# Comparing Supply Function Equilibria of Pay-as-Bid and Uniform-Price Auctions[1]

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**Abstract**

This paper derives a Supply Function Equilibrium (SFE) of a pay-as-bid auction (discriminatory auction), such as the balancing market for electric power in Britain. It is shown that a SFE always exists if the hazard rate of the perfectly inelastic demand is monotonically decreasing and marginal costs are non-decreasing. With demand following a Pareto distribution of the second kind, the SFE of a pay-as-bid auction is compared to the SFE of a uniform-price auction, the auction form in most electricity markets. The demand-weighted average price in the former is found to be (weakly) lower than in the latter.

Keywords: supply function equilibrium, pay-as-bid auction, uniform-price auction, discriminatory auction, divisible good auction, oligopoly, capacity constraint, electricity market

JEL codes: C62, D43, D44, L11, L13, L94

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## 1. INTRODUCTION

Most electric power markets are organised as uniform-price auctions (UPAs). One exception, however, is the balancing market for electricity trade in England and Wales, which in 2001 switched from a UPA to a pay-as-bid auction (PABA), also known as a discriminatory auction. It was the belief of the British regulatory authority (Ofgem) that the reform, which also replaced the day-ahead auction with bilateral contracting, would decrease mark-ups in wholesale electricity prices [7,8]. Before the collapse of the California Power Exchange, a similar switch was also considered for this market [17].

The balancing market allows the system operator to buy or sell last-minute power from power producers to keep a continuous balance of demand and supply. This paper focuses on market situations where more supply is needed, i.e. the system operator buys power as in a procurement auction. It is straightforward nonetheless to draw analogous conclusions for market situations where less supply is needed and the system operator sells power in the balancing market, as in a sales auction.

In a UPA, all accepted bids are paid the marginal bid. Thus in its procurement version, all infra-marginal bids are accepted at a price above their bid. In a PABA, all accepted bids are paid their bid. A natural, but naive, first thought is that switching to a pay-as-bid auction would drastically reduce mark-ups for infra-marginal units and thereby decrease the average electricity price. However, firms change their bidding strategy after switching to a PABA. Based on intuition and experience from classical auction theory, many researchers actually argue in favour of electricity markets being organized as UPAs, see Kahn et al. [17] and Wolfram [29] for example. An experiment by Rassenti et al. [22] also suggests that average prices are higher in PABAs.[3]

In classical auction theory, comprising the *private-value*, *common-value* and *affiliated-value* models, demand of the auctioneer is certain whereas uncertainties relate to costs [20]. In contrast, electricity markets are generally characterised by known production costs. It is common to make the approximation that producers can forecast electricity demand without uncertainty, see e.g. [7,24]. However, even if the uncertainty is small, the standard deviation of the system operator's demand in the balancing market is positive, as the system imbalance

[3] The demand in the experiment is not revealed to the players, but is certain in each period and the players can deduce it while playing. As in SFE with certain demand, this set-up would lead to an enormous range of equilibria [18]. Thus the experimental results are very much driven by the equilibrium selection process. Further, it is not certain that the experiments are long enough to allow the players find an equilibrium, especially as they have to find out the certain demand by themselves.

depends on unpredicted consumer behaviour, temperature shocks, and unexpected outages in machines and transmission-lines during the delivery period. Thus in a model of strategic bidding in balancing markets, costs can be assumed to be certain and demand uncertain. To date, most theoretical studies of electric power auctions have been devoted to the UPA, see e.g. [1-3,9,11,13-15,23]. Three recent exceptions [6,8,24], study bidding behaviour in electric power markets organised as PABAs and compare prices and welfare in PABAs and UPAs. All of these studies and a study of treasury auctions [27] indicate that electricity consumers should prefer PABA. This paper comes to the same conclusion with a model that is particularly relevant for balancing markets and which is more general in terms of number of firms and/or production costs.

The model developed in this paper is very much related to the Supply Function Equilibrium (SFE) under uncertainty, which was introduced by Klemperer & Meyer [18]. In the Nash equilibrium of the static game, each producer commits to the supply function that maximises his expected profit given the bids of competitors. The set-up of their model is similar to the organisation of many electricity markets with firms submitting supply functions to a uniform-price auction with uncertain (or time-varying) demand, and SFE is an often used model of strategic bidding in electric power markets organised as UPA [1-3,11,13-15,23]. In this paper, the fundamental assumptions of the SFE model are employed to derive a similar model for a pay-as-bid auction. It is assumed that demand is perfectly inelastic. Further, to rule out multiple equilibria, demand is assumed to exceed market capacity with a positive probability. Such events are unlikely but occur once per year or once per decade in real electricity markets. In the model the risk of power shortage is allowed to be arbitrarily small. Moreover, the standard deviation of demand can be made arbitrarily small. Thus the risk of power shortage assumption does not necessarily contradict the common assumption that producers can forecast demand with a high accuracy. To facilitate a closed-form solution for general probability distributions, only symmetric equilibria are considered. As for UPAs, it should be possible to extend the analysis to consider asymmetric producers [14,15].

Another contribution of this paper is the comparison of the two SFE models for procurement auctions. When demand follows the Pareto distribution of the second kind [16], the demand-weighted average price is weakly lower in PABA than in UPA.[4] This probability distribution is not unreasonable for the balancing market, for which large imbalances are less likely than small imbalances. In a one-shot game with perfectly inelastic demand and symmetric firms, mark-ups have no implications for social efficiency. Large mark-ups do,

[4] Analogously, demand-weighted average prices in sales auctions would be higher in a PABA than a UPA.

however, imply substantial redistribution of income from power consumers to power producers, itself of social interest. Furthermore, large mark-ups lead to welfare losses in the long term as firms entering the market invest unnecessarily in additional capacity [11].

The supply function equilibrium is related to the divisible good auction literature, which was originally started by Wilson [28]. In particular, the study of treasury auctions by Wang & Zender [27] has similarities with this paper. Especially their analysis of sales auctions with random non-competitive bids — which has priority over competitive bids — and risk-neutral bidders with symmetric information. For this special case, they conclude that an auctioneer would prefer the pay-as-bid auction. For an equivalent procurement auction, their conclusion is restricted to constant marginal costs. Further, their choice of density function is extreme as it induces oligopoly bidders in the pay-as-bid auction to bid along the constant marginal cost, as if under perfect competition.

Federico and Rahman [8] use a supply function equilibrium model to compare UPA and PABA for two polar cases — perfect competition and monopoly — assuming that demand is elastic and follows a uniform probability distribution. They show that expected output decreases and expected consumer surplus increases after switching to a PABA. On the other hand, welfare is reduced in the competitive case. Under monopoly bidding, welfare is larger in PABAs if and only if marginal costs are sufficiently flat and demand uncertainty sufficiently low.

Fabra et al. [6] derive a Nash equilibrium for an oligopoly market where firms compete with stepped supply functions. Firms have heterogeneous constant marginal costs, and in the duopoly case, also asymmetric capacities. Demand is perfectly inelastic and known with certainty by the producers. Under these circumstances they prove that average prices are lower in the PABA than in a UPA, and numerical examples suggest that the difference might be substantial. If demand is sufficiently high, the PABA has no pure strategy equilibria and only a mixed strategy equilibrium. Son et al. [24] analyse a duopoly model, which is similar to the duopoly model of Fabra et al., except that one of the two firms has two production units with different marginal costs. Son et al. also conclude that average prices are lower in the PABA than in a UPA if demand is certain and perfectly inelastic. Simulations by Fabra et al. and Son et al. suggest that the conclusion may hold also for elastic demand.

This paper is structured as follows. Notation and assumptions are presented in Section 2 and the unique SFE of a PABA is derived in Section 3. It is shown that the first-order condition implies that the bid of each production unit is chosen to maximise the unit's expected profit, given the bids of competitors. The risk of power shortage provides an end-condition for the supply functions. A unique equilibrium candidate exists that satisfies both the first-order condition and the end-condition. Next, a second-order condition is derived. A unique equilibrium always exists if the demand's probability distribution has a downward sloping hazard rate and marginal costs are non-decreasing. In Section 4, average prices in the two procurement auctions are compared. Section 5 illustrates the two supply function equilibria with a simple example and Section 6 presents the conclusions.

## 2. NOTATION AND ASSUMPTIONS

Assume that there are $N \geq 2$ symmetric producers. The bid of each producer $i$ consists of a monotonically increasing supply function $S_i(p)$, where $p$ is the price.[5] The inverse of the supply function is denoted by $p_i(S_i)$. $S_{-i}(p)$ and $S(p)$ denote the combined supply of firm $i$'s competitors' and total supply in the marketplace, respectively. The marginal bid as a function of total supply is denoted $p(S)$. The average accepted bid for a given total supply, $\hat{p}(S)$, is called the equilibrium price. In a UPA, all accepted bids are paid the marginal bid, i.e. $\hat{p}_U(S) \equiv p_U(S)$, while $\hat{p}(S) \equiv \int_0^S p(x)dx / S$ in the PABA. As in Klemperer & Meyer's original work [18], only equilibria with twice continuously differentiable supply functions are considered. Thus in a symmetric equilibrium, $p_i(S_i)$ is smooth for $S_i \in (0, \varepsilon^* / N)$, where $\varepsilon^*$ is the total offered supply of all producers. If firms are withholding capacity, $\varepsilon^*$ is less than $\bar{\varepsilon}$, the total capacity of all producers.

Denote perfectly inelastic demand by $\varepsilon$, its probability density function by $f(\varepsilon)$ and its distribution function by $F(\varepsilon)$. The density function is continuously differentiable and has support on the interval $[0, \hat{\varepsilon}]$, where $\hat{\varepsilon}$ is maximum demand. It is assumed that $\hat{\varepsilon} \geq \bar{\varepsilon}$, i.e. the capacity constraints of all producers will bind with a positive probability, which is allowed to be arbitrarily small. Demand is zero above the reservation price (price cap) $\bar{p}$. Therefore, in the extreme case where demand exceeds market capacity, the market price of the uniform-price auction equals the price cap.

[5] Electricity auctions do not normally accept decreasing supply functions.

All firms have identical cost functions $C(S_i)$, which are increasing, convex, twice continuously differentiable, and fulfil $C'(\bar{\varepsilon}/N) < \bar{p}$. $R$ denotes the sum of firms' expected revenues and $\pi_i$ denotes the expected profit of firm *i*.

## 3. THE UNIQUE SYMMETRIC SFE OF A PAY-AS-BID AUCTION

In the SFE of a UPA, a firm chooses, conditional on residual demand, a supply function that maximises profit for each demand outcome [18]. This section derives necessary and sufficient conditions for the SFE of a PABA. In Section 3.1, the first-order condition of a PABA is derived. It implies that each firm chooses a supply function that maximises expected profit for each of its production units, given residual demand. The first-order condition is a differential equation which can be solved for general cost functions and the solution has one integration constant. In Section 3.2, this constant is identified by considering the risk of power shortage; the symmetric equilibrium bids of all firms must reach the price cap exactly when the aggregate capacity constraint binds. This forms the end-condition.

The first-order condition and the end-condition must necessarily be satisfied in equilibrium. In Section 3.3, a sufficient second-order condition is also derived. Unlike a UPA, it is not possible to prove that all increasing, smooth supply functions satisfying the necessary conditions are supply function equilibria of a PABA. In particular, it turns out that if the hazard rate of demand is locally increasing and marginal costs are sufficiently flat, then pure strategy equilibria of a PABA do not exist. On the other hand, I show that a pure strategy equilibrium always exists if the hazard rate is monotonically decreasing and marginal costs are non-decreasing. This is fulfilled by the Pareto distribution of the second kind, which is used in the comparison of the UPA and PABA. Moreover, this choice simplifies the algebraic manipulations, as the inverse of its hazard rate is linear.

### 3.1. The first-order condition

It is assumed that firm *i*'s competitors follow a symmetric equilibrium candidate. The first-order condition derived below must necessarily be fulfilled if the strategy implied by the symmetric equilibrium candidate locally maximises firm *i*'s expected profit. To avoid differentiability problems, all considered deviations of firm *i* satisfy $p_i(0) = p_j(0)$ and

$p_i\left(\varepsilon^* / N\right) = p_j\left(\varepsilon^* / N\right)\ \forall j \neq i.$ The profit from an accepted bid of an infinitesimally small unit is $\left[p(S_i) - C'(S_i)\right]dS_i$. Thus the expected profit of firm $i$ is

$$\pi_i = \int_0^{\varepsilon^*} f(\varepsilon) \int_0^{\varepsilon - S_{-i}(p(\varepsilon))} [p_i(S_i) - C'(S_i)] dS_i d\varepsilon + \int_{\varepsilon^*}^{\hat{\varepsilon}} f(\varepsilon) \int_0^{\varepsilon^* / N} [p_i(S_i) - C'(S_i)] dS_i d\varepsilon.$$

The second term of this expression represents the contribution from demand outcomes exceeding market supply. By changing the order of integration the following can be shown:[6]

$$\begin{aligned} \pi_i[p_i(S_i)] &= \int_0^{\varepsilon^* / N} [p_i(S_i) - C'(S_i)] \int_{S_i + S_{-i}[p_i(S_i)]}^{\varepsilon^*} f(\varepsilon)\, d\varepsilon dS_i + \\ &+ \int_0^{\varepsilon^* / N} [p_i(S_i) - C'(S_i)] \int_{\varepsilon^*}^{\hat{\varepsilon}} f(\varepsilon) d\varepsilon dS_i = \int_0^{\varepsilon^* / N} [p_i(S_i) - C'(S_i)] \int_{S_i + S_{-i}[p_i(S_i)]}^{\hat{\varepsilon}} f(\varepsilon)\, d\varepsilon dS_i = \\ &= \int_0^{\varepsilon^* / N} \underbrace{[p_i(S_i) - C'(S_i)][1 - F(S_i + S_{-i}[p_i(S_i)])]}_{\varphi_i[S_i, p_i(S_i)]} dS_i. \end{aligned} \tag{1}$$

Firm $i$ chooses the bid function $p_i(S_i)$ such that its expected profit is maximised. That is, the firm faces a calculus of variation problem with the fixed terminal points $p_i(0) = p_j(0)$ and $p_i\left(\varepsilon^* / N\right) = p_j\left(\varepsilon^* / N\right)$. As $p_i'(S_i)$ does not enter the integral, the Euler equation degenerates to the following equation [5]:

$$\begin{aligned} &\frac{\partial \varphi_i}{\partial p_i} = 1 - F[S_{-i}(p_i(S_i)) + S_i] - S_{-i}'(p_i(S_i))(p_i(S_i) - C'(S_i)) f[S_{-i}(p_i(S_i)) + S_i] = 0, \\ &\forall\, S_i \in \left[0, \varepsilon^* / N\right]. \end{aligned} \tag{2}$$

The functional $\varphi_i[S_i, p_i(S_i)]$ represents the contribution to expected profit from an infinitesimally small unit. Thus, the Euler equation implies that expected profit from each unit is maximised independently, conditional on residual demand. Because only equilibria with smooth and increasing supply functions are considered, (2) can be written as follows:

$$\begin{aligned} &1 - F[S_{-i}(p) + S_i(p)] - S_{-i}'(p)(p - C'(S_i(p))) f[S_{-i}(p) + S_i(p)] = 0, \\ &\forall\, p : S_i(p) \in \left(0, \varepsilon^* / N\right) \end{aligned} \tag{3}$$

In addition, only symmetric SFE are considered, i.e. $S_{-i}(p) \equiv (N-1) S_i(p)$. Thus (3) can be further simplified to:

[6] Note that the limits of the integrals may change when the order of integration is changed [25]. It is straightforward to verify the new integration limits by plotting the integrated area.

$$1-F[NS_i(p)]-(N-1)S_i'(p)(p-C'(S_i(p)))f[NS_i(p)]=0, \forall\, p: S_i(p)\in\left(0,\varepsilon^*/N\right) \quad (4)$$

This first-order condition of a PABA corresponds to the first-order condition of a UPA derived by Klemperer & Meyer [18]. Note that (4) implies that $p>C'$ if and only if $S_i'(p)>0$.

In order to solve the differential equation, it is transformed into an equation in terms of $p(\varepsilon)$, the price of the marginal unit as a function of the demand, instead of $S_i(p)$. The same transformation is applied when solving the differential equation associated with the SFE of a UPA [1,23]. In the symmetric equilibrium, $\varepsilon = NS_i(p(\varepsilon))$ and $S_i' = \frac{1}{Np'(\varepsilon)}$, if $\varepsilon \le \varepsilon^*$. Thus

$$1-F(\varepsilon)-\frac{(N-1)}{Np'(\varepsilon)}[p(\varepsilon)-C'(\varepsilon/N)]f(\varepsilon)=0 \quad \forall \varepsilon\in\left[0,\varepsilon^*\right].$$

and

$$\frac{(N-1)p(\varepsilon)f(\varepsilon)}{N}-p'(\varepsilon)[1-F(\varepsilon)]=\frac{(N-1)C'(\varepsilon/N)f(\varepsilon)}{N}.$$

This differential equation can be solved by means of the integrating factor, $[1-F(\varepsilon)]^{\frac{N-1}{N}-1}$, to yield

$$p(\varepsilon)=\frac{A-\int^{\varepsilon}(N-1)C'(u/N)f(u)[1-F(u)]^{\frac{N-1}{N}-1}du}{N[1-F(\varepsilon)]^{\frac{N-1}{N}}} \quad \forall \varepsilon\in\left[0,\varepsilon^*\right], \quad (5)$$

where $A$ is an integration constant.

### 3.2. Determining the integration constant

Analogous to the derivation of the SFE of a UPA, the integration constant $A$ allows for a continuum of potential equilibria [18]. In this section, I argue that the integration constant can be uniquely determined if, as in [13], the capacity constraint binds with a positive probability, which can be arbitrarily small. This unlikely event may occur after large demand shocks and/or unexpected simultaneous multiple generator failures.[7] In this case, the price of the marginal unit must reach the price cap exactly when the capacity constraint starts to bind. Note that this is true even if the standard deviation of demand is made arbitrarily small.

[7] To avoid inconsistencies in the model, one can limit attention to production uncertainties for firms who exclusively have must-run production. These firms cannot bid strategically in the balancing market. Two examples of such firms in the British market are British Energy and British nuclear group, both of whom exclusively produce nuclear power.

If $\varepsilon^* < \bar{\varepsilon}$, some capacity is withheld from the auction. However, it cannot be optimal to withhold power. A producer will find it profitable to offer previously withheld units at or just below the price cap. Bidding with his whole capacity will increase the contribution to expected profit of demand outcomes $\varepsilon > \varepsilon^*$, while the possible profit reduction associated with demand outcomes $\varepsilon \le \varepsilon^*$ can be made arbitrarily small.

The highest bid in the auction must equal the price cap. Otherwise, the highest bid could be increased without lowering the probability that its associated unit is accepted. Moreover, as noted in Section 2, the analysis is confined to equilibria with twice continuously differentiable supply functions. Hence $S_i'(p) < \infty$, implying that $p'(\varepsilon) > 0$.[8] Thus by construction supply functions cannot have horizontal segments at the price cap.

In summary, the price of the marginal unit must reach the price cap but not before the capacity constraint binds. Hence, the integration constant can be pinned down by the end-condition $p(\bar{\varepsilon}) = \bar{p}$.[9] It follows from (5) that

$$p(\varepsilon) = \frac{N\left[1 - F(\bar{\varepsilon})\right]^{\frac{N-1}{N}} \bar{p} + \int_{\varepsilon}^{\bar{\varepsilon}} (N-1) C'(u/N) f(u) \left[1 - F(u)\right]^{\frac{N-1}{N} - 1} du}{N\left[1 - F(\varepsilon)\right]^{\frac{N-1}{N}}} \quad \forall \varepsilon \in \left[0, \bar{\varepsilon}\right]. \qquad (6)$$

### 3.3. The second-order condition

The only remaining equilibrium candidate is given by (6) which fulfils both the first-order condition and the end-condition of the PABA. For this candidate, let $\breve{p}(\varepsilon)$ be the price of the marginal unit as a function of demand. The symmetric supply functions of the candidate are designated by $\breve{S}_i(p)$. If the aggregate supply of competitors equals $\breve{S}_{-i}$, then — as shown by (2) — the expected profit of any unit of firm *i* is at a local extremum. By studying the second-order condition, it can be verified that, under certain conditions, expected profit is globally maximised for each production unit of firm *i*. Because firms and the equilibrium candidate are both symmetric, this argument is true for any firm and offers a sufficient condition for a SFE.

It follows from (1) that for a given $S_i$, the expected profit from the marginal unit of firm *i* is

[8] The assumption simplifies the proof but is not critical. Allowing for perfectly elastic bids does not change the result because, as in a Bertrand game, it is profitable to slightly undercut competitors' horizontal bids [13].

[9] The same end-condition is used to derive a unique SFE for UPAs [13]. Baldick & Hogan have suggested the same end-condition for UPAs but offer a weaker motivation; the price cap and capacity constraints can be viewed as public signals that coordinate the bids of producers [2].

$$\varphi_i(S_i, p)dS_i = [p - C'(S_i)][1 - F(S_i + S_{-i}(p))]dS_i, \tag{7}$$

where $p=p_i(S_i)$ is the bid of the marginal unit. Because competitors follow $\breve{S}_i$, (2) can be rewritten as

$$\frac{\partial \varphi_i(S_i, p)}{\partial p} = 1 - F[\breve{S}_{-i}(p) + S_i] - \breve{S}_{-i}{}'(p)(p - C'(S_i)) f[\breve{S}_{-i}(p) + S_i]$$

or

$$\frac{\partial \varphi_i(S_i, p)}{\partial p} = f[\breve{S}_{-i}(p) + S_i] \underbrace{\left\{ \overbrace{\left( \frac{1 - F[\breve{S}_{-i}(p) + S_i]}{f[\breve{S}_{-i}(p) + S_i]} \right)}^{G[\breve{S}_{-i}(p)+S_i]=1/H[\breve{S}_{-i}(p)+S_i]} - \breve{S}_{-i}{}'(p)(p - C'(S_i)) \right\}}_{\eta(p, S_i)}, \tag{8}$$

where $G(x)$ is the inverse of the hazard rate $H(x)$. Let $p^* = \breve{p}_i(S_i)$. The first-order condition of the PABA in (4) ensures that $\left.\frac{\partial \varphi_i(S_i, p)}{\partial p}\right|_{p=p^*} = 0.$ As $f>0$ and $\eta(p^*, S_i) = 0,$ the following two conditions would ensure that $\varphi_i(S_i, p)$ is globally maximised at the price $p^*$ :

$\eta(p, S_i) > 0$ for $p \in [\breve{p}(0), p^*)$[10] and $\eta(p, S_i) < 0$ for $p \in (p^*, \bar{p}]$. A SFE is guaranteed if they are both fulfilled for all $S_i \in [0, \bar{\varepsilon}/N]$. On the other hand, if $\left.\frac{\partial \eta(p, S_i)}{\partial p}\right|_{p=p^*} > 0$ for some $S_i$, then $\varphi_i(S_i, p)$ is locally minimised at the price $p^* = \breve{p}_i(S_i)$ and there exists a profitable deviation. The conditions for the global maximum and local minimum can be used to show the following theorem:

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[10] It is never profitable to offer a unit below $\breve{p}(0)$, as the unit is always accepted at this price.

**Theorem 1.**

i) $\left[p(\varepsilon)-C'(\varepsilon/N)\right]G'(\varepsilon)+G(\varepsilon)C''(\varepsilon/N)>0\ \forall\varepsilon\in\left(0,\bar{\varepsilon}\right)$ is necessary for the existence of a SFE, as the condition ensures that there are no local profitable unilateral deviations from the candidate in (6).

ii) If $G'(\varepsilon)>0\ \forall\varepsilon\in\left(0,\bar{\varepsilon}\right)$, then the equilibrium candidate in (6) is a SFE.

iii) If $\left[p(\varepsilon)-C'(\varepsilon/N)\right]G'(\varepsilon)+G(\varepsilon)C''(\varepsilon/N)<0$ for some $\varepsilon\in\left(0,\bar{\varepsilon}\right)$, then the equilibrium candidate in (6) is not a SFE, and a smooth, symmetric SFE does not exist.

Proof: See Appendix.

As $H'(x)=\frac{-G'(x)}{G^2(x)}$, it follows from Theorem 1 that a downward sloping hazard function ensures a SFE. On the other hand, if the hazard function is locally upward sloping and marginal costs sufficiently flat, then smooth symmetric equilibria can be ruled out. Both of these two properties are true for a procurement version of an auction studied by Wang & Zender [27]; the auction with random non-competitive bids and risk-neutral bidders with symmetric information. Still Wang & Zender deduce an equilibrium where firms have perfectly elastic supply at the marginal cost. This equilibrium is possible, as the auctioneer is never short of bids in their model, otherwise this equilibrium would also be ruled out.

There is some intuition behind the non-existent equilibria. In the case of a monopolist or Cournot player, a similar problem occurs when demand or residual demand is sufficiently convex [10]. In the PABA, it follows from (7) that $1-F\left[\breve{S}_{-i}(p)+S_i\right]$ can be interpreted as the residual demand of the marginal unit when firm $i$ supplies $S_i$ units of power.[11] Now, differentiate $1-F\left[\breve{S}_{-i}(p)+S_i\right]$ twice with respect to $p$. Eliminate $\breve{S}''_{-i}(p)$ by differentiating the first-order condition of the PABA in (4). Consider the case $f'(\varepsilon)\geq 0$, which implies an upward sloping hazard rate (see (8)). Then it can be shown that the residual demand, $1-F\left[\breve{S}_{-i}(p)+S_i\right]$, is convex, if marginal costs are sufficiently flat.

The existence of equilibria is easier to guarantee in UPAs, as supply function equilibria of UPAs are independent of $f(\varepsilon)$. A symmetric SFE of a UPA exists as long as the demand function is concave [18].

[11] Previously Bulow and Klemperer [4] have noted that the probability that a bid is accepted (1-F) can be interpreted as residual demand.

### 3.4. The Pareto distribution of the second kind

As shown below, the Pareto distribution of the second kind has $G'(x) = const > 0$, which according to Theorem 1 guarantees a SFE. Moreover, due to the linearity of the inverse hazard rate, it will turn out that the first-order condition becomes particularly simple. The Pareto distribution of the second kind has the probability distribution

$$F(x) = 1 - \beta^{\frac{1}{\alpha}} (\alpha x + \beta)^{\frac{-1}{\alpha}} \tag{9}$$

and the probability density

$$f(x) = \beta^{\frac{1}{\alpha}} (\alpha x + \beta)^{\frac{-1}{\alpha} - 1}. \tag{10}$$

Hence, the inverse of its hazard rate is

$$G(x) = \frac{1 - F(x)}{f(x)} = \alpha x + \beta, \tag{11}$$

where $\alpha, \beta > 0$.

The parameter $\beta$ determines $f(0)$ as illustrated in Figure 1. When α is large, $f$ has a steep negative slope for small arguments and a thick tail for large arguments, vice versa for small α. The density function is decreasing and strictly convex for all $\alpha, \beta > 0$. Thus the Pareto distribution of the second kind captures the important characteristic that small imbalances are more likely than large imbalances in a balancing market.

With the Pareto distribution of the second kind, the first-order condition of the PABA in (4) can be simplified to

$$\alpha N S_i(p) + \beta - (N-1) S_i'(p)(p - C'(S_i(p))) \equiv 0,$$

which resembles the first-order condition of the UPA [18],

$$S_i(p) - (N-1) S_i'(p)(p - C'(S_i(p))) \equiv 0.$$

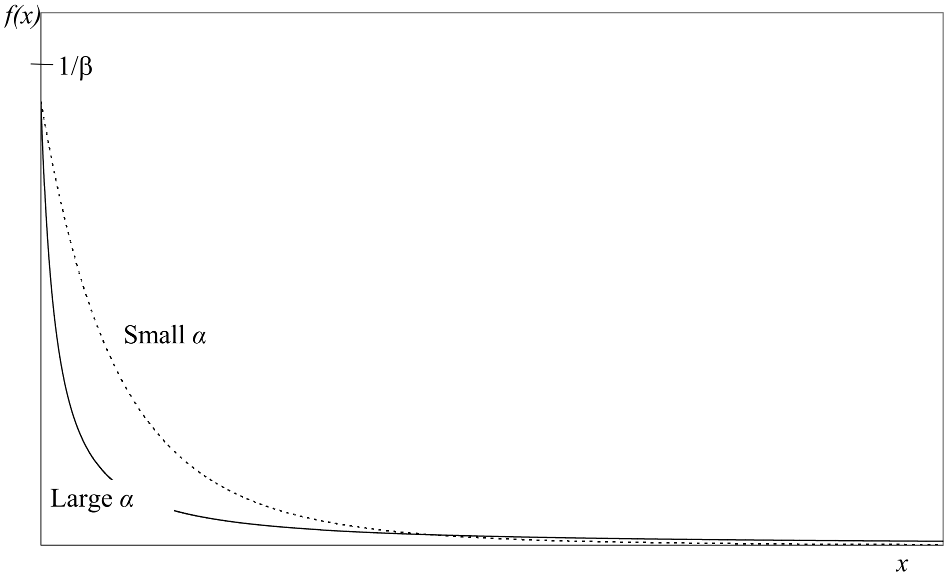


**Figure 1.** *The effect of α and β on the probability density function f(x).*

It follows from (6) that the equilibrium marginal bid in a PABA with a Pareto distribution of the second kind is

$$p(\varepsilon)=\frac{N\left(\alpha\bar{\varepsilon}+\beta\right)^{\frac{1-N}{\alpha N}}\bar{p}+\int_{\varepsilon}^{\bar{\varepsilon}}(N-1)C'(u/N)(\alpha u+\beta)^{\frac{1-N}{\alpha N}-1}du}{N(\alpha\varepsilon+\beta)^{\frac{1-N}{\alpha N}}}. \qquad (12)$$

## 4. Comparing pay-as-bid and uniform-price auctions

Demand is assumed to be perfectly inelastic. As a result, total production in a pay-as-bid and uniform-price auction are equivalent. Furthermore, only symmetric equilibria are considered. This means that for every demand outcome, the most cost-effective generators will be accepted in both procurement auctions. Hence, production costs are also the same in both auctions for all outcomes. Average prices and mark-ups will differ, however, and the extent of the difference is investigated by comparing firms' total expected revenue in the two auctions. To ensure a SFE in both procurement auctions, demand is assumed to follow the Pareto distribution of the second kind. In the first subsection, expected revenues in the PABA are shown to be equal to or lower than expected revenues in the UPA when marginal costs are constant. This result is then used in the next subsection to prove the same inequality for non-decreasing marginal costs.

### 4.1. Constant marginal costs

It follows from (1) that total expected revenue for all firms in a PABA is

$$R_p = \int_0^{\bar{\varepsilon}} (1 - F(S))p(S)dS = \int_0^{\bar{\varepsilon}} (1 - F(\varepsilon))p(\varepsilon)d\varepsilon. \tag{13}$$

By means of (9) and (12) it can be shown that

$$R_P = \int_0^{\bar{\varepsilon}} \frac{\beta^{\frac{1}{\alpha}} (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}} \left\{ \bar{p}N\left(\alpha\bar{\varepsilon} + \beta\right)^{\frac{1-N}{\alpha N}} + (N-1)\int_{\varepsilon}^{\bar{\varepsilon}} C'(u/N)(\alpha u + \beta)^{\frac{1-N}{\alpha N}-1} du \right\}}{N(\alpha\varepsilon + \beta)^{\frac{1-N}{\alpha N}}} d\varepsilon . \tag{14}$$

Constant marginal costs are assumed in this section, i.e. $C'(u/N) \equiv c.$ For this case, straightforward integration yields

$$\begin{aligned} R_P &= \frac{\beta^{\frac{N-1}{\alpha N}}\left(\bar{p} - c\right)}{\left(\alpha\bar{\varepsilon} + \beta\right)^{\frac{N-1}{\alpha N}}} \int_0^{\bar{\varepsilon}} \left(1 + \frac{\alpha\varepsilon}{\beta}\right)^{\frac{-1}{\alpha N}} d\varepsilon + \beta^{\frac{1}{\alpha}} c \int_0^{\bar{\varepsilon}} (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}} d\varepsilon = \\ &= \left(\bar{p} - c\right)\bar{\varepsilon} g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) + \beta^{\frac{1}{\alpha}} c \int_0^{\bar{\varepsilon}} (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}} d\varepsilon, \end{aligned} \tag{15}$$

where

$$g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) = \frac{\left(1 + \frac{\alpha\bar{\varepsilon}}{\beta}\right)^{\frac{-1}{\alpha N}+1} - 1}{\left(1 - \frac{1}{\alpha N}\right)\frac{\alpha\bar{\varepsilon}}{\beta}\left(\frac{\alpha\bar{\varepsilon}}{\beta} + 1\right)^{\frac{N-1}{\alpha N}}}. \tag{16}$$

The following can be shown by means of integration by parts:

$$\beta^{\frac{1}{\alpha}} \int_0^{\bar{\varepsilon}} (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}} d\varepsilon = \beta^{\frac{1}{\alpha}} \left[ (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}} \varepsilon \right]_0^{\bar{\varepsilon}} + \beta^{\frac{1}{\alpha}} \int_0^{\bar{\varepsilon}} (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}-1} \varepsilon d\varepsilon = \bar{\varepsilon}\left[1 - F\left(\bar{\varepsilon}\right)\right] + \int_0^{\bar{\varepsilon}} f(\varepsilon)\varepsilon d\varepsilon.$$

Thus the second term in (15), $\beta^{\frac{1}{\alpha}} c \int_0^{\bar{\varepsilon}} (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}} d\varepsilon$, is the expected production cost, and the first term, $\left(\bar{p} - c\right)\bar{\varepsilon} g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)$, is a measure of the mark-up.

The equilibrium marginal bid for symmetric firms in a UPA can be calculated from [13],

$$p_U(\varepsilon) = \frac{\bar{p}\varepsilon^{N-1}}{\bar{\varepsilon}^{N-1}} + (N-1)\varepsilon^{N-1} \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C'(u/N)du}{u^N} \quad \forall \left[0, \bar{\varepsilon}\right]. \tag{17}$$

The total expected revenue for firms in a UPA is[12]

$$R_U = \int_0^{\bar{\varepsilon}} f(\varepsilon)\varepsilon p_U(\varepsilon)d\varepsilon + \left(1 - F\left(\bar{\varepsilon}\right)\right)\bar{\varepsilon}\bar{p}.$$

The second term is the contribution from demand outcomes exceeding market capacity. Combining (9), (10) and (17) yields

$$R_U = \int_0^{\bar{\varepsilon}} \beta^{\frac{1}{\alpha}} (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}-1} \varepsilon \left[ \frac{\bar{p}\varepsilon^{N-1}}{\bar{\varepsilon}^{N-1}} + (N-1)\varepsilon^{N-1} \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C'(u/N)du}{u^N} \right] d\varepsilon + \beta^{\frac{1}{\alpha}} \left(\alpha\bar{\varepsilon} + \beta\right)^{\frac{-1}{\alpha}} \bar{\varepsilon}\bar{p}. \quad (18)$$

Assuming constant marginal costs, the expression can be simplified by means of integration by parts:

$$\begin{aligned} R_U &= \frac{N(\bar{p}-c)}{\bar{\varepsilon}^{N-1}} \int_0^{\bar{\varepsilon}} \left(\frac{\alpha\varepsilon}{\beta}+1\right)^{\frac{-1}{\alpha}} \varepsilon^{N-1}\,\mathrm{d}\varepsilon + \beta^{\frac{1}{\alpha}} c \int_0^{\bar{\varepsilon}} (\alpha\varepsilon+\beta)^{\frac{-1}{\alpha}} d\varepsilon = \\ &= (\bar{p}-c)\bar{\varepsilon} g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) + \beta^{\frac{1}{\alpha}} c \int_0^{\bar{\varepsilon}} (\alpha\varepsilon+\beta)^{\frac{-1}{\alpha}} d\varepsilon, \end{aligned} \quad (19)$$

where

$$g_U\left(\alpha, N, \tfrac{\alpha\bar{\varepsilon}}{\beta}\right) = \frac{N}{\left(\frac{\alpha\bar{\varepsilon}}{\beta}\right)^N} \int_0^{\frac{\alpha\bar{\varepsilon}}{\beta}} (1+t)^{\frac{-1}{\alpha}} t^{N-1}\,\mathrm{d}t. \quad (20)$$

The integral can be solved by repeated use of integration by parts. Subtracting (15) from (19) yields the following:

$$R_U - R_P = (\bar{p}-c)\bar{\varepsilon}\left[g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) - g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)\right].$$

The contour plot of $\frac{g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) - g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)}{g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)}$ in Figure 2 illustrates the relative decrease of mark-ups when switching from a UPA to a PABA. The plot is not very sensitive to the number of firms. As the ratio is positive over a wide range of parameters, it seems that $R_U - R_P \geq 0$. This inequality can indeed be proven mathematically.

[12] Analogously $R_p = \int_0^{\bar{\varepsilon}} f(\varepsilon)\varepsilon\hat{p}(\varepsilon)d\varepsilon + \left(1 - F\left(\bar{\varepsilon}\right)\right)\bar{\varepsilon}\hat{p}\left(\bar{\varepsilon}\right)$. By means of integration by parts and the definition of $\hat{p}(\varepsilon)$ it is straightforward to verify that this expression is equal to (13)

**Theorem 2.** With perfectly inelastic demand given by the Pareto distribution of the second kind and constant marginal costs, the expected revenue of symmetric firms in a pay-as-bid procurement auction is weakly lower than their expected revenue in a uniform-price procurement auction.

Proof: See [12].

Figure 2 shows that switching from a UPA to a PABA almost eliminates mark-ups in the area, for which both $\bar{\varepsilon} >> \beta$ and $\alpha < 1$. As can be seen in Figure 3, this area correspond to a very low risk of power shortage. In contrast, mark-ups are nearly unchanged for either large $\alpha$ (fat tail of the probability density function) or small $\bar{\varepsilon}$ (small capacity), both of which imply a high risk of power shortage.

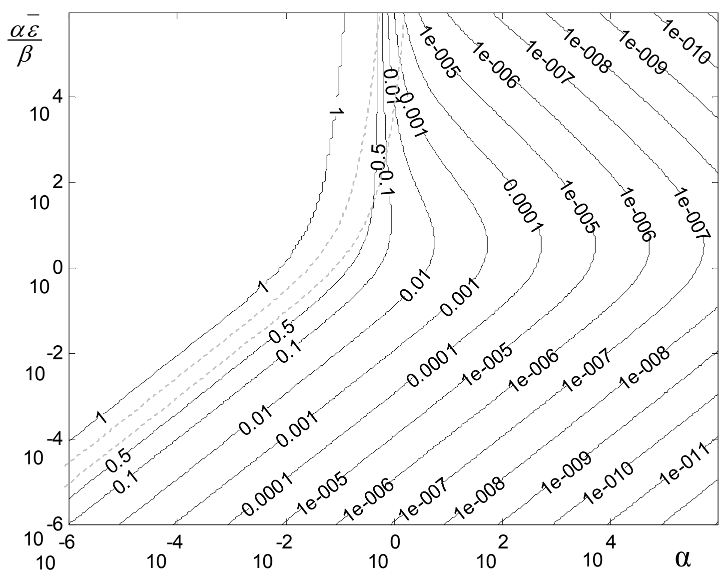


Figure 2. *Contour plot of* $\frac{g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) - g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)}{g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)}$ *when N=2. The grey dotted lines indicate a region with a risk of power shortage realistic for electric power markets.*

In most electric power markets, reasonable assumptions for the likelihood of power shortages range from once every hundred years to 100 times per year. This range roughly correspond to the per hour probability of a power shortage being $10^{-6}$ to 0.01; one hour is the normal length of the delivery period. This region is indicated in Figure 2. Switching to a pay-as-bid auction in an electric power market can reduce average mark-ups by 60 to 99 percent if $\alpha < 0.1$ and $N$=2; the lower the risk of power shortage, the larger the impact. The impact is somewhat reduced if the number of symmetric firms increases. For $N$=10 and $\alpha < 0.1$, switching to a pay-as-bid auction reduces average mark-ups between 20 and 90 percent. For $\alpha > 1$, which corresponds to a more convex probability density function, there is little gain from switching to a pay-as-bid auction in the electric power market, regardless of the number of symmetric firms.

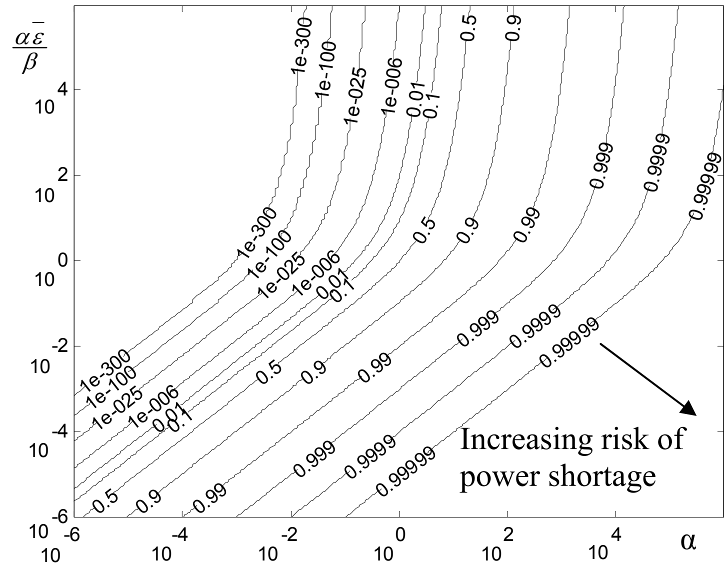


Figure 3. *Contour plot of 1-* $F(\bar{\varepsilon}) = \left(\frac{\alpha\bar{\varepsilon}}{\beta} + 1\right)^{\frac{-1}{\alpha}}$, *the probability of a power shortage.*

The intuition underlying the role of *α* in the comparison of PABAs and UPAs is as follows. Equilibrium bids in UPAs are not influenced by the probability distribution of demand [13,18]. Bids in PABAs are, however, sensitive to α. In particular, a smaller α makes low demand outcomes more likely. Intuitively, this increases the elasticity of residual demand for small $S_i$. [13] Thus mark-ups are lower for these units in a PABA in accordance with the inverse elasticity rule [26]. For small values of *α*, two effects drive down firms' expected revenues in the PABA; (i) lower mark-ups for low demand outcomes and (ii) an increased probability of low demand outcomes. In the UPA, only the second effect drives down firms' expected revenues. The same intuition may also explain why firms' expected revenues are lower in PABAs than UPAs for the Pareto distribution of the second kind, which is characterised by a decreasing probability density.

[13] Recall that $1 - F\left[\breve{S}_{-i}(p) + S_i\right]$ can be interpreted as residual demand of the marginal unit when firm *i* supplies $S_i$ units of power.

### 4.2 Non-decreasing marginal costs

In Section 4.1 it was shown that switching from a UPA to PABA reduces firms' revenues if marginal costs are constant and demand follows a Pareto distribution of the second kind. Using Theorem 2, this section demonstrates that the conclusion can be generalised to non-decreasing marginal costs.

The expected revenue in (14) is valid for non-decreasing marginal costs. The term related to the price cap can be rewritten in the same manner as (15):

$$R_P = \overline{\overline{p\varepsilon}} g_P\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right) + \int_0^{\overline{\varepsilon}} \frac{\beta^{\frac{1}{\alpha}}(\alpha\varepsilon+\beta)^{\frac{-1}{\alpha}}(N-1)\int_{\varepsilon}^{\overline{\varepsilon}} C'(u/N)(\alpha u+\beta)^{\frac{1-N}{\alpha N}-1}du}{N(\alpha\varepsilon+\beta)^{\frac{1-N}{\alpha N}}} d\varepsilon.$$

By reversing the order of integration [25], it can be shown that

$$R_p = \overline{\overline{p\varepsilon}} g_P\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right) + \frac{\beta^{\frac{1}{\alpha}}(N-1)}{N}\int_0^{\overline{\varepsilon}} C'(u/N)(\alpha u+\beta)^{\frac{1-N}{\alpha N}-1}\int_0^{u}(\alpha\varepsilon+\beta)^{\frac{-1}{\alpha N}} d\varepsilon du =$$

$$= \overline{\overline{p\varepsilon}} g_P\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right) + (N-1)\int_0^{\overline{\varepsilon}} C'(u/N)\underbrace{\left(\frac{\alpha u}{\beta}+1\right)^{\frac{1-N}{\alpha N}-1}\frac{\left(\frac{\alpha u}{\beta}+1\right)^{1-\frac{1}{\alpha N}}-1}{\alpha N-1}}_{h_P\left(\alpha, N, \frac{\alpha u}{\beta}\right)} du. \quad (21)$$

Similarly it follows from (18) and (19) that

$$R_U = \overline{\overline{p\varepsilon}} g_U\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right) + \beta^{\frac{1}{\alpha}}(N-1)\int_0^{\overline{\varepsilon}}(\alpha\varepsilon+\beta)^{\frac{-1}{\alpha}-1}\varepsilon^N\int_{\varepsilon}^{\overline{\varepsilon}}\frac{C'(u/N)du}{u^N} d\varepsilon =$$

$$= \overline{\overline{p\varepsilon}} g_U\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right) + \beta^{\frac{1}{\alpha}}(N-1)\int_0^{\overline{\varepsilon}}\frac{C'(u/N)}{u^N}\int_0^{u}(\alpha\varepsilon+\beta)^{\frac{-1}{\alpha}-1}\varepsilon^N d\varepsilon du = \quad (22)$$

$$= \overline{\overline{p\varepsilon}} g_U\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right) + (N-1)\int_0^{\overline{\varepsilon}} C'(u/N)\underbrace{\left(\frac{\alpha u}{\beta}\right)^{-N}\int_0^{\frac{\alpha u}{\beta}}\frac{(t+1)^{\frac{-1}{\alpha}-1}t^N}{\alpha} dt}_{h_U\left(\alpha, N, \frac{\alpha u}{\beta}\right)} du.$$

The integral in $h_U\left(\alpha, N, \frac{\alpha u}{\beta}\right)$ can be solved analytically by repeated integration by parts. Let $h_{\alpha N}(x) = h_U(\alpha, N, x) - h_P(\alpha, N, x)$. Equations (21) and (22) now imply that

$$\Delta R = R_U - R_P = \overline{\overline{p\varepsilon}}\left[g_U\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right) - g_P\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right)\right] + (N-1)\int_0^{\overline{\varepsilon}} C'(u/N) h_{\alpha N}\left(\frac{\alpha u}{\beta}\right) du. \quad (23)$$

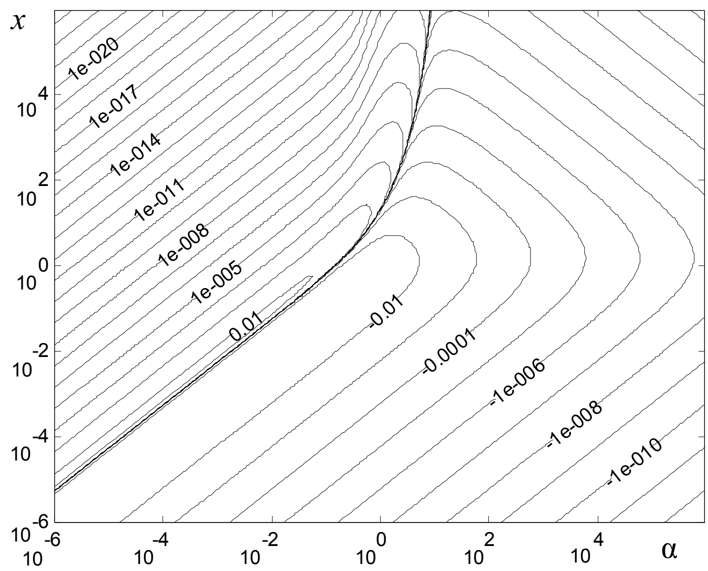


Figure 4. *Contour plot of* $h_{\alpha N}(x)$ for *N*=2.

Figure 4 presents a contour plot of $h_{\alpha N}(x)$. The levels in the contour plot are very sensitive to *N,* whilst the pattern is comparatively stable. The function $h_{\alpha N}(x)$ appears to have profile -|+ for *N≥2* and *x*≥0, which is verified mathematically in [12]. If $h_{\alpha N}\left(\frac{\alpha u}{\beta}\right)$ changes sign for $u < \bar{\varepsilon}$, let this point be denoted $u^*$, otherwise set $u^* = \bar{\varepsilon}$. Use (23) to calculate $\Delta R_1$ for the non-decreasing cost function $C_1(\varepsilon / N)$. Next calculate $\Delta R_2$ for the constant marginal cost $c_2 = C_1'(u^* / N)$. Compared to $C_1'(\varepsilon / N)$, $c_2$ puts a (weakly) higher weight on negative $h_{\alpha N}(x)$ and a (weakly) lower weight on positive $h_{\alpha N}(x)$. Thus

$$\Delta R_1 \geq \Delta R_2.$$

From Theorem 2 it follows that $\Delta R_2 \geq 0$. Thus $\Delta R_1 \geq 0$ and $R_U \geq R_P$ is true also for non-decreasing marginal costs. From the reasoning above we can also conclude that $\Delta R_1$ increases if the slope of $C_1''$ is increased while $C_1'(u^* / N)$ is kept constant.

**Theorem 3.** With perfectly inelastic demand given by the Pareto distribution of the second kind and non-decreasing marginal costs, the expected revenue of symmetric firms in a pay-as-bid procurement auction is weakly lower than their expected revenue in a uniform-price procurement auction.

Proof: See [12].

Recall that demand is assumed to be perfectly inelastic and, accordingly, independent of the auction design. Thus Theorem 3 implies that the demand-weighted average price is weakly lower in PABAs than in UPAs. Furthermore, because only symmetric equilibria are considered, the most cost-effective generators will be accepted in both auctions for any level of demand. Thus production costs are the same in both procurement auctions and average mark-ups are weakly lower in PABAs than UPAs.

It is obvious that $R_U$=$R_P$=0 when $\bar{\varepsilon}=0$, i.e. when market capacity is zero. It can also be shown that firms' total expected revenues are the same in both auctions under perfect competition and monopoly. The former is also proven by Federico & Rahman [8], but for a uniformly distributed demand.

**Theorem 4.** With perfectly inelastic demand given by the Pareto distribution of the second kind, non-decreasing marginal costs, and $N \to \infty$ or $N$=1, the expected revenue of symmetric firms in a pay-as-bid auction is identical to their expected revenue in a uniform-price auction.

Proof: See Appendix.

**5. EXAMPLE**

Assume $N$=2, $\alpha$=1, and linear marginal costs, $C'(x) \equiv \gamma x$. The marginal bid in the PABA for these parameter values can be calculated by means of integration by parts and (12):

$$\frac{p(\varepsilon)}{\beta} = \frac{\left(\frac{2\bar{p}}{\beta} + \frac{\gamma\bar{\varepsilon}}{\beta} + 2\gamma\right)\left(\frac{\bar{\varepsilon}}{\beta} + 1\right)^{-1/2}}{2\left(\frac{\varepsilon}{\beta} + 1\right)^{-1/2}} + \frac{\gamma\varepsilon}{2\beta} - \gamma\left(\frac{\varepsilon}{\beta} + 1\right)$$

The demand and price are normalised with respect to *β*. In the PABA, the average price as a function of demand (the equilibrium price) is:

$$\frac{\hat{p}(\varepsilon)}{\beta} = \frac{\int_0^{\varepsilon} p(x)dx}{\beta\varepsilon} = \left(\frac{2\bar{p}}{\beta} + \frac{\gamma\bar{\varepsilon}}{\beta} + 2\gamma\right)\left(\frac{\bar{\varepsilon}}{\beta}+1\right)^{-1/2} \frac{\left(\frac{\varepsilon}{\beta}+1\right)^{3/2} - 1}{3\varepsilon/\beta} + \frac{\gamma\varepsilon}{4\beta} - \gamma\left(\frac{\varepsilon}{2\beta}+1\right).$$

The equilibrium price in the UPA can be calculated by means of (17):

$$p_U(\varepsilon) = \frac{\bar{p}\varepsilon}{\bar{\varepsilon}} + \frac{\varepsilon\gamma}{2}\ln\left(\frac{\bar{\varepsilon}}{\varepsilon}\right) = \beta\left[\frac{\bar{p}\varepsilon/\beta^2}{\bar{\varepsilon}/\beta} + \frac{\varepsilon\gamma/\beta}{2}\ln\left(\frac{\bar{\varepsilon}/\beta}{\varepsilon/\beta}\right)\right].$$

Figure 5 shows $p_U(\varepsilon)$, $p(\varepsilon)$, and $\hat{p}(\varepsilon)$ for $\gamma$=0.02, $\frac{\bar{p}}{\beta} = 10^3$ and $\frac{\bar{\varepsilon}}{\beta} = 10^4$. The latter corresponds to a risk of power shortage $\approx 10^{-4}$. The equilibrium price in the uniform-price auction equals *C'(0)* at zero demand. This is true in general for symmetric SFE of UPAs [13]. The unit with the lowest marginal cost still contributes to profits, as it is paid the marginal bid for $\varepsilon$>0. It is also true in general that the lowest bid in the pay-as-bid auction is higher than *C'(0)*. If not, then the cheapest unit would not contribute to profits because accepted bids are always paid their bid. Therefore, the equilibrium price is higher in the PABA when demand is sufficiently small. In both procurement auctions, all units except for the one with the highest bid are offered below the price cap. Thus the equilibrium price in a PABA is always below the price cap. In the uniform-price auction, on the other hand, the equilibrium price equals the price cap when demand equals or exceeds the market capacity. Hence, the equilibrium price is lower in the PABA when demand is sufficiently high.

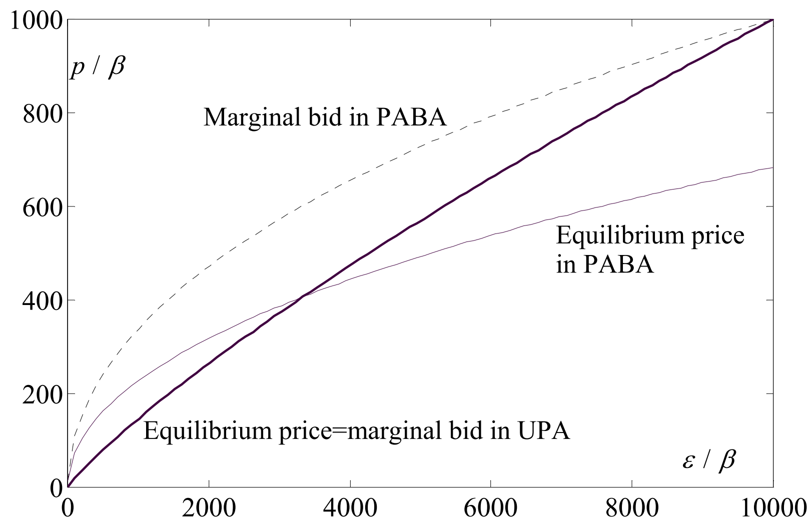


Figure 5. *Example for duopoly: prices as a function of demand are compared for the uniform-price auction (UPA) and pay-as-bid auction (PABA).*

## 6. CONCLUSIONS

The supply function equilibrium (SFE) framework for uniform-price auctions (UPAs) is similar to the organisation of most electricity markets and is often employed to model strategic bidding in such markets. SFE are also useful when analysing other divisible good auctions, such as treasury auctions with random non-competitive bids [27].

This paper derives a closed-form solution for the SFE of a pay-as-bid auction (PABA), the auction used in the balancing market of Britain. In the analysis, demand is assumed to be perfectly inelastic and firms symmetric. To rule out multiple equilibria, demand is assumed to exceed market capacity with a positive probability. Such events are unlikely but occur once per year or once per decade in real electricity markets. In the model, the risk of power shortage is allowed to be arbitrarily small. Moreover, the standard deviation of demand can be made arbitrarily small. Thus the risk of power shortage assumption does not necessarily contradict the common assumption that producers can forecast demand with a high accuracy. Unlike supply function equilibria of UPAs, pure strategy equilibria of PABAs do not always exist. In particular, it can be shown that a SFE of a PABA does not exist if there is a demand interval in which the hazard rate is locally upward sloping and marginal costs are sufficiently flat. However, a SFE always exists if the hazard rate of demand is monotonically decreasing and marginal costs are non-decreasing.

The equilibrium of a PABA is compared to the SFE of a UPA. In the comparison it is assumed that demand is given by the Pareto distribution of the second kind, for which the inverse of the hazard rate is linear and increasing. It can then be shown that the demand-weighted average price in the PABA is equal to or lower than the price in the UPA. Average prices are equal in the cases of a monopoly or perfect competition. An analogous calculation would show that the demand-weighted average price in a pay-as-bid sales auction, in which the supply of the auctioneer follows a Pareto distribution of the second kind, is (weakly) higher compared to a uniform-price sales auction. Thus the auctioneer would weakly prefer the pay-as-bid auction for positive as well as negative demand. For a probability density function with a low degree of convexity, switching from a UPA to a PABA will substantially reduce the average mark-up in electricity procurement auctions. With a high degree of convexity, the change in the average mark-up is negligible. That mark-ups are lower and consumer surplus higher in PABAs is in line with previous theoretical studies based on other assumptions [7,8,24,27]. The result contradicts the findings of an experimental study [22]. However, that study did not consider uncertain demand.

The equilibrium price — the average price as a function of demand — is higher in PABAs compared to UPAs when demand is sufficiently low, but lower when demand is sufficiently high. This seems to be in agreement with the experimental finding that price volatility is lower in PABAs than UPAs [22].

A general assumption of the analysis is that firms are risk-neutral. Introducing risk aversion does not change the SFE of a UPA, as firms receive the best price for every demand outcome, given the bids of competitors. A risk-averse firm in a PABA, however, would put less weight on high-demand outcomes when profits are high and more weight on low-demand outcomes when profits are low. Hence, given the bids of competitors, risk-averse firms decrease their bids to increase profits for low-demand outcomes. Intuitively this would also be true in equilibrium. It appears that with risk-averse bidders, the advantages of PABAs are likely to increase. Another advantage of PABAs is that the risk for tacit collusion is lowered compared to UPAs. This is shown by both Fabra [6] and Klemperer [19].

The paper focuses on the case of symmetric firms. Analogous to [14], it should be possible to analytically derive supply function equilibria of PABAs for firms with identical constant marginal costs but asymmetric capacities. The unique equilibrium is expected to be piece-wise symmetric. For general cost functions, asymmetric equilibria could be calculated numerically as in [15]. Results by Fabra et al. [7] and Son et al. [24] suggest that the auctioneer would prefer PABA to UPA also for asymmetric firms.

This paper and Fabra et al. [7] show that there is a larger risk to lack a pure strategy equilibrium in the PABA. This could be a disadvantage for both consumers and producers, as prices might vary unpredictably in a market without a pure strategy equilibrium. Kahn et al. [17] also point out that in a UPA it is optimal for small firms to simply bid their marginal costs while in a PABA, all firms must forecast market prices if they are to receive any contributions to profits. This introduces an additional fixed cost for small firms which could be disadvantageous to competition in the long-run.

Lastly, as small imbalances are more likely than large imbalances, the Pareto distribution of the second kind is a more reasonable representation of the uncertain demand in balancing markets than the uniform distribution employed by Federico & Rahman [8]. Nonetheless, an interesting topic for future research is to compare PABAs and UPAs for other distributions. The normal distribution is a natural choice. However, as its hazard rate is increasing, one has to make sure that marginal costs are sufficiently steep to ensure the existence of a SFE.

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## APPENDIX

**Proof of Theorem** 1

It follows from (8) that

$$\eta(p,S_i) = G\left[\breve{S}_{-i}(p) + S_i\right] - \breve{S}_{-i}{}'(p)\left(p - C'(S_i)\right). \tag{24}$$

The equality in (4) is valid for an interval of prices. Thus

$$G\left[\breve{S}(p)\right] - \breve{S}_{-i}{}'(p)\left[p - C'\left(\breve{S}_i(p)\right)\right] \equiv 0.$$

The expression above can be used to eliminate $\breve{S}_{-i}{}'(p)$ from (24). Accordingly, (24) can be written on the following form:

$$\eta(p,S_i)=\frac{\left(p-C'\left(\breve{S}_i(p)\right)\right)G\left[\breve{S}_{-i}(p)+S_i\right]-G\left[\breve{S}(p)\right]\left(p-C'(S_i)\right)}{p-C'\left(\breve{S}_i(p)\right)}=$$
$$=\frac{\left[p-C'\left(\breve{S}_i(p)\right)\right]\left\{G\left[\breve{S}_{-i}(p)+S_i\right]-G\left[\breve{S}(p)\right]\right\}+G\left[\breve{S}(p)\right]\left(C'(S_i)-C'\left(\breve{S}_i(p)\right)\right)}{p-C'\left(\breve{S}_i(p)\right)}. \tag{25}$$

It follows from (4) that $p>C'\left(\breve{S}_i\right)$ for increasing supply functions. Further, marginal costs are non-decreasing and *G(x)*>0, as the hazard rate is never negative. To prove claim ii) consider the case when *G(x)* is monotonically increasing. If $p\in\left[\breve{p}(0),p^*\right)$ then $S_i>\breve{S}_i(p)\Rightarrow$ $C'(S_i)>C'\left(\breve{S}_i(p)\right)$ and $G\left(\breve{S}_{-i}(p)+S_i\right)>G\left(\breve{S}(p)\right)$. Thus it follows from (25) that $\eta(p,S_i)>0$ for $p\in\left[\breve{p}(0),p^*\right)$. Analogously, it can be proven that $\eta(p,S_i)<0$ for $p\in\left(p^*,\bar{p}\right]$ The two conditions are fulfilled for all $S_i\in\left[0,\bar{\varepsilon}/N\right]$, which proves claim ii.

There is a local minimum if $\eta\left(p^*-,S_i\right)<0$ and $\eta\left(p^*+,S_i\right)>0$. It follows from (25) that a local minimum occurs if $\left[p-C'\left(\breve{S}_i(p)\right)\right]G'\left[\breve{S}(p)\right]+G\left[\breve{S}(p)\right]C''\left(\breve{S}_i(p)\right)<0$, which proves claim iii). Analogously there is a local maximum if $\left[p-C'\left(\breve{S}_i(p)\right)\right]G'\left[\breve{S}(p)\right]+G\left[\breve{S}(p)\right]C''\left(\breve{S}_i(p)\right)>0$, which proves claim i) □

**Proof of Theorem 4**

It is known from (23) that

$$R_U-R_P=\overline{\overline{p\varepsilon}}\left[g_U\left(\alpha,N,\frac{\alpha\bar{\varepsilon}}{\beta}\right)-g_P\left(\alpha,N,\frac{\alpha\bar{\varepsilon}}{\beta}\right)\right]+$$
$$+(N-1)\int_0^{\bar{\varepsilon}}C'(u/N)\left[h_U\left(\alpha,N,\tfrac{\alpha u}{\beta}\right)-h_P\left(\alpha,N,\tfrac{\alpha u}{\beta}\right)\right]du.$$

Thus the auctions have the same expected revenue under perfect competition, if $\lim_{N\to\infty}g_U\left(\alpha,N,\frac{\alpha\bar{\varepsilon}}{\beta}\right)=\lim_{N\to\infty}g_P\left(\alpha,N,\frac{\alpha\bar{\varepsilon}}{\beta}\right)$ and $\lim_{N\to\infty}(N-1)h_U\left(\alpha,N,\frac{\alpha u}{\beta}\right)=\lim_{N\to\infty}(N-1)h_P\left(\alpha,N,\frac{\alpha u}{\beta}\right)$.

These two equalities are proven below.

It follows from (16) that

$$\lim_{N\to\infty} g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) = \lim_{N\to\infty} \frac{\left(1+\frac{\alpha\bar{\varepsilon}}{\beta}\right)^{\frac{-1}{\alpha N}+1} - 1}{\left(1-\frac{1}{\alpha N}\right)\frac{\alpha\bar{\varepsilon}}{\beta}\left(\frac{\alpha\bar{\varepsilon}}{\beta}+1\right)^{\frac{N-1}{\alpha N}}} = \frac{1}{\left(\frac{\alpha\bar{\varepsilon}}{\beta}+1\right)^{\frac{1}{\alpha}}}.$$

Now consider $\lim_{N\to\infty} g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)$. It is known that $\lim_{x\to\infty}\frac{x^p}{a^x} = 0$ if $a > 1$ [21], which implies that $\lim_{N\to\infty}\frac{Nt^{-1}}{\left(\frac{\alpha\bar{\varepsilon}}{\beta t}\right)^N} = 0$ if $t < \frac{\alpha\bar{\varepsilon}}{\beta}$. Thus only $t$ infinitesimally close to $\frac{\alpha\bar{\varepsilon}}{\beta}$ will contribute to the value of the integral in (20).

$$\lim_{N\to\infty} g_U\left(\alpha, N, \tfrac{\alpha\bar{\varepsilon}}{\beta}\right) = \lim_{N\to\infty}\frac{N\int_0^{\frac{\alpha\bar{\varepsilon}}{\beta}}(1+t)^{\frac{-1}{\alpha}}t^{N-1}dt}{\left(\frac{\alpha\bar{\varepsilon}}{\beta}\right)^N} = \lim_{N\to\infty}\frac{N\left(1+\frac{\alpha\bar{\varepsilon}}{\beta}\right)^{\frac{-1}{\alpha}}}{\left(\frac{\alpha\bar{\varepsilon}}{\beta}\right)^N}\int_0^{\frac{\alpha\bar{\varepsilon}}{\beta}}t^{N-1}dt = \left(1+\tfrac{\alpha\bar{\varepsilon}}{\beta}\right)^{\frac{-1}{\alpha}} =$$

$$= \lim_{N\to\infty} g_P\left(\alpha, N, \tfrac{\alpha\bar{\varepsilon}}{\beta}\right).$$

Using (21) it can also be shown that

$$\lim_{N\to\infty}(N-1)h_P\left(\alpha, N, \frac{\alpha u}{\beta}\right) = \lim_{N\to\infty}(N-1)\left(\frac{\alpha u}{\beta}+1\right)^{\frac{1-N}{\alpha N}-1}\frac{\left(\frac{\alpha u}{\beta}+1\right)^{1-\frac{1}{\alpha N}}-1}{\alpha N-1} = \frac{\left(\frac{\alpha u}{\beta}+1\right)^{\frac{-1}{\alpha}-1}u}{\beta}.$$

Now consider $\lim_{N\to\infty}(N-1)h_U\left(\alpha, N, \frac{\alpha u}{\beta}\right)$. With the same argument as above, $\lim_{N\to\infty}\frac{(N-1)t^N}{\left(\frac{\alpha u}{\beta}\right)^N} = 0$ if $t < \frac{\alpha u}{\beta}$. Thus only $t$ infinitesimally close to $\frac{\alpha u}{\beta}$ will contribute to the value of the integral in (22).

$$\lim_{N\to\infty}(N-1)h_U\left(\alpha, N, \frac{\alpha u}{\beta}\right) = \lim_{N\to\infty}(N-1)\left(\frac{\alpha u}{\beta}\right)^{-N}\int_0^{\frac{\alpha u}{\beta}}\frac{(t+1)^{\frac{-1}{\alpha}-1}t^N dt}{\alpha} =$$

$$= \lim_{N\to\infty}(N-1)\left(\tfrac{\alpha u}{\beta}+1\right)^{\frac{-1}{\alpha}-1}\left(\frac{\alpha u}{\beta}\right)^{-N}\int_0^{\frac{\alpha u}{\beta}}\frac{t^N dt}{\alpha} = \frac{\left(\frac{\alpha u}{\beta}+1\right)^{\frac{-1}{\alpha}-1}u}{\beta} = \lim_{N\to\infty}(N-1)h_p\left(\alpha, N, \frac{\alpha u}{\beta}\right).$$

This analysis has assumed that $N \geq 2$, which excludes the case of a monopoly. In such a case, the inelastic auctioneer must buy from the monopolist who will offer all units at the price cap in both auctions (or arbitrarily close to the price cap).[14] Thus, expected revenue is identical in the two auctions. □

[14] Supply functions are assumed to fulfil $S'(p) < \infty$.