Risk in multistage stochastic programmes

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EPOC

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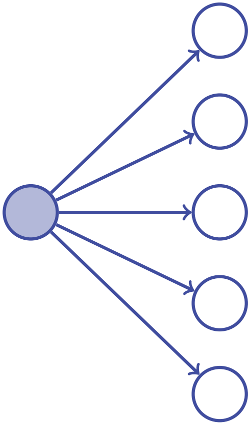
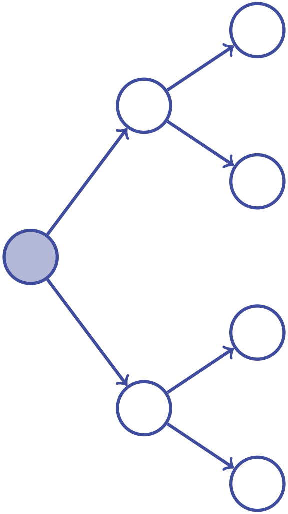




Two stage risk

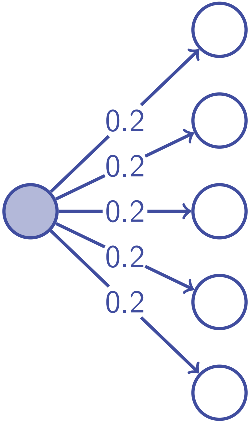
Multistage risk

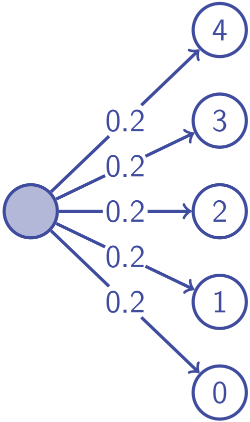
Algorithm

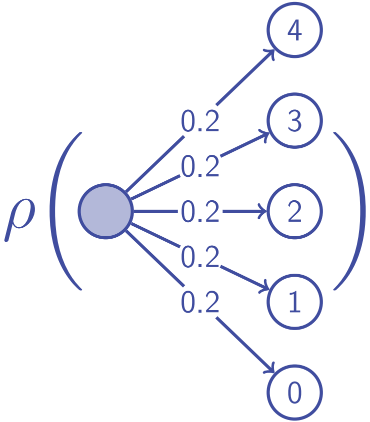
Results

Risk is present in a random variable if it has undesirable scenarios.

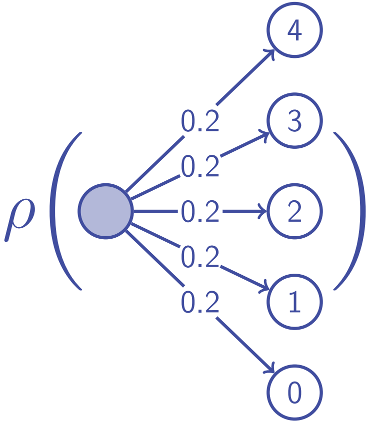
$$\rho : \mathcal{Z} \mapsto \mathbb{R}$$

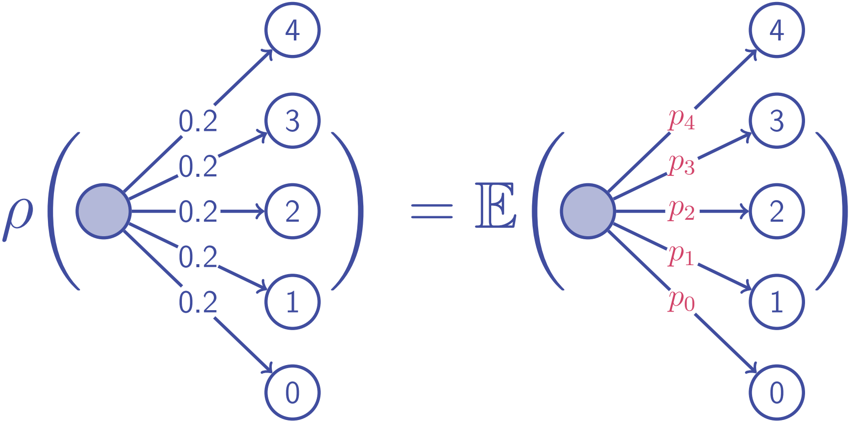

0.2
0.2
0.2
0.2
0.2


4
3
2
1
0
0.2
0.2
0.2
0.2
0.2


ρ
0.2
0.2
0.2
0.2
0.2
4
3
2
1
0

$$\rho(Z) = \min_{\mathbb{P} \in \mathbb{Q}_\rho} \mathbb{E}_{\mathbb{P}}[Z]$$


ρ
0.2
0.2
0.2
0.2
0.2
4
3
2
1
0


ρ
0.2
0.2
0.2
0.2
0.2
4
3
2
1
0
= 𝔼
p4
p3
p2
p1
p0
4
3
2
1
0

$$\min(Z) = \min_{\mathbb{P} \in \mathbb{Q}_{\min}} \mathbb{E}_{\mathbb{P}}[Z]$$

where $\mathbb{Q}_{\min}$ is every distribution

$$\mathbb{E}(Z) = \min_{\mathbb{P} \in \mathbb{Q}_{\mathbb{E}}} \mathbb{E}_{\mathbb{P}}[Z]$$

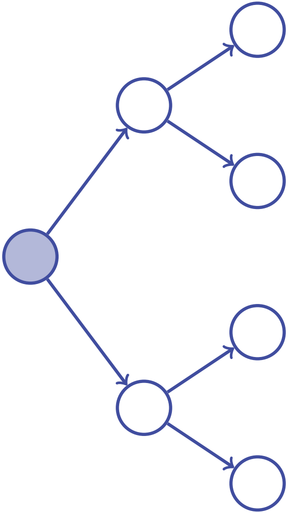
where $\mathbb{Q}_{\mathbb{E}}$ is the singleton

Coherent risk measures are a set of rules about how to modify probabilities to perceive the worst cases as more likely.

$$\rho(Z(x)) = \min_{\mathbb{P} \in \mathbb{Q}_\rho} \mathbb{E}_{\mathbb{P}}[Z(x)]$$

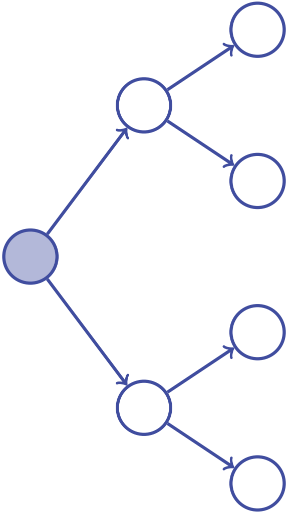
$$\max_{x} \rho(Z(x)) = \max_{x} \min_{\mathbb{P} \in \mathbb{Q}_\rho} \mathbb{E}_{\mathbb{P}}[Z(x)]$$

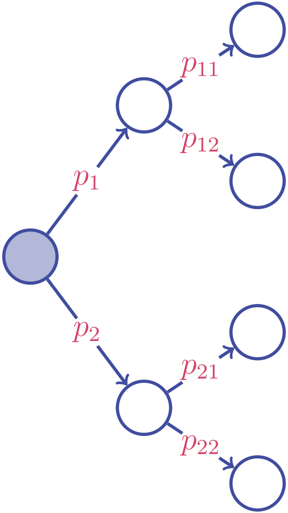
How can these ideas be extended into multistage problems?



$$\Psi : \mathcal{Z}_1 \times \mathcal{Z}_2 \mapsto \mathbb{R}$$

We can conceive a notion of **multistage coherency**.




p_1
p_2
p_{11}
p_{12}
p_{21}
p_{22}

$$\Psi(Z_1, Z_2|Z_1) =$$

$$\min_{\mathbb{P}_0 \in \mathbb{Q}_0, \mathbb{P}_{Z_1} \in \mathbb{Q}_{Z_1}} \mathbb{E}_{\mathbb{P}_0}\left[Z_1 + \mathbb{E}_{\mathbb{P}_{Z_1}}[Z_2|Z_1]\right]$$

We must also have the property of time consistency.

Plenty of debate about what time consistency actually is!

$$(\bar{x}_1, \bar{x}_2) = \arg\min \mathbb{O}(x_1, x_2)$$

$$(\bar{x}_1, \bar{x}_2) = \arg\min \mathbb{O}(x_1, x_2)$$

$$(x_1^*, x_2^*) = \arg\min \mathbb{O}(\bar{x}_1, x_2)$$

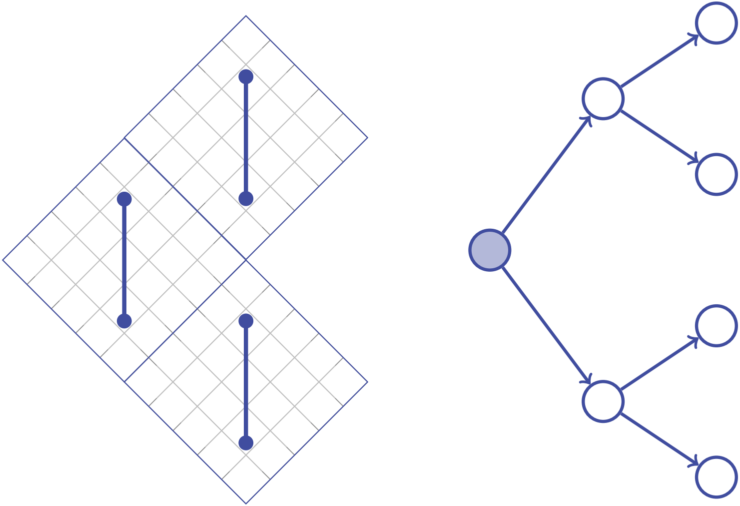
Dynamic programming equations give us time consistency.

Nested risk measures are coherent and time consistent.

Slight modification to SDDP to incorporate nested risk measures.

$$\Psi(Z_1, Z_2|Z_1) =$$

$$\min_{\mathbb{P}_0 \in \mathbb{Q}_0} \mathbb{E}_{\mathbb{P}_0}\Big[Z_1 + \min_{\mathbb{P}_{Z_1} \in \mathbb{Q}_{Z_1}} \mathbb{E}_{\mathbb{P}_{Z_1}}[Z_2|Z_1]\Big]$$

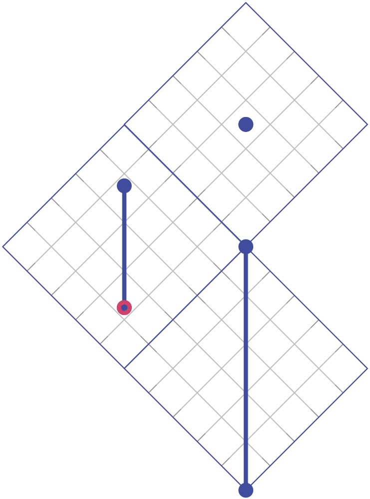


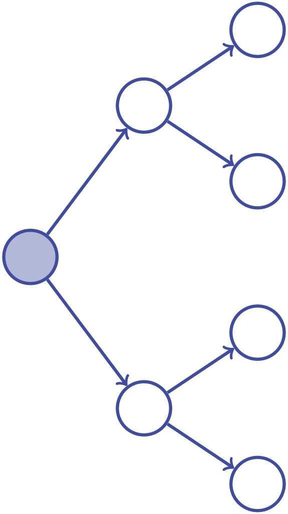
Multistage risk | Nested risk measures

$$\Psi(Z_1, Z_2|Z_1) = \min_{\mathbb{P}\in\mathscr{Q}} \mathbb{E}_{\mathbb{P}}[Z_1 + Z_2|Z_1]$$

$$\mathscr{Q} = \mathbb{Q}_0 \circ \mathbb{Q}_{Z_1}$$

Earlier work showed how $\mathscr{Q}$ can be constructed so $\min_{\mathbb{P} \in \mathscr{Q}} \mathbb{E}_{\mathbb{P}}[Z_1 + Z_2 | Z_1] = \mathrm{CVaR}(Z_1 + Z_2 | Z_1)$.





$$V_{\omega}^{t}(\beta, x) = \max_{a \in \mathcal{X}(x)} \min_{\mathbb{P} \in \mathbb{Q}(\beta)} g(a) + \mathbb{E}_{\mathbb{P}}\left[V_{\omega'}^{t+1}(\beta_{\omega'}, x_{\omega'}^{t+1})\right]$$

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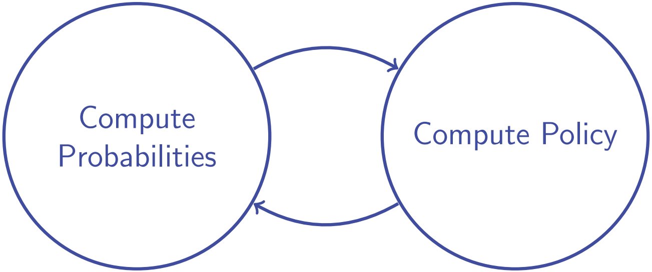
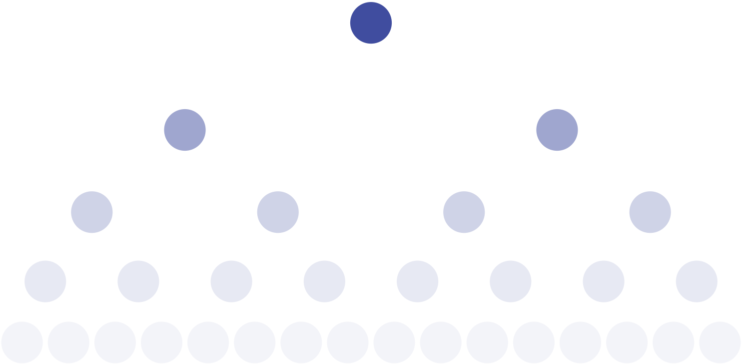
$$V_{\omega}^{t}(\beta, x) = \max_{a \in \mathcal{X}(x)} \min_{\mathbb{P} \in \mathbb{Q}(\beta)} g(a) + \mathbb{E}_{\mathbb{P}}\left[V_{\omega'}^{t+1}(\beta_{\omega'}, x_{\omega'}^{t+1})\right]$$

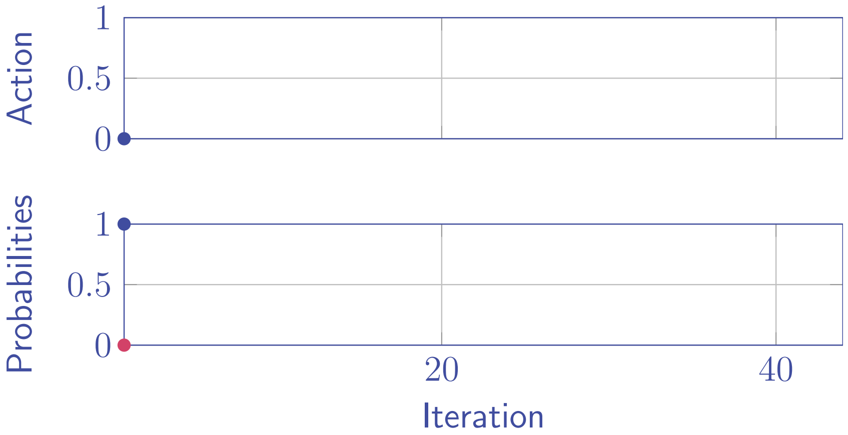

Compute
Probabilities
Compute Policy

Converge on policy and probabilities at the same time!

Sampling algorithm that constructs linear estimates of value functions.

Useful for: multistage zero-sum games, multistage robust optimisation problems.





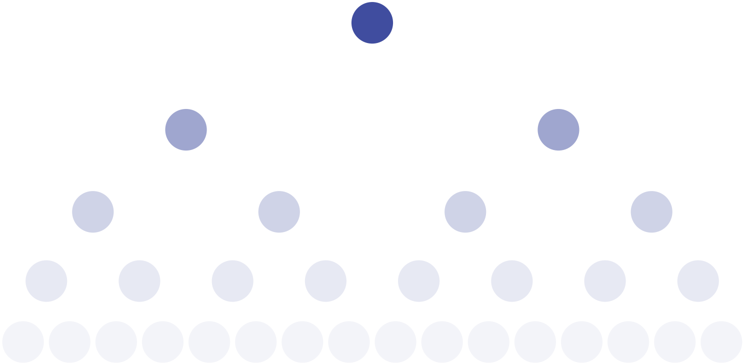
Action
1
0.5
0
Probabilities
1
0.5
0
20
40
Iteration

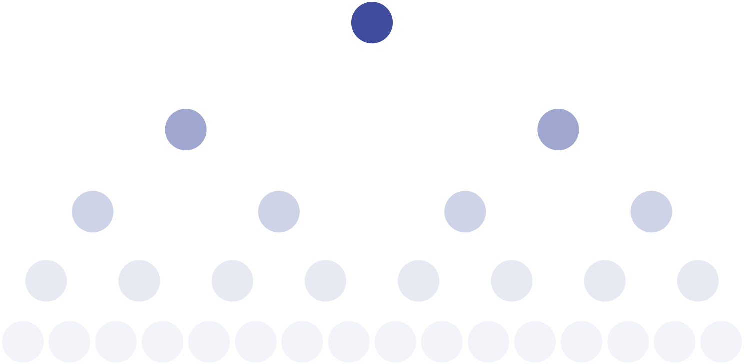
Two stage risk

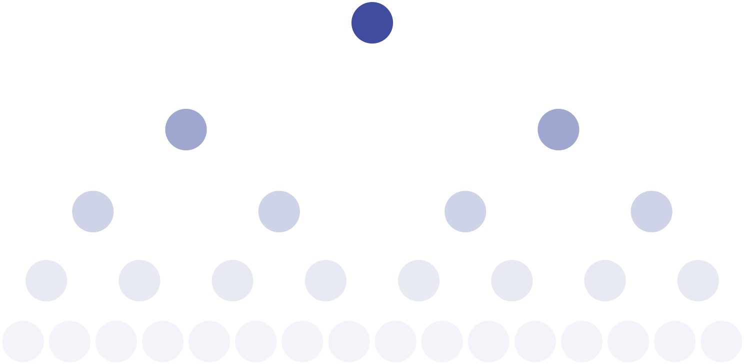
Multistage risk

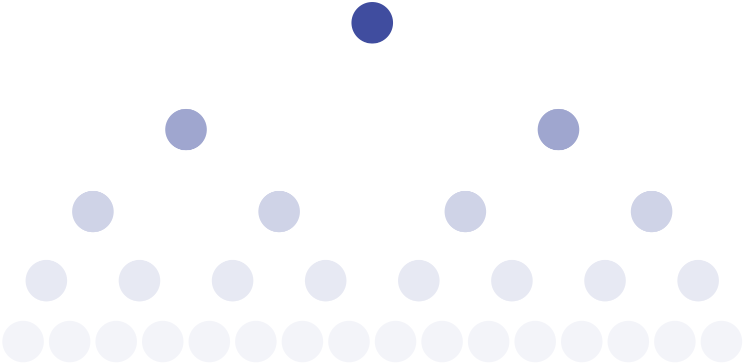
Algorithm

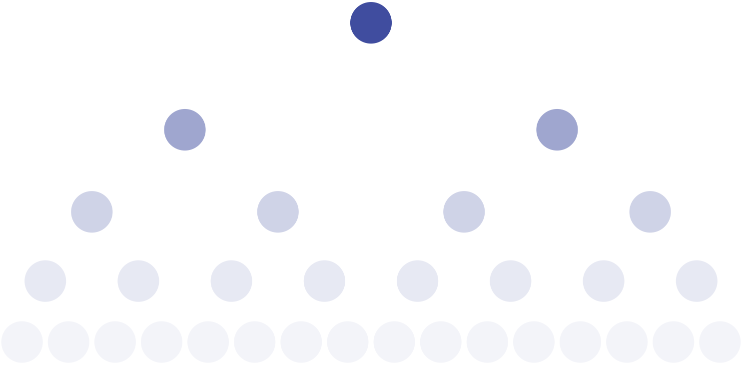
Results











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