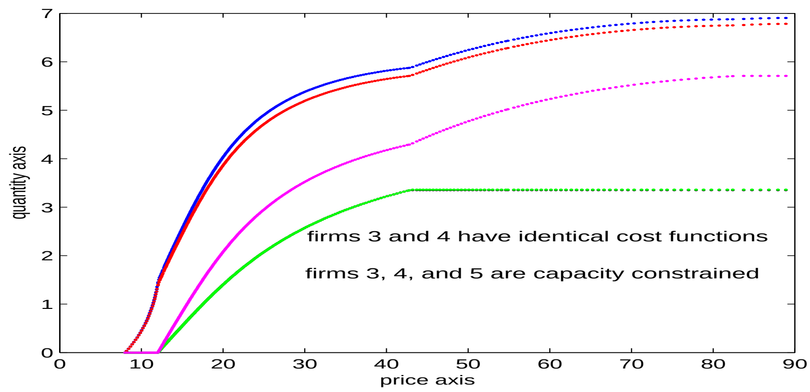

Figure 6: Five firms with three capacity-constrained

Note that firm 3 and firm 4 have identical cost functions. The load duration curve is $0.8+34.2(1-t)$ and price elasticity is 0.1. Hence we have a demand function $D(p) = 0.8-0.1p$ and a demand shock uniformly distributed over $[0, 34.2]$. To solve this problem we use (21) with an objective function

$$\sum_{i,k}(\xi_{ik} - 0.5)^2 + \sum_{k}(\varepsilon_k - \tilde{\varepsilon}_k)^2 + \sum_{i,k}\beta_{ik}$$

and $K = 300$. We use $0 \le \beta_{ik} \le 100$ and $0.01 \le \xi_{ik} \le 0.99$ here. The capacitated supply function equilibrium is given in Figure 6, where three firms are actually capacity-constrained at the highest prices. This is one of a family of possible solutions.

5 Conclusions

Supply function equilibria are known to be the appropriate equilibrium concept in some circumstances, but their application has been limited by difficulties in dealing with asymmetric cases. In this paper, we characterize the form of supply function equilibria when both capacity limits and cost functions vary from one firm to another. Our main result is that under certain conditions supply function equilibria are ordered with at most one such equilibria in which all but one firm reaches its capacity limit. At first sight this does not seem to deal with the problem of equilibrium selection. However our numerical results suggest that for many problems the range of possible solutions is very small. In particular it is very hard to find examples for which the behaviour of Figure 1 occurs, i.e. a single problem with both a family of equilibria having two or more firms not reaching their capacity constraints at the highest prices, and another solution in which all but one firm reaches their capacity constraints. In fact there are many problems which seem to have reasonable data, but which do not have any equilibrium solutions.

We also propose a novel scheme to approximate supply function equilibria through a discretization of the demand distribution. This results in a piecewise linear supply function which can always be found (given a fine enough discretization) if a strong supply function equilibrium exists. Moreover we provide convergence results and demonstrate that the scheme works well in practice.

There are a number of advantages of the scheme comparing with existing computational methods of finding equilibrium supply functions. The scheme does not need the specification of an initial point (i.e. a price-quantity pair) as is the case with ODE methods. Also the range of realizable prices in equilibrium is determined endogenously by the demand profile and firms' profit maximization strategies. This avoids one source of problems in methods which require a price range to be specified at the outset, which may lead to wrong conclusions.

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